

THE (R,S)-SYMMETRIC AND (R,S)-SKEW SYMMETRIC SOLUTIONS OF THE PAIR OF MATRIX EQUATIONS $A_1XB_1 = C_1$ AND $A_2XB_2 = C_2$

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ABSTRACT. Let $R \in \mathbf{C}^{m \times m}$ and $S \in \mathbf{C}^{n \times n}$ be nontrivial involution matrices; i.e., $R = R^{-1} \neq \pm I$ and $S = S^{-1} \neq \pm I$. An $m \times n$ complex matrix A is said to be an (R, S) -symmetric ((R, S) -skew symmetric) matrix if $RAS = A$ ($RAS = -A$). The (R, S) -symmetric and (R, S) -skew symmetric matrices have a number of special properties and widely used in engineering and scientific computing. Here, we introduce the necessary and sufficient conditions for the solvability of the pair of matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$, over (R, S) -symmetric and (R, S) -skew symmetric matrices, and give the general expressions of the solutions for the solvable cases. Finally, we give necessary and sufficient conditions for the existence of (R, S) -symmetric and (R, S) -skew symmetric solutions and representations of these solutions to the pair of matrix equations in some special cases.

1. Introduction

Throughout, the notation $\mathbf{C}^{m \times n}$ represents the vector space of all $m \times n$ matrices over the complex field $\mathbf{C}$. By A^T , we denote the transpose matrix of A . The conjugate transpose of the matrix $A \in \mathbf{C}^{m \times n}$ is

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denoted by A^* . We define a conditional inverse of $A \in \mathbf{C}^{m \times n}$, denoted by A^- , to be any matrix $B \in \mathbf{C}^{n \times m}$ satisfying $ABA = A$. The symbol $A \otimes B$ denotes the Kronecker matrix product, $(a_{ij}B)$. The symbol $\text{vec}(A)$ denotes the $mn \times 1$ vector formed by the vertical concatenation of the matrix A . For a given $mn \times 1$ vector w , we use $\text{Invec}^{m,n}(w)$ to denote the $m \times n$ matrix $W \in \mathbf{C}^{m \times n}$ such that $\text{vec}(W) = w$. Now, we define (R, S) -symmetric and (R, S) -skew symmetric matrices as follows.

Definition 1.1. [22] Assume that $R \in \mathbf{C}^{m \times m}$ and $S \in \mathbf{C}^{n \times n}$ are non-trivial involution matrices, that is, $R = R^{-1} \neq \pm I$ and $S = S^{-1} \neq \pm I$. An $m \times n$ matrix A is said to be an (R, S) -symmetric ((R, S) -skew symmetric) matrix if $RAS = A$ ($RAS = -A$). $\mathcal{S}_{R,S}^{m \times n}$ and $\tilde{\mathcal{S}}_{R,S}^{m \times n}$ represent the set of $m \times n$ (R, S) -symmetric and (R, S) -skew symmetric matrices, respectively.

The (R, S) -symmetric and (R, S) -skew symmetric matrices have wide applications in information theory, linear estimate theory and numerical analysis [1, 16, 27, 28]. Solving matrix equations is a topic of very active research in computational mathematics, and has been widely applied in various areas such as principal component analysis, biology, electricity, solid mechanics, automatics control theory, vibration theory, and so on. A large number of papers have presented several methods for solving matrix equations [9, 8, 20, 23, 25, 26]. Dai [3] and Chu [2] studied the linear matrix equation

$$AXB = C,$$

with a symmetric condition on the solution X . Ramadan and El-Sayed [21] proposed a simple method for generating a nonsingular solution of the matrix equation $XH = HX$, where the matrix H is in an unreduced lower Hessenberg form. Mitra [17, 18] proposed conditions for the existence of a solution and a representation of a general common solution to the pair of individually consistent simultaneous linear matrix equations

$$(1.1) \quad A_1XB_1 = C_1 \text{ and } A_2XB_2 = C_2.$$

Also Navarra et al. [19] studied a representation of the solution X to the pair of matrix equations (1.1). In [24], Wang considered (1.1) over an arbitrary regular ring with identity and derived the necessary and sufficient conditions for the existence and the expression for the general solution of the pair of matrix equations. In [4, 5, 7], some iterative

algorithms were proposed to solve the Sylvester matrix equation and the generalized coupled Sylvester matrix equations over reflexive and anti-reflexive matrices. In [6], an iterative algorithm was constructed for solving the second-order Sylvester matrix equation

$$EVF^2 - AVF - CV = BW.$$

Zhou et al. [31] proposed an iterative method for finding weighted least squares solutions to the coupled Sylvester matrix equations. Ding and Chen [10, 11] presented the hierarchical gradient iterative (HGI) algorithms for general matrix equations and hierarchical least-squares-iterative (HLSI) algorithms for the generalized coupled Sylvester matrix equation and general coupled matrix equations [12, 13]. The HGI algorithms [10, 11] and HLSI algorithms [14, 11, 13] for solving general (coupled) matrix equations are two innovational and computationally efficient numerical ones and were proposed based on the hierarchical identification principle [12, 15], which regards the unknown matrix as the system parameter matrix to be identified.

The reminder of our work is organized as follows. In Section 2, we first review some structure properties of the (R, S) -symmetric and (R, S) -skew symmetric matrices, subsets $\mathcal{S}_{R,S}^{m \times n}$ and $\tilde{\mathcal{S}}_{R,S}^{m \times n}$. Then, we will present the necessary and sufficient conditions for the existence of (R, S) -symmetric and (R, S) -skew symmetric solutions of (1.1), respectively. Some special cases of the pair of matrix equations (1.1) over (R, S) -symmetric and (R, S) -skew symmetric matrices are considered in Section 3.

2. Main Results

In this section, we first discuss the structure and properties of the (R, S) -symmetric and (R, S) -skew symmetric matrices and subsets $\mathcal{S}_{R,S}^{m \times n}$ and $\tilde{\mathcal{S}}_{R,S}^{m \times n}$. Assume that $R \in \mathbf{C}^{m \times m}$ and $S \in \mathbf{C}^{n \times n}$ are nontrivial involution matrices. From “an involution is diagonalizable”, there are positive numbers r, k and matrices $P \in \mathbf{C}^{m \times r}$, $Q \in \mathbf{C}^{m \times (m-r)}$, $U \in \mathbf{C}^{n \times k}$ and $V \in \mathbf{C}^{n \times (n-k)}$ such that [22]

$$P^*P = I, \quad Q^*Q = I, \quad RP = P, \quad RQ = -Q,$$

and

$$U^*U = I, \quad V^*V = I, \quad SU = U, \quad SV = -V.$$

Also, if we consider

$$(2.1) \quad \widehat{U} = \frac{U^*(I+S)}{2}, \quad \widehat{V} = \frac{V^*(I-S)}{2}, \quad \widehat{P} = \frac{P^*(I+R)}{2}, \quad \widehat{Q} = \frac{Q^*(I-R)}{2},$$

then we have

$$(2.2) \quad \widehat{P}\widehat{P} = I, \widehat{P}\widehat{Q} = 0, \widehat{Q}\widehat{P} = 0, \widehat{Q}\widehat{Q} = I, \widehat{U}\widehat{U} = I, \widehat{U}\widehat{V} = 0, \widehat{V}\widehat{U} = 0, \widehat{V}\widehat{V} = I.$$

This implies:

$$(2.3) \quad \begin{bmatrix} U & V \end{bmatrix}^{-1} = \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} P & Q \end{bmatrix}^{-1} = \begin{bmatrix} \widehat{P} \\ \widehat{Q} \end{bmatrix}.$$

In the following lemmas, we will give characterizations of (R, S) -symmetric and (R, S) -skew symmetric matrices.

Lemma 2.1. [22] *A is (R, S) -symmetric if and only if*

$$(2.4) \quad A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix},$$

where, $A_1 \in \mathbf{C}^{r \times k}$ and $A_4 \in \mathbf{C}^{(m-r) \times (n-k)}$.

Lemma 2.2. [22] *A is (R, S) -skew symmetric if and only if*

$$(2.5) \quad A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix},$$

where, $A_2 \in \mathbf{C}^{r \times (n-k)}$ and $A_3 \in \mathbf{C}^{(m-r) \times k}$.

Without loss of generality, in the rest of this paper we assume that the matrices $A_i, B_i, C_i \in \mathbf{C}^{n \times n}$; $i = 1, 2$; have the following decompositions:

$$(2.6) \quad \begin{cases} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix} A_i \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} A_{i,1} & A_{i,2} \\ A_{i,3} & A_{i,4} \end{bmatrix}, \\ \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix} B_i \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} B_{i,1} & B_{i,2} \\ B_{i,3} & B_{i,4} \end{bmatrix}, \\ C'_i = \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix} C_i \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} C_{i,1} & C_{i,2} \\ C_{i,3} & C_{i,4} \end{bmatrix}, \end{cases}$$

where, $A_{i,1} \in \mathbf{C}^{r \times r}$, $B_{i,1} \in \mathbf{C}^{k \times k}$, $C_{i,1} \in \mathbf{C}^{r \times k}$, $A_{i,4} \in \mathbf{C}^{(n-r) \times (n-r)}$, $B_{i,4} \in \mathbf{C}^{(n-k) \times (n-k)}$ and $C_{i,4} \in \mathbf{C}^{(n-r) \times (n-k)}$, for $i = 1, 2$. Also, we let

$$(2.7) \quad A'_i = (A_{i,1}^T \ A_{i,3}^T)^T, \quad B'_i = (B_{i,1}, B_{i,2}),$$

$$A''_i = (A_{i,2}^T \ A_{i,4}^T)^T \quad \text{and} \quad B''_i = (B_{i,3}, B_{i,4}),$$

for $i = 1, 2$. The following theorems provide the general conditions for the existence of the (R, S) -symmetric and (R, S) -skew symmetric solutions to the pair of matrix equations (1.1).

Theorem 2.3. *Let $A_i, B_i, C_i \in \mathbf{C}^{n \times n}$, for $i = 1, 2$, be given matrices. Then, the following conditions are equivalent.*

(1) *The pair of matrix equations (1.1) has a common solution $X \in \mathcal{S}_{R,S}^{n \times n}$.*

(2) *The following pair of matrix equations has the solutions X_1 and X_4 :*

$$(2.8) \quad A'_1 X_1 B'_1 + A''_1 X_4 B''_1 = C'_1 \text{ and } A'_2 X_1 B'_2 + A''_2 X_4 B''_2 = C'_2.$$

(3) *The following system of matrix equations has the solutions X_1 and X_4 :*

$$(2.9) \quad \begin{cases} A_{1,1} X_1 B_{1,1} + A_{1,2} X_4 B_{1,3} = C_{1,1}, & A_{2,1} X_1 B_{2,1} + A_{2,2} X_4 B_{2,3} = C_{2,1}, \\ A_{1,1} X_1 B_{1,2} + A_{1,2} X_4 B_{1,4} = C_{1,2}, & A_{2,1} X_1 B_{2,2} + A_{2,2} X_4 B_{2,4} = C_{2,2}, \\ A_{1,3} X_1 B_{1,1} + A_{1,4} X_4 B_{1,3} = C_{1,3}, & A_{2,3} X_1 B_{2,1} + A_{2,4} X_4 B_{2,3} = C_{2,3}, \\ A_{1,3} X_1 B_{1,2} + A_{1,4} X_4 B_{1,4} = C_{1,4}, & A_{2,3} X_1 B_{2,2} + A_{2,4} X_4 B_{2,4} = C_{2,4}, \end{cases}$$

in which case, the common solution $X \in \mathcal{S}_{R,S}^{n \times n}$ is represented by

$$X = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix},$$

where, $X_1 \in \mathbf{C}^{r \times k}$ and $X_4 \in \mathbf{C}^{(n-r) \times (n-k)}$.

Proof. First we show that (2) $\Leftrightarrow$ (3). By substituting (2.7) into (2.8), we can obtain the system of matrix equations (2.9). This implies (2) $\Leftrightarrow$ (3). (1) $\Leftrightarrow$ (3). Suppose that the system of matrix equations (1.1) has a

common solution $X \in \mathcal{S}_{R,S}^{n \times n}$. By Lemma 2.1, there exist $X_1 \in \mathbf{C}^{r \times k}$ and $X_4 \in \mathbf{C}^{(n-r) \times (n-k)}$ so that

$$X = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}.$$

Now, from $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ and using the decompositions (2.6), we get

$$\begin{bmatrix} A_{1,1}X_1B_{1,1} + A_{1,2}X_4B_{1,3} & A_{1,1}X_1B_{1,2} + A_{1,2}X_4B_{1,4} \\ A_{1,3}X_1B_{1,1} + A_{1,4}X_4B_{1,3} & A_{1,3}X_1B_{1,2} + A_{1,4}X_4B_{1,4} \end{bmatrix} = \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{1,3} & C_{1,4} \end{bmatrix},$$

and

$$\begin{bmatrix} A_{2,1}X_1B_{2,1} + A_{2,2}X_4B_{2,3} & A_{2,1}X_1B_{2,2} + A_{2,2}X_4B_{2,4} \\ A_{2,3}X_1B_{2,1} + A_{2,4}X_4B_{2,3} & A_{2,3}X_1B_{2,2} + A_{2,4}X_4B_{2,4} \end{bmatrix} = \begin{bmatrix} C_{2,1} & C_{2,2} \\ C_{2,3} & C_{2,4} \end{bmatrix}.$$

Conversely, if the system of matrix equations (2.9) has solutions X_1 and X_4 , then it not difficult to get

$$A_1XB_1 = C_1 \text{ and } A_2XB_2 = C_2,$$

where,

$$X = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix} \in \mathcal{S}_{R,S}^{n \times n}$$

□

Similar to the proof of Theorem 2.3, we can prove the following theorem.

Theorem 2.4. Let $A_i, B_i, C_i \in \mathbf{C}^{n \times n}$, for $i = 1, 2$, be given matrices. Then the following conditions are equivalent.

(1) The pair of matrix equations (1.1) has a common solution $X \in \widetilde{\mathcal{S}}_{R,S}^{n \times n}$.

(2) The following system of matrix equations has the solutions X_2 and X_3 :

$$(2.10) \quad A_1''X_3B_1' + A_1'X_2B_1'' = C_1' \text{ and } A_2''X_3B_2' + A_2'X_2B_2'' = C_2'.$$

(3) The following system of matrix equations has the solutions X_2 and X_3 :

$$(2.11) \quad \begin{cases} A_{1,2}X_3B_{1,1} + A_{1,1}X_2B_{1,3} = C_{1,1}, & A_{2,2}X_3B_{2,1} + A_{2,1}X_2B_{2,3} = C_{2,1}, \\ A_{1,2}X_3B_{1,2} + A_{1,1}X_2B_{1,4} = C_{1,2}, & A_{2,2}X_3B_{2,2} + A_{2,1}X_2B_{2,4} = C_{2,2}, \\ A_{1,4}X_3B_{1,1} + A_{1,3}X_2B_{1,3} = C_{1,3}, & A_{2,4}X_3B_{2,1} + A_{2,3}X_2B_{2,3} = C_{2,3}, \\ A_{1,4}X_3B_{1,2} + A_{1,3}X_2B_{1,4} = C_{1,4}, & A_{2,4}X_3B_{2,2} + A_{2,3}X_2B_{2,4} = C_{2,4}, \end{cases}$$

in which case, the common solution $X \in \tilde{\mathcal{S}}_{R,S}^{n \times n}$ is represented by

$$X = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} 0 & X_2 \\ X_3 & 0 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix},$$

where, $X_2 \in \mathcal{C}^{r \times (n-k)}$ and $X_3 \in \mathcal{C}^{(n-r) \times k}$.

3. Some Special Cases

In the next theorems, we will consider the special cases when $B_{1,1} = B_{2,1} = B_{1,2} = B_{2,2} = 0$ or $A_{1,1} = A_{2,1} = A_{1,3} = A_{2,3} = 0$. We find necessary and sufficient conditions for the existence of the solution $X \in \mathcal{S}_{R,S}^{n \times n}$ ($X \in \tilde{\mathcal{S}}_{R,S}^{n \times n}$) and give the expression for the general solution.

Theorem 3.1. Let $A_i, B_i, C_i \in \mathcal{C}^{n \times n}$, for $i = 1, 2$, be given matrices. If $B_{1,1} = B_{2,1} = B_{1,2} = B_{2,2} = 0$ or $A_{1,1} = A_{2,1} = A_{1,3} = A_{2,3} = 0$, then the pair of matrix equations (1.1) has a common solution $X \in \mathcal{S}_{R,S}^{n \times n}$ if and only if $A_1''A_1''-C_1'B_1''-B_1'' = C_1'$ and $\text{Invec}^{n,n}(GG^-\text{vec}(F)) = F$, where,

$$\begin{aligned} G &= B_2''^* \otimes A_2'' + E^* \otimes D, & D &= -A_2''A_1''-A_1'', \\ E &= B_1''B_1''-B_2'', & F &= C_2' - A_2''A_1''-C_1'B_1''-B_2'', \end{aligned}$$

In which case, the general common solution is given by

$$X = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix},$$

where,

$$\begin{aligned} Z &= A_1''-C_1'B_1''- \text{Invec}^{(n-r),(n-k)}(G^-\text{vec}(F) + (I - G^-G)\text{vec}(W)) \\ &\quad - A_1''-1A_1''[\text{Invec}^{(n-r),(n-k)}(G^-\text{vec}(F) + (I - G^-G)\text{vec}(W))]B_1''B_1''-, \\ X_1 &\in \mathcal{C}^{r \times k}, \text{ and } W \in \mathcal{C}^{(n-r) \times (n-k)} \text{ are arbitrary matrices.} \end{aligned}$$

Proof. Let $B_{1,1} = B_{2,1} = B_{1,2} = B_{2,2} = 0$ or $A_{1,1} = A_{2,1} = A_{1,3} = A_{2,3} = 0$, and let $X \in \mathcal{S}_{R,S}^{n \times n}$ be a solution of (1.1). We can assume that X is represented by

$$X = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}.$$

Now, it follows from (2.8) that

$$(3.1) \quad A_1'' X_4 B_1'' = C_1', \quad A_2'' X_4 B_2'' = C_2',$$

where,

$$(3.2) \quad A_i'' = (A_{i,2}^T \ A_{i,4}^T)^T \text{ and } B_i'' = (B_{i,3} \ B_{i,4}), \text{ for } i = 1, 2.$$

From the obtained results in [19], the pair of matrix equations $A_1'' X_4 B_1'' = C_1'$ and $A_2'' X_4 B_2'' = C_2'$ has a common solution if and only if $A_1'' A_1''^{-} C_1' B_1''^{-} B_1'' = C_1'$ and $\text{Invec}^{n,n}(GG^{-}\text{vec}(F)) = F$, where,

$$A_i'' = (A_{i,2}^T \ A_{i,4}^T)^T, \quad B_i'' = (B_{i,3} \ B_{i,4}) \quad i = 1, 2,$$

$$G = B_2''^* \otimes A_2'' + E^* \otimes D, \quad D = -A_2'' A_1''^{-} A_1'',$$

$$E = B_1'' B_1''^{-} B_2'', \quad F = C_2' - A_2'' A_1''^{-} C_1' B_1''^{-} B_2'',$$

In that case, the general solution is given by:

$$X_4 = A_1''^{-} C_1' B_1''^{-} + \text{Invec}^{(n-r),(n-k)}(G^{-}\text{vec}(F) + (I - G^{-}G)\text{vec}(W)) \\ - A_1''^{-} A_1'' [\text{Invec}^{(n-r),(n-k)}(G^{-}\text{vec}(F) + (I - G^{-}G)\text{vec}(W))] B_1'' B_1''^{-},$$

where, $W \in \mathbf{C}^{(n-r) \times (n-k)}$ is an arbitrary matrix. $\square$

By a similar proof to the proof of Theorem 3.1, we can prove the following theorem.

Theorem 3.2. Let $A_i, B_i, C_i \in \mathbf{C}^{n \times n}$, for $i = 1, 2$, be given matrices. If $B_{1,1} = B_{2,1} = B_{1,2} = B_{2,2} = 0$ or $A_{1,2} = A_{2,2} = A_{1,4} = A_{2,4} = 0$, then the pair of matrix equations (1.1) has a common solution $X \in \widetilde{\mathcal{S}}_{R,S}^{n \times n}$ if and only if $A_1' A_1'^{-} C_1' B_1''^{-} B_1'' = C_1'$ and $\text{Invec}^{n,n}(GG^{-}\text{vec}(F)) = F$, where,

$$G = B_2''^* \otimes A_2' + E^* \otimes D, \quad D = -A_2' A_1'^{-} A_1',$$

$$E = B_1'' B_1''^{-} B_2'', \quad F = C_2' - A_2' A_1'^{-} C_1' B_1''^{-} B_2'',$$

in which case, the general common solution is given by

$$X = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} 0 & Z \\ X_3 & 0 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix},$$

where,

$$\begin{aligned} Z &= A_1'^- C_1' B_1''^- + \text{Invec}^{r, (n-k)}(G^- \text{vec}(F) + (I - G^- G) \text{vec}(W)) \\ &\quad - A_1'^- A_1' [\text{Invec}^{r, (n-k)}(G^- \text{vec}(F) + (I - G^- G) \text{vec}(W))] B_1'' B_1''^-, \\ X_3 &\in \mathbf{C}^{(n-r) \times k} \text{ and } W \in \mathbf{C}^{r \times (n-k)} \text{ are arbitrary matrices.} \end{aligned}$$

4. Conclusions

We have considered the (R, S) -symmetric and (R, S) -skew symmetric solutions of the pair of matrix equations $A_1 X B_1 = C_1$ and $A_2 X B_2 = C_2$. By making use of the decompositions (2.6), we presented general analytic formulae, and gave necessary and sufficient conditions for guaranteeing the existence of these solutions. Also, we derived necessary and sufficient conditions for the existence and the expressions for the general (R, S) -symmetric and (R, S) -skew symmetric solutions to the pair of matrix equations in some special cases.

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