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SYMMETRIC CURVATURE TENSOR

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ABSTRACT. Recently, we have used the symmetric bracket of vector fields, and developed the notion of the symmetric derivation. Using this machinery, we have defined the concept of symmetric curvature. This concept is natural and is related to the notions divergence and Laplacian of vector fields. This concept is also related to the derivations on the algebra of symmetric forms which has been discussed by the authors. We introduce a new class of geometric vector fields and prove some basic facts about them. We call these vector fields affinewise. By contraction of the symmetric curvature, we define two new curvatures which have direct relations to the notions of divergence, Laplacian, and the Ricci tensor.

1. Introduction

Symmetric and alternating tensors are in parallel, and have essential roles in differential geometry. For instance, the symmetric bracket was introduced and named *symmetric product* by Crouch [1]. It also arises in the work of Lewis and Murray [5] on a class of mechanical control systems. Alternating tensors have a well developed theory and most of the theorems in geometry can be stated in the language of differential forms. In [4], by definition of a whole string of new geometric concepts, such as derivations of symmetric forms, the symmetric Lie derivative, the

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symmetric differential of symmetric forms and the Frölicher-Nijenhuis bracket of symmetric forms, we have developed a convenient calculus for symmetric tensors in parallel to the calculus of differential forms.

The curvature of a connection is a well developed notion and depends on the calculus of differential forms. Now, using the calculus of symmetric tensors, it is natural to define a similar concept of curvature which we call "symmetric curvature".

This concept is associated with the notions of "divergence" and "Laplacian" of vector fields. These notions are also related to the derivations on the algebra of symmetric forms which have been discussed in [4].

Furthermore, we define a new type of vector fields, the so-called affinewise vector fields, and show that if the value of any affinewise vector filed in a point and its covariant derivatives in any direction at the point are given, then it is determined everywhere.

Next, by contraction of the symmetric curvature, we define the form curvature and the vector curvature along a vector field which have direct relations to the notions of divergence, Laplacian, and the Ricci tensor.

Using these new concepts, we give interesting characterizations of the harmonic vector fields, Killing vector fields, affine vector fields and geodesic vector fields.

2. Symmetric Forms and Associated Concepts

In this section, we define the k-symmetric forms, k-symmetric forms with values in a vector bundle, the insertion operator and other related concepts.

Let M be a C^{∞} manifold and TM be its tangent bundle. Let also $\bigvee^k (TM)^*$ be the vector bundle of symmetric covariant tensors of degree k over M. The sections of $\bigvee^k (TM)^*$ are called k-symmetric forms and they span a space denoted by $S^k(M)$. The set of all symmetric forms, i.e., $S(M) := \bigoplus_{k>0} S^k(M)$, with the symmetric product \lor given by

$$(\omega \lor \eta)(X_1, \dots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}),$$

where, $\omega \in S^k(M), \eta \in S^l(M)$, is a graded algebra.

If E is a vector bundle on M, then the sections of the vector bundle $\bigvee^k (TM)^* \bigotimes E$ are called k-symmetric forms with values in E and are denoted by $S^k(M, E)$. The set of all symmetric forms with values in E,

i.e., $S(M, E) := \bigoplus_{k \ge 0} S^k(M, E)$, with the symmetric product \lor defined above, in which $\omega \in S^k(M)$ and $\eta \in S^l(M, E)$, is a (graded) S(M)-module.

Let $U \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the space of vector fields on M. The insertion operator $i_U : S^k(M) \longrightarrow S^{k-1}(M)$ is a linear map given by

$$i_U\omega(X_1,\ldots,X_{k-1})=\omega(U,X_1,\ldots,X_{k-1}),$$

where, $\omega \in S^k(M)$ and $X_1, \ldots, X_{k-1} \in \mathfrak{X}X(M)$.

This operator can be defined on vector valued symmetric forms as follows:

$$i_U \Phi(X_1, \ldots, X_{k-1}) = \Phi(U, X_1, \ldots, X_{k-1}),$$

where, $\Phi \in S^k(M, E)$ and $X_1, \ldots, X_{k-1} \in \mathfrak{X}(M)$.

For any decomposable vector valued symmetric form $\omega \otimes X \in S^k(M, E)$, we have

$$i_U(\omega\otimes X)=(i_U\omega)\otimes X.$$

A linear map $D : S(M) \longrightarrow S(M)$ is said to be of degree k, if $D(S^{l}(M)) \subset S^{k+l}(M)$, and D is said to be a derivation of degree k, if furthermore,

$$D(\omega \lor \eta) = D\omega \lor \eta + \omega \lor D\eta,$$

for any $\omega, \eta \in S(M)$.

Let $Der_k(S(M))$ be the linear space of all derivations of degree k and let $Der(S(M)) := \bigoplus_{k\geq 0} Der_k(S(M))$. A derivation D is called algebraic, if $D|_{S^0(M)} = 0$.

If D_1 and D_2 are derivations of degrees k and l, respectively, then $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ is a derivation of degree k+l. A derivation is completely determined by its effect on $S^0(M) = C^{\infty}(M)$ and $S^1(M)$.

Definition 2.1. For any $\Phi \in S^{k+1}(M, TM)$, the insertion operator is the linear map $i(\Phi) : S^{l}(M) \to S^{k+l}(M)$, defined by

$$(i(\Phi)\omega)(X_1,...,X_{k+l}) = \frac{1}{(l-1)!(k+1)!} \sum_{\sigma \in S_{k+l}} \omega(\Phi(X_{\sigma(1)},...,X_{\sigma(k+1)}),X_{\sigma(k+2)},...,X_{\sigma(k+l)}),$$

where, $l \ge 1$, and $i(\Phi)f = 0$, for any $f \in C^{\infty}(M)$.

We note that $i(\Phi)\omega = \omega \circ \Phi$, for any $\omega \in S^1(M)$. Hence, if $i(\Phi) = 0$, then $\Phi = 0$. It is not difficult to show that for $\eta \otimes U \in S^{k+1}(M, TM)$ and $\omega \in S^l(M)$, we have

$$i(\eta \otimes U)\omega = \eta \vee i_U\omega.$$

Hence, $i(\Phi)$ (for $\Phi \in S^k(M, TM)$) is a derivation of degree k - 1 on S(M). Moreover, every algebraic derivation of degree k on S(M) is an insertion of a unique TM-valued (k + 1)-symmetric form [4].

Let ∇ be a torsion-free connection on M. Since $2\nabla_X Y$ is a bilinear map with respect to vector fields X and Y, it can be written as the sum of its symmetric and antisymmetric parts as follows:

$$2\nabla_X Y = (\nabla_X Y + \nabla_Y X) + (\nabla_X Y - \nabla_Y X) = \nabla_X Y + \nabla_Y X + [X, Y].$$

The symmetric bracket of the two vector fields X and Y on M is denoted by $[X, Y]^s$ and is defined as follows:

$$[X,Y]^s = \nabla_X Y + \nabla_Y X.$$

Note that for $X, Y \in \mathfrak{X}(M)$ and $f \in C^{\infty}(M)$, we have

$$[fX,Y]^s = f[X,Y]^s + Y(f)X.$$

Definition 2.2. Let ∇ be a linear connection on M. A vector field X is called a geodesic vector field if its integral curves are geodesics.

Locally, geodesic vector fields exist on any manifold. In fact, for every point $p \in M$ and $v \in T_p M$, there exists a local geodesic vector field X that is defined on a neighborhood of p in which $X_p = v$.

A vector field X is a geodesic field if and only if $[X, X]^s = 2\nabla_X X = 0$. For example, geodesic vector fields on \mathbb{R}^n are constant vector fields. On a Lie group with the connection $\nabla_X Y = \frac{1}{2}[X, Y]$ for left-invariant vector fields X and Y, the left-invariant vector fields are geodesic vector fields.

The symmetric Lie derivative along a vector field X is the linear map $L_X^s : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$, defined by $L_X^s Y = [X, Y]^s$. For $f \in C^{\infty}(M)$, $\omega \in S^k(M)$, and $X_1, \ldots, X_k \in \mathfrak{X}(M)$, we set

$$(L_X^s\omega)(X_1,\ldots,X_k) = X\omega(X_1,\ldots,X_k) - \sum_{i=1}^k \omega(X_1,\ldots,L_X^sX_i,\ldots,X_k),$$

and $L_X^s f = X(f)$. Then, it is obvious that $L_X^s \in Der_0(S(M))$.

Proposition 2.3. [4] Let L_X and L_X^s be respectively the Lie derivative and the symmetric Lie derivative along the vector field X with respect to a connection ∇ on M. Then, $2\nabla_X = L_X + L_X^s$.

Let ∇ be a torsion free connection on M. The symmetric differential is the derivation $d^s: S(M) \longrightarrow S(M)$ of degree 1, defined by

$$(d^{s}\omega)(X_{1},\ldots,X_{k+1}) = \sum_{i=1}^{k+1} X_{i}\omega(X_{1},\ldots,\hat{X}_{i},\ldots,X_{k+1}) - \sum_{i$$

and $d^s f = df$, where, $f \in C^{\infty}(M)$, $\omega \in S^k(M)$ and $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$.

From the above definition, we deduce the following results.

Lemma 2.4. [4] Let d^s be the symmetric differential of a torsion free connection ∇ . Suppose that $\omega \in S^k(M)$ and $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$. Let $\{E_i\}_{i=1}^n$ be a local basis of vector fields and $\{\omega^i\}_{i=1}^n$ be its dual basis. Then,

(i) $(d^s\omega)(X_1,\ldots,X_{k+1}) = \sum_{i=1}^{k+1} (\nabla_{X_i}\omega)(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}),$ (ii) $d^s\omega = \sum_{i=1}^n \omega^i \vee \nabla_{E_i}\omega.$

The following results were proved in [4].

1) If (M, g) is a Riemannian manifold with the Levi-Civita connection ∇ , then 1-form ω is Killing if and only if $d^s \omega = 0$ (ω is Killing if the vector field ω^{\sharp} is Killing, where $g(\omega^{\sharp}, X) = \omega(X)$).

2) If ∇ and $\overline{\nabla}$ are two torsion-free connections with symmetric differentials d^s and \overline{d}^s , respectively, and $\overline{\nabla} = \nabla + \Phi$, for $\Phi \in S^2(M, TM)$, then $\overline{d}^s = d^s - 2i(\Phi)$.

3) If ∇ is a torsion-free connection on M with the symmetric differential d^s and X is a vector field, then on the algebra of the symmetric forms S(M), we have $[i_X, d^s] = L_X^s$.

Now, we give the following theorem and corollaries for d^s .

Theorem 2.5. Let ω be a 1-form on M. Then, $d^s\omega = 0$ if and only if, for all geodesic α , $\omega(\alpha'(t))$ is constant.

Proof. Let $d^s \omega = 0$. If α is an integral curve of geodesic vector field U, then we obtain:

$$0 = d^{s}\omega(\alpha', \alpha') = d^{s}\omega(U, U) = 2U.\omega(U) - \omega([U, U]^{s})$$
$$= 2U.\omega(U) = 2\alpha'.\omega(\alpha') = 2\frac{d}{dt}\omega(\alpha').$$

Conversely, let $p \in M$, $v \in T_pM$ and α be a geodesic on M such that $\alpha(0) = p$, $\alpha'(0) = v$. Then, we have

$$(d^{s}\omega)_{p}(v,v) = 2U.\omega(U) - \omega([U,U]^{s}) = (2\alpha'(t).\omega(\alpha'(t)))|_{t=0} = 0.$$

Corollary 2.6. Let $f \in C^{\infty}(M)$ be a function. Then, $d^{s}(d^{s}f) = 0$ if and only if, for all geodesic α , we have $f(\alpha(t)) = at + b$.

Proof. From the above theorem, $d^s(d^s f) = 0$, if and only if, for all geodesic α , we have $(d^s f)(\alpha'(t)) = a$, where a is constant. Also, we have $(d^s f)(\alpha'(t)) = (f \circ \alpha)'(t)$. Hence, there exists a constant b such that $f(\alpha(t)) = at + b$.

The following corollary is also deduced from Theorem 2.5.

Corollary 2.7. A vector field X on a manifold M is Killing if and only if, for all geodesic α , $\langle X, \alpha'(t) \rangle$ is constant.

By considering a fixed connection on M, we can define the *symmetric* Lie derivative, along a TM-valued symmetric form, as follows:

$$L^s_{\Phi} = [i_{\Phi}, d^s], \quad \forall \Phi \in S(M, TM).$$

In [4], we proved that if $\eta \otimes X$ is a decomposable symmetric form and $\omega \in S(M)$, then

$$L^s_{\eta\otimes X}\omega = \eta \vee L^s_X\omega - d^s\eta \vee i_X\omega.$$

Theorem 2.8. [4] Let ∇ be a torsion-free connection on M. Every derivation $D \in Der_k(S(M))$ can be uniquely written in the form $D = i(\Phi) + L_{\Psi}^s$, for some $\Phi \in S^{k+1}(M, TM)$ and $\Psi \in S^k(M, TM)$. Moreover, Ψ is independent of ∇ .

It is not difficult to show that D is algebraic if and only if $\Psi = 0$ and $D = d^s$ if and only if $\Psi = 1_{TM}$ and $\Phi = 0$ (see [4]).

Let $\Phi \in S^k(M, TM)$ and $\Psi \in S^l(M, TM)$ be two symmetric forms. Then, $[L^s_{\Phi}, L^s_{\Psi}]$ is a derivation of degree k + l on S(M). By Theorem 2.8, there exist a unique $\Theta \in S^{k+l}(M, TM)$ and $\Omega \in S^{k+l+1}(M, TM)$ such that $[L^s_{\Phi}, L^s_{\Psi}] = i(\Omega) + L^s_{\Theta}$. We define that the Frölicher-Nijenhuis bracket of Φ and Ψ is equal to Θ and we denote it by $[\Phi, \Psi]$.

From the above definition, we have the following proposition.

Proposition 2.9. [4] Let $\Phi = \phi \otimes X$ and $\Psi = \psi \otimes Y$ be two decomposable symmetric forms. Then,

$$\begin{split} [\phi \otimes X, \psi \otimes Y] = & \phi \lor \psi \otimes [X, Y] + \phi \lor (L_X^s \psi) \otimes Y - (L_Y^s \phi) \lor \psi \otimes X - \\ & d^s \phi \lor i_X \psi \otimes Y + d^s \psi \lor i_Y \phi \otimes X. \end{split}$$

3. Main results

Let E be a vector bundle with the connection ∇ over M and $\overline{\nabla}$ be a torsion-free linear connection on M. For every section $Z \in \Gamma(E)$, the bilinear map

$$\nabla \nabla Z : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \Gamma(E),$$

defined by

$$\nabla \nabla Z(X,Y) = \nabla_X \nabla_Y Z - \nabla_{\bar{\nabla}_X Y} Z$$

can be written as the sum of its symmetric and antisymmetric parts as follows:

$$\nabla \nabla Z(X,Y) = \frac{1}{2} (\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z - \nabla_{\bar{\nabla}_X Y} Z - \nabla_{\bar{\nabla}_Y X} Z) + \frac{1}{2} (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\bar{\nabla}_X Y} Z + \nabla_{\bar{\nabla}_Y X} Z)$$
$$= \frac{1}{2} (\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z - \nabla_{[X,Y]^s} Z) + \frac{1}{2} (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z).$$

The last term in the parentheses is the antisymmetric part of $\nabla \nabla Z$ and is the curvature of ∇ , which is denoted by R(X,Y)Z. The first expression is the symmetric part of $\nabla \nabla Z$ and we call it the symmetric curvature of ∇ and denote it by $R_Z^s(X,Y)$. So,

$$R_Z^s(X,Y) = \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z - \nabla_{[X,Y]^s} Z.$$

Note that $R_Z^s(X, Y)$ is not tensorial in argument Z, but it is tensorial and symmetric in two arguments X, Y. Moreover, R does not depend on the choice of $\overline{\nabla}$, but R^s does.

Remark 3.1. The concept of symmetric curvature tensor is used already in mathematics, but with a completely different meaning than is given here. Let ∇ be a connection on E, and $\overline{\nabla}$ be a torsion-free linear connection on M. We define the symmetric differential $d^s : S^k(M, E) \longrightarrow S^{k+1}(M, E)$ by

$$(d^{s}\Phi)(X_{1},\ldots,X_{k+1}) = \sum_{i} \nabla_{X_{i}}\Phi(X_{1},\ldots,\hat{X}_{i},\ldots,X_{k+1}) - \sum_{i$$

and $(d^s Z)(X) = \nabla_X Z, Z \in \Gamma E, X \in \mathfrak{X}(M)$. Note that this definition is well defined and d^s depends on the two connections ∇ and $\overline{\nabla}$. Now, let X and Y be two vector fields on M and Z be a section of E. Then,

$$(d^s \circ d^s Z)(X, Y) = R^s_Z(X, Y).$$

Hence, $d^s \circ d^s Z = 0$ if and only if $R_Z^s = 0$.

As was mentioned before, for every vector fields X and Y on M, the derivation $[L_X^s, L_Y^s]$ is of degree 0, and by Theorem 2.8, it can be written in the form of $L_{\Psi}^s + i(\Phi)$, for some $\Phi \in S^1(M, TM)$ and $\Psi \in S^0(M, TM)$. We show that

$$\begin{split} [L_X^s, L_Y^s] &= L_{[X,Y]}^s + i(2\nabla[X,Y] - R(X,Y) + R_X^s(\cdot,Y) - R_Y^s(X,\cdot)).\\ \text{Let } f \in C^\infty(M), \, \omega \in S^1(M), \, \text{and } U \in \mathfrak{X}(M), \, \text{then}\\ [L_X^s, L_Y^s](f) &= L_X^sY(f) - L_Y^sX(f) = XY(f) - YX(f) = L_{[X,Y]}^s(f). \end{split}$$

For $\omega \in S^1(M)$, we have

$$\begin{split} ([L_X^s, L_Y^s]\omega)(U) &= (L_X^s \circ L_Y^s \omega)(U) - (L_Y^s \circ L_X^s \omega)(U) \\ &= X(Y(\omega(U)) - X(\omega([Y, U]^s)) - Y(\omega([X, U]^s)) \\ &+ \omega([Y, [X, U]^s]^s) - Y(X(\omega(U)) + Y(\omega([X, U]^s)) + \\ X(\omega([Y, U]^s)) - \omega([X, [Y, U]^s]^s) \\ &= [X, Y](\omega(U)) + \omega \Big(\nabla_Y \nabla_X U - \nabla_X \nabla_Y U + \nabla_Y \nabla_U X \\ &- \nabla_X \nabla_U Y + \nabla_{[X, U]^s} Y - \nabla_{[Y, U]^s} X \Big). \end{split}$$

On the other hand,

$$\begin{split} &((L^s_{[X,Y]} + i(2\nabla[X,Y] - R(X,Y) + R^s_X(\cdot,Y) - R^s_Y(X,\cdot)))\omega)(U) = \\ &[X,Y](\omega(U)) - \omega([[X,Y],U]^s) + \omega(2\nabla_U[X,Y]) - \omega(R(X,Y)U) \\ &-R^s_X(U,Y) + R^s_Y(X,U)) = [X,Y](\omega(U)) + \omega(\nabla_Y\nabla_XU - \nabla_X\nabla_YU) \\ &+\nabla_Y\nabla_UX - \nabla_X\nabla_UY + \nabla_{[X,U]^s}Y - \nabla_{[Y,U]^s}X). \end{split}$$

We now state and prove the following theorem.

Theorem 3.2. Let $\Phi = \phi \otimes X$ and $\Psi = \psi \otimes Y$ be two decomposable symmetric forms. Then, $[L^s_{\Phi}, L^s_{\Psi}] = L^s_{[\Phi, \Psi]} + i(\Omega)$, where,

$$\begin{split} \Omega = & \phi \lor \psi \lor (2\nabla[X,Y] - R(X,Y) + R_X^s(\cdot,Y) - R_Y^s(X,\cdot)) + \\ & 2d^s \phi \lor \psi \otimes \nabla_X Y - 2\phi \lor d^s \psi \otimes \nabla_Y X - \phi \lor [L_X^s,d^s] \psi \otimes Y + \\ & \psi \lor [L_Y^s,d^s] \phi \otimes X + d^s d^s \phi \lor i_X \psi \otimes Y - d^s d^s \psi \lor i_Y \phi \otimes X. \end{split}$$

Proof. Let ω be a 1-form. Then, we have

$$([L^{s}_{\Phi}, L^{s}_{\Psi}] - L^{s}_{\Theta})\omega = L^{s}_{\phi \otimes X}(L^{s}_{\psi \otimes Y}\omega) - L^{s}_{\psi \otimes Y}(L^{s}_{\phi \otimes X}\omega) - L^{s}_{\Theta}\omega$$
$$= L^{s}_{\phi \otimes X}(\psi \vee L^{s}_{Y}\omega - d^{s}\psi \vee i_{Y}\omega) - L^{s}_{\psi \otimes Y}(\phi \vee L^{s}_{X}\omega - d^{s}\phi \vee i_{X}\omega) - L^{s}_{\Theta}\omega.$$

Using the definition of $L^s_{\Theta}\omega$ and the above relation, we conclude that:

$$\begin{split} ([L_{\Phi}^{s}, L_{\Psi}^{s}] - L_{\Theta}^{s})\omega &= \\ \phi \lor L_{X}^{s}(\psi \lor L_{Y}^{s}\omega - d^{s}\psi \lor i_{Y}\omega) - d^{s}\phi \lor i_{X}(\psi \lor L_{Y}^{s}\omega - d^{s}\psi \lor i_{Y}\omega) - \\ \psi \lor L_{Y}^{s}(\phi \lor L_{X}^{s}\omega - d^{s}\phi \lor i_{X}\omega) + d^{s}\psi \lor i_{Y}(\phi \lor L_{X}^{s}\omega - d^{s}\phi \lor i_{X}\omega) - \\ (\phi \lor \psi \lor L_{X}^{s})\omega - d^{s}\phi \lor \psi \lor i_{[X,Y]}\omega - \phi \lor d^{s}\psi \lor i_{[X,Y]}\omega + \\ \phi \lor L_{X}^{s}\psi \lor L_{Y}^{s}\omega - d^{s}\phi \lor L_{X}^{s}\psi \lor i_{Y}\omega - \phi \lor d^{s}L_{X}^{s}\psi \lor i_{Y}\omega - \\ L_{Y}^{s}\phi \lor \psi \lor L_{X}^{s}\omega + d^{s}(L_{Y}^{s}\phi) \lor \psi \lor i_{X}\omega + L_{Y}^{s}\phi \lor d^{s}\psi \lor i_{X}\omega - \\ d^{s}\phi \lor i_{X}\psi \lor L_{Y}^{s}\omega + d^{s}d^{s}\phi \lor i_{X}\psi \lor i_{Y}\omega + d^{s}\phi \lor d^{s}i_{X}\psi \lor i_{Y}\omega + \\ d^{s}\psi \lor i_{Y}\phi \lor L_{X}^{s}\omega - d^{s}d^{s}\psi \lor i_{Y}\phi \lor i_{X}\omega - d^{s}\psi \lor d^{s}i_{Y}\phi \lor i_{X}\omega) \\ = i(\Omega)\omega. \end{split}$$

Definition 3.3. A section $Z \in \Gamma E$ is called affinewise, if its symmetric curvature tensor vanishes, i.e., $R_Z^s = 0$. In particular, affinewise section of E = TM is called an affinewise vector field.

The set of affinewise sections is a linear subspace of ΓE . In particular, the zero section is an affinewise. We now state three examples of the affinwise sections.

Example 3.4. Let Z be a parallel section of the vector bundle E. Since for every vector field V, $\nabla_V Z = 0$, we find $R_Z^s = 0$, thus all parallel sections are affinewise.

Example 3.5. Consider a trivial vector bundle $F = \mathbb{R}^n \times V$ with the trivial connection on it and trivial connection on \mathbb{R}^n . A section of F is a smooth map $Z : \mathbb{R}^n \longrightarrow V$. By a routine calculation, we find $R_Z^s = 0$ if and only if Z is an affine map. So, the affinewise sections of F are the same as the affine maps.

Example 3.6. For a manifold M with connection ∇ , consider a trivial line bundle $F = M \times \mathbb{R}$ with the trivial connection on it. Sections of F are smooth functions $f : M \longrightarrow \mathbb{R}$. Since, $R_f^s = d^s(d^s f)$, then f is affinewise if and only if $d^s(d^s f) = 0$.

For geodesic vector fields, we have a relation for computing the symmetric curvature. If X is a geodesic vector field on M, then for any section Z of E, we have

$$R_Z^s(X,X) = 2\nabla_X \nabla_X Z.$$

In the remainder of our work, let I be an interval containing 0. For any curve $\gamma : I \longrightarrow M$ we denote the parallel translation along γ by \mathbb{P}_{γ} . For every $\xi \in E_{\gamma(0)}$ and $t \in I$, $(\mathbb{P}_{\gamma}\xi)(t)$ is the parallel transport of ξ along γ to $E_{\gamma(t)}M$.

Lemma 3.7. Let Z be an affinewise section of E. Assume that γ : $I \longrightarrow M$ is a geodesic of $\overline{\nabla}$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then, for every $q = \gamma(t_0)$, we have

$$Z_q = (\mathbb{P}_{\gamma}(Z_p + t_0 \nabla_v Z))(t_0).$$

Proof. Define the curve $h : I \longrightarrow E_p$ by $(\mathbb{P}_{\gamma}h(t))(t) = Z_{\gamma(t)}$. Note that h is a smooth map and $(\mathbb{P}_{\gamma}h'(t))(t) = \nabla_{\gamma'(t)}Z$. By repeating this procedure, we find $(\mathbb{P}_{\gamma}h''(t))(t) = \nabla_{\gamma'(t)}\nabla_{\gamma'(t)}Z$. If X is a local geodesic vector field such that $X_p = \gamma'(0)$, then

$$0 = R_Z^s(\gamma'(t), \gamma'(t)) = 2(\nabla_X \nabla_X Z)_{\gamma(t)} = 2(\nabla_{\gamma'(t)} \nabla_{\gamma'(t)} Z) = 2(\mathbb{P}_{\gamma} h''(t))(t).$$

So, for every $t \in I$, h''(t) = 0. This means that h(t) = at + b, in which $a = h'(0) = \nabla_v Z$ and $b = h(0) = Z_p$. So,

$$Z_q = (\mathbb{P}_{\gamma}(Z_p + t_0 \nabla_v Z))(t_0).$$

Conversely, every section of E with the above property is an affinewise section.

Lemma 3.8. Let Z and Z' be two affinewise sections of E such that, for some $p \in M$, $Z_p = Z'_p$, and for all $v \in T_pM$, $\nabla_v Z = \nabla_v Z'$. If V is a convex open neighborhood of $o \in T_pM$ such that the map exp is defined on it, then Z = Z' on $\exp(V)$.

Proof. Let q be a point of $\exp(V)$. For some $v \in T_pM$, we can define the geodesic $\gamma(t) = \exp(tv)$. We have $\gamma(0) = p$ and $\gamma'(0) = v$, and for some $t_0, \gamma(t_0) = q$. From Lemma 3.7, we have

$$Z_q = (\mathbb{P}_{\gamma}(Z_p + t_0 \nabla_v Z))(t_0), \quad Z'_q = (\mathbb{P}_{\gamma}(Z'_p + t_0 \nabla_v Z'))(t_0).$$
$$Z_p = Z'_p \text{ and } \nabla_v Z = \nabla_v Z', \text{ we obtain } Z_q = Z'_q.$$

Since $Z_p = Z'_p$ and $\nabla_v Z = \nabla_v Z'$, we obtain $Z_q = Z'_q$. **Lemma 3.9.** Let ∇ be a connection on M. Given $p \in M$, there ex-

Lemma 5.9. Let ∇ be a connection on M. Given $p \in M$, there exists ists an open neighborhood U of p such that for all $q \in U$, there exists an open neighborhood $W_q \subseteq T_q M$ of $0 \in T_q M$ such that $\exp |_{W_q}$ is a diffeomorphism and $p \in \exp(W_q)$.

Proof. This lemma is a standard one. We only give a sketch of the proof. Let an open set \widetilde{TM} of TM be the domain of exp. Define the map $F: \widetilde{TM} \longrightarrow M \times M$ by

$$F(v) = (\pi(v), \exp(v)).$$

We know that $(F_*)_{0_p}$ is an isomorphism. From the inverse map theorem, the proof can be completed.

Theorem 3.10. Let Z be an affinewise section of E. If M is connected and, for some $p \in M$, Z_p , and for every $v \in T_pM$, the $\nabla_v Z$ are given, then Z is determined everywhere on M.

Proof. Suppose that Z' is another affinewise vector field, such that $Z'_p = Z_p$ and, for all $v \in T_pM$, $\nabla_v Z' = \nabla_v Z$. We must prove that Z' = Z. Let $B = \{x \in M \mid Z_x = Z'_x, \forall v \in T_xM \ \nabla_v Z = \nabla_v Z'\}$. Since $p \in B$, then B is non-empty. We show that B is open and closed.

Suppose that $x \in B$. There exists an open convex neighborhood V of $o \in T_x M$ such that exp is a diffeomorphism on it. The set $U = \exp(V)$ is an open set of M and by Lemma 3.8, Z = Z' on U. Since U is open, for all $q \in U$, and $v \in T_q M$, we have $\nabla_v Z = \nabla_v Z'$. Thus, $U \subseteq B$, i.e., B is open.

Let x be a limit point of B. Consider the open neighborhood U of x in Lemma 3.9. Since x is a limit point of B, there exists a point q in $B \cap U$. Let W_q be the open neighborhood of T_qM such that exp is a diffeomorphism and $x \in \exp(W_q)$. From Lemma 3.8, Z = Z' on

 $U = \exp(W_q)$. Since U is open and $x \in U$, for all $\in T_x M$, we have $\nabla_v Z = \nabla_v Z'$. Thus, $x \in B$, i.e., B is closed.

Therefore, B is a non-empty open and closed subset of M and since M is connected, then we have B = M and Z = Z'.

Corollary 3.11. If M is connected, then the set of all affinewise sections of E is a linear subspace of ΓE of dimension less than or equal to $m + m \cdot n$, where, $n = \dim M$ and $m = \operatorname{rank} E$.

Proposition 3.12. Suppose E is a Riemannian vector bundle and ∇ is a Riemannian connection, and M is geodesically complete. If Z is a bounded affinewise section of E, then Z is parallel, and so its length is constant.

Proof. Let p be a point of M and $v \in T_p M$. Suppose $\gamma : \mathbb{R} \longrightarrow M$ is the geodesic of M with $\gamma(0) = p$ and $\gamma'(0) = v$. From Lemma 3.7, we have

$$Z_{\gamma(t)} = (\mathbb{P}_{\gamma}(Z_p + t\nabla_v Z))(t).$$

Since \mathbb{P}_{γ} is an isometry, we have

$$|t| \|\nabla_v Z\| - \|Z_p\| \le \|Z_p + t\nabla_v Z\| = \|(\mathbb{P}_{\gamma}(Z_p + t\nabla_v Z))(t)\| = \|Z_{\gamma(t)}\|.$$

If $\nabla_v Z \neq 0$, then the left hand side of the above inequality tends to infinity, as $t \to \infty$. This is in contradiction with the boundedness of Z. Hence, $\nabla_v Z = 0$, i.e., Z is parallel.

Corollary 3.13. On compact Riemannian manifolds, only parallel vector fields are affinewise.

Corollary 3.14. For the sphere S^n of dimension $n \ge 2$, there is no non-zero affinewise vector field.

Proposition 3.15. For the sphere S^2 , there is no non-zero affinewise local vector field.

Proof. Suppose that Z be a non-zero affinewise local vector field on S^2 , defined on an open set U. Let p = (0, 0, 1) be the north pole of S^2 . By rotation and multiplication to a scalar, we can assume $p \in U$ and $Z_p = (1, 0, 0)$. Let (φ, θ) be the sphere coordinate on S^2 , $0 < \varphi < \pi$, $0 < \theta < 2\pi$. Hence, we have

$$\nabla_{\frac{\partial}{\partial \phi}} \frac{\partial}{\partial \phi} = 0 \ , \ \nabla_{\frac{\partial}{\partial \phi}} \frac{\partial}{\partial \theta} = \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \phi} = \cot \phi \frac{\partial}{\partial \theta} \ , \ \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta} = -\frac{1}{2} \sin 2\phi \frac{\partial}{\partial \phi}.$$

Let $Z = Z^1 \frac{\partial}{\partial \phi} + Z^2 \frac{\partial}{\partial \theta}$. Then the condition $R_Z^s = 0$ is satisfied if and only if

(3.1)
$$\frac{\partial^2 Z^1}{\partial \phi^2} = 0,$$

(3.2)
$$\frac{\partial^2 Z^2}{\partial \phi^2} + 2 \frac{\partial Z^2}{\partial \phi} \cot \phi - Z^2 = 0,$$

(3.3)
$$\frac{\partial^2 Z^1}{\partial \theta^2} - Z^1 \cos^2 \phi - \frac{\partial Z^2}{\partial \theta} \sin 2\phi = 0,$$

(3.4)
$$2\frac{\partial Z^1}{\partial \theta}\cot\phi - Z^2\cos^2\phi + \frac{\partial^2 Z^2}{\partial \theta^2} = 0,$$

(3.5)
$$2\frac{\partial^2 Z^1}{\partial \phi \partial \theta} - \frac{\partial Z^2}{\partial \phi} \sin 2\phi - \frac{1}{2}Z^2 \sin 2\phi - Z^2 \cos^2 \phi = 0,$$

(3.6)
$$2\frac{\partial^2 Z^2}{\partial \phi \partial \theta} + 2\frac{\partial Z^1}{\partial \phi} \cot \phi - Z^1 + 2\frac{\partial Z^2}{\partial \theta} \cot \phi = 0.$$

Let q be a point of U distinct from p, and its coordinate be (φ, θ) . Then, the geodesic joining p and q are given by

 $\gamma(t) = (\sin t \cos \theta, \sin t \sin \theta, \cos t)$, where, $\gamma(0) = p$, $\gamma(\varphi) = q$. For $v \in T_p S^2$, put $L(v) = \nabla_v Z$. Then, by Lemma 3.4, we have

$$Z_q = (\mathbb{P}_{\gamma}(Z_p + \varphi L(\gamma'(0))))(\varphi)$$

Suppose that the matrix L in the basis $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0)\}$ of $T_p S^2$ is:

$$L = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right).$$

Then, by simple calculations, we get

$$Z^{1} = 1 + \phi(a\cos^{2}\theta + d\sin^{2}\theta + (b+c)\sin\theta\cos\theta),$$

$$Z^{2} = \frac{1}{\sin\phi}(\sin\theta + \phi(c\cos^{2}\theta - b\sin^{2}\theta + (d-a)\sin\theta\cos\theta)).$$

It is easy to see that these functions do not satisfy the equations (1)-(6). For example, consider equation (6). By calculation, for arbitrary ϕ and $\theta = \frac{\pi}{2}$, we obtain:

$$\frac{2d\cos\phi - \sin\phi - d\phi\sin\phi + 2a - 2d}{\sin\phi} = 0.$$

But, for the above equation, we can consider two cases: for a = 0, if $\phi \to 0$, then the left hand side tends to -1 and in the case $a \neq 0$, if $\phi \to 0$, then the left hand side tends to infinity. Thus, in both cases, we have a contradiction.

Note. By the same argument, it follows that, there are no local affinewise vector fields on the *n*-dimensional sphere S^n , with $n \ge 2$.

Lemma 3.16. Let E be a Riemannian vector bundle, and Z be an affinewise section of E. If ||Z|| is constant on an open set U, then Z is parallel on U.

Proof. Suppose $p \in U$ and $v \in T_pM$, for the geodesic $\gamma : I \longrightarrow U$, which satisfies $\gamma(0) = p$ and $\gamma'(0) = v$. We have $Z_{\gamma(t)} = (\mathbb{P}_{\gamma}(Z_p + t\nabla_v Z))(t)$. Since \mathbb{P}_{γ} is an isometry, we have

 $\forall t \in I \ \|Z_p\| = \|Z_{\gamma}(t)\| = \|(\mathbb{P}_{\gamma}(Z_p + t\nabla_v Z))(t)\| = \|Z_p + t\nabla_v Z\|.$

So,

$$t^2 \|\nabla_v Z\|^2 + 2tZ_p \cdot \nabla_v Z = 0.$$

If $\nabla_v Z \neq 0$, this equation can not hold for all $t \in I$. Therefore, $\nabla_v Z = 0$.

Lemma 3.17. Suppose E is a Riemannian vector bundle and ∇ is a Riemannian connection. If Z is an affinewise section of E and, for some point $p \in M$, for all $v \in T_pM$, $\nabla_v Z = 0$, then for every open set $W \subseteq T_pM$, where, $0_p \in W$ and exp is a diffeomorphism on it, Z is parallel on $\exp(W)$.

Proof. For every $q \in \exp(W)$, there exists a geodesic $\gamma : I \longrightarrow M$ such that $\gamma(t_0) = q$. Consequently,

$$||Z_q|| = ||\mathbb{P}_{\gamma}(Z_p + t\nabla_v Z)|| = ||\mathbb{P}_{\gamma}Z_p|| = ||Z_p||.$$

So, Z has constant length on exp(W), and by Lemma 3.16, Z is parallel on exp(W).

Theorem 3.18. Let E be a Riemannian vector bundle, ∇ be a Riemannian connection, and M be connected. Let Z be a section of E. If for some $p \in M$, for all $v \in T_pM$, $\nabla_v Z = 0$, then Z is parallel.

Proof. Define

$$B = \{ q \in M \mid \forall v \in T_q M \ \nabla_v Z = 0 \}.$$

By assumption, $p \in B$, and so B is non-empty. By Lemma 3.8, B is open. Let q be a limit point of B. By Lemma 3.8, we consider an open neighborhood U of q in such a way that, for all $q' \in U$, there exists an open set $W \subseteq T_{q'}M$, with $0 \in W$, and $\exp : W \longrightarrow \exp(W)$ is diffeomorphism and $q \in \exp(W)$. Since $U \cap B \neq \phi$, we can choose $q' \in U \cap B$. By Lemma 3.17, Z is parallel on $\exp(W)$, and so for every

 $v \in T_q M$, $\nabla_v Z = 0$, $q \in B$. Hence, B is an open and closed non-empty subset of connected space M, and thus B = M and Z is parallel. \Box

Corollary 3.19. Let M be a connected manifold and $f \in C^{\infty}(M)$. If $d^{s}(d^{s}f) = 0$ and df is zero in some point of M, then f is constant.

Proof. Consider the trivial bundle $E = M \times \mathbb{R}$. If $d^s(d^s f) = 0$, then f is an affinewise section of E that satisfies conditions of Theorem 3.18. Therefore, f is a parallel section, i.e., $\nabla f = df = 0$. Consequently, f is constant.

Theorem 3.20. Let (M, g) be a Riemannain manifold. If $d^{s}(d^{s}f) = 0$, f is not a constant function and M is connected, then for every $c \in f(M)$, $f^{-1}(c)$ is a flat submanifold, i.e., the second fundamental form of $f^{-1}(c)$ vanishes.

Proof. Since f is a non-constant function and $d^s(d^s f) = 0$, it is easy to see that the rank of f is 1. Therefore, for all constant $c \in f(M)$, $f^{-1}(c)$ is the submanifold of M. Assume $t_0 \in I$, $p \in f^{-1}(c)$ and $\alpha : I \longrightarrow M$ be a geodesic such that $\alpha(t_0) = p$ and $\alpha'(t_0) \in T_p f^{-1}(c)$. From Corollary 2.6, we have

$$f(\alpha(t)) = at + b \quad \forall t \in I,$$

where, $a = df(\alpha'(t))$. From the above equation, it is not difficult to show that a = 0 and $f(\alpha(t)) = b$, for all $t \in I$. Since $\alpha(t_0) = p \in f^{-1}(c)$, then $f(\alpha(t_0)) = c$. Hence, we have

$$b = f(\alpha(t_0)) = c.$$

Consequently, $f(\alpha(t)) = c$, for all $t \in I$, i.e., $\alpha(t) \in f^{-1}(c)$.

Let
$$\nabla$$
 be a connection on a manifold M . An affine vector field on M is a vector field X such that for all vector fields Y and Z , we have

$$L_X(\nabla_Y Z) = \nabla_{L_X Y} Z + \nabla_Y L_X Z.$$

Proposition 3.21. Let ∇ be a connection on a manifold M. If X is an affine vector field and Z is an affinewise vector field, then [X, Z] is an affinewise vector field.

Proof. Simple calculations show that for arbitrary vector fields Y, Z, and W, we have

$$L_X(R_Z^s(Y,W)) = R_{L_XZ}^s(Y,W) + R_Z^s(L_XY,W) + R_Z^s(Y,L_XW).$$

Now, in the above equality, if Z is affinewise, then we find $R^s_{L_XZ}(Y,W) = 0$. Hence, [X, Z] is affinewise.

Affine and affinewise vector fields are two distinct concepts. An affinewise vector field is not necessarily an affine vector field. For example, in \mathbb{R}^n , the vector field $Z_q = (q, 2q)$ is affinewise, but it is not an affine vector field. On S^n , with $n \ge 2$, there exists no nonzero affinewise vector field, but there exist nonzero Killing vector fields which are affine.

By contracting symmetric curvature, we can find new curvatures. This contraction can be done in two ways.

Definition 3.22. Let $\{E_i\}$ be a basis of local vector fields on M with the dual basis $\{\omega^i\}$. For every vector field Z, we assign a 1-form ω_Z as follows, calling it the form curvature along Z,

$$\omega_Z(X) = \sum_i \omega^i (R_Z^s(E_i, X))$$

For example, if Z is a vector field on \mathbb{R}^n , then $\omega_Z = 2d(Div(Z))$.

Theorem 3.23. Let Z be a vector field on a Riemannian manifold (M, g). Then, we have

$$\omega_Z = 2d(Div(Z)) + Ric(., Z).$$

Proof. Let $\{E_i\}$ be a basis of locally vector field with dual $\{\omega^i\}$ and $[E_i, E_j] = 0$. Then,

$$d(DivZ)(E_l) = E_l(Div(Z)) = E_l(\omega^i(\nabla_{E_i}Z))$$
$$= (\nabla_{E_l}\omega^i)(\nabla_{E_i}Z) + \omega^i(\nabla_{E_l}\nabla_{E_i}Z).$$

On the other hand, we have

$$Ric(E_l, Z) = \omega^i (R(E_i, E_l)Z) = \omega^i (\nabla_{E_i} \nabla_{E_l} Z - \nabla_{E_l} \nabla_{E_i} Z)$$
$$= \omega^i (\nabla_{E_i} \nabla_{E_l} Z) - \omega^i (\nabla_{E_l} \nabla_{E_i} Z).$$

So,

$$2d(Div(Z))(E_l) + Ric(E_l, Z) = 2(\nabla_{E_l}\omega^i)(\nabla_{E_i}Z) + \omega^i(\nabla_{E_l}\nabla_{E_i}Z) + \omega^i(\nabla_{E_i}\nabla_{E_l}Z).$$

Now,

$$\omega_Z(E_l) = \omega^i(R_Z^s(E_l, E_i)) = \omega^i(\nabla_{E_l}\nabla_{E_i}Z + \nabla_{E_i}\nabla_{E_l}Z - \nabla_{[E_i, E_l]^s}Z).$$

Since $[E_i, E_l] = 0$, we have $[E_i, E_l]^s = 2\nabla_{E_l}E_i$. Therefore,

$$\omega_Z(E_l) = \omega^i (\nabla_{E_l} \nabla_{E_i} Z) + \omega^i (\nabla_{E_i} \nabla_{E_l} Z) - 2\omega^i (\nabla_{\nabla_{E_l} E_i} Z).$$

To complete the proof, we must prove the following equality:

$$\omega^{i}(\nabla_{\nabla_{E_{l}}E_{i}}Z) = -(\nabla_{E_{l}}\omega^{i})(\nabla_{E_{i}}Z).$$

Let $Z = Z^k E_k$. Hence,

$$\begin{split} \omega^{i}(\nabla_{\nabla_{E_{l}}E_{i}}Z) &= \omega^{i}(\nabla_{\nabla_{E_{l}}E_{i}}Z^{k}E_{k}) = \omega^{i}(\nabla_{\Gamma_{il}^{r}E_{r}}Z^{k}E_{k}) \\ &= \omega^{i}(\Gamma_{il}^{r}E_{r}(Z^{k})E_{k} + Z^{k}\Gamma_{il}^{r}\Gamma_{rk}^{s}E_{s}) \\ &= \Gamma_{il}^{r}E_{r}(Z^{i}) + Z^{k}\Gamma_{il}^{r}\Gamma_{rk}^{i}. \end{split}$$

Now, compute the right hand side of the equality:

$$(\nabla_{E_l}\omega^i)(\nabla_{E_i}Z) = E_l(\omega^i(\nabla_{E_i}Z^kE_k)) - \omega^i(\nabla_{E_l}\nabla_{E_i}Z^kE_k)$$

$$= E_l(\omega^i(E_i(Z^k)E_k + Z^k\Gamma^s_{ki}E_s))$$

$$- \omega^i(\nabla_{E_l}(E_i(Z^k)E_k) + \nabla_{E_l}(Z^k\Gamma^r_{ki}E_r))$$

$$= E_l(E_i(Z^i)) + E_l(Z^k\Gamma^i_{ki}) - E_l(E_i(Z^i))$$

$$- E_i(Z^k)\Gamma^i_{kl} - E_l(Z^k\Gamma^i_{ki}) - Z^k\Gamma^r_{ki}\Gamma^i_{rl}$$

$$= -\Gamma^i_{kl}E_i(Z^k) - Z^k\Gamma^r_{ki}\Gamma^i_{rl}.$$

Definition 3.24. Let $\{E_i\}$ be an orthonormal basis of local vector fields on the Reimannian manifold M. For every vector field Z, we assign a vector field X_Z as follows, calling it the vector curvature along Z,

$$X_Z = \sum_i R_Z^s(E_i, E_i).$$

Theorem 3.25. Let Z be a vector field on a Riemannian manifold M. Then, $X_Z = 2tr\nabla^2 Z$, where, $tr\nabla^2 Z = \sum \nabla^2 Z(E_i, E_i)$.

Proof. By simple calculations, we have

$$tr\nabla^2 Z = \sum \nabla^2 Z(E_i, E_i) = \sum \frac{1}{2} R_Z^s(E_i, E_i) + \frac{1}{2} R(E_i, E_i)(Z) = \frac{1}{2} X_Z.$$

Now, we can restate some results about geometric vector fields from [7] as follows.

Theorem 3.26. Let Z be a vector field on a Riemannian manifold (M, g) with the the Ricci tensor Ric of g.

i) Z is harmonic if and only if $\langle X_Z, . \rangle = 2Ric(Z, .)$.

- ii) If Z is a Killing field, then $\langle X_Z, . \rangle = -2Ric(Z, .)$. The converse holds if M is compact and DivZ = 0.
- iii) If Z is an affine field, then divZ is locally constant and $\langle X_Z, . \rangle + 2Ric(Z, .) = 0.$

From Theorem 3.23 and Theorem 3.26, we can deduce the following corollary.

Corollary 3.27. Let Z be a vector field on a Riemannian manifold M.

- i) Z is harmonic if and only if $\operatorname{div}(Z) = 0$ and $\langle X_Z, . \rangle = 2\omega_Z(.)$.
- ii) If Z is Killing, then $\operatorname{div}(Z) = 0$ and $\langle X_Z, . \rangle = -2\omega_Z(.)$. The converse is true if M is compact.
- iii) If Z is affine, then $\operatorname{div}(Z)$ is locally constant and $\langle X_Z, . \rangle = -2\omega_Z(.)$. If M is compact, conversely we can deduce Z is affine.

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 ${\it Symmetric\ curvature\ tensor}$

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