# ON THE NILPOTENCY CLASS OF THE AUTOMORPHISM GROUP OF SOME FINITE $p$-GROUPS 

S. FOULADI* AND R. ORFI

Communicated by Saeid Azam


#### Abstract

Let $G$ be a $p$-group of order $p^{n}$ and $\Phi=\Phi(G)$ be the Frattini subgroup of $G$. It is shown that the nilpotency class of Aut ${ }^{\Phi}(G)$, the group of all automorphisms of $G$ centralizing $G / \Phi(G)$, takes the maximum value $n-2$ if and only if $G$ is of maximal class. We also determine the nilpotency class of Aut $^{\Phi}(G)$ when $G$ is a finite abelian $p$-group.


## 1. Introduction

It is well known [3, III, Satz 3.17] that if $G$ is a finite $p$-group with the Frattini subgroup $\Phi=\Phi(G)$, then Aut ${ }^{\Phi}(G)$ is a finite $p$-group. Liebeck [5] found an upper bound for the nilpotency class of $\mathrm{Aut}^{\Phi}(G)$.

Here, we find the nilpotency class of Aut ${ }^{\Phi}(G)$ in some cases. A straightforward consequence of the result in [5] shows that the nilpotency class of $\operatorname{Aut}^{\Phi}(G)$ is less than or equal to $n-2$, for all non-cyclic $p$ groups of order $p^{n}$. Here, we show that the nilpotency class of Aut ${ }^{\Phi}(G)$ takes the maximum value $n-2$ if and only if $G$ is of maximal class. Moreover, we find the nilpotency class of Aut ${ }^{\Phi}(G)$ for a finite abelian $p$-group $G$ in terms of its invariants, where $p$ is an odd prime.

[^0]Throughout, the following notation is used. The terms of the lower central series of $G$ are denoted by $\Gamma_{i}=\Gamma_{i}(G)$. The center of $G$ is denoted by $Z=Z(G)$. The nilpotency class of a group $G$ is denoted by $\operatorname{cl}(G)$. If $\alpha$ is an automorphism of $G$ and $x$ is an element of $G$, we write $x^{\alpha}$ for the image of $x$ under $\alpha$. The inner automorphism induced by the element $g$ is denoted by $\sigma_{g}$. For a normal subgroup $N$ of $G$, we let Aut ${ }^{N}(G)$ denote the group of all automorphisms of $G$ centralizing $G / N$. We write $d(G)$ for the minimal number of generators of $G$. An extra-special $p$-group is a $p$-group $G$ with $\Phi(G)=Z(G)=G^{\prime} \cong \mathbb{Z}_{p}$. A non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian. Also, $\mathbb{Z}_{n}$ is the cyclic group of order $n$. All unexplained notation is standard and follows that of [4].

## 2. Maximum Value of $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)$

Let $G$ be a non-cyclic $p$-group of order $p^{n}$. In this section, we prove that $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)$ takes the maximum value $n-2$ if and only if $G$ is of maximal class. First, we give some basic results that are needed for the main results of this section. In [5], Liebeck proved the following theorems which play important roles in our proofs.
Theorem 2.1. [5, Theorem 2] Let $G$ be a finite d-generator p-group with lower central series $G=\Gamma_{1}>\cdots>\Gamma_{c}>\Gamma_{c+1}=1$ and $\Phi(G) \neq 1$. Let $\Gamma_{c}$ have exponent $p^{m}$. If $N=\Gamma_{c}^{p^{m-1}}$, the group generated by all $p^{m-1}$ th powers of elements of $\Gamma_{c}$, then
(i) $N$ is elementwise fixed by all automorphisms in $\operatorname{Aut}^{\Phi}(G)$,
(ii) $\operatorname{Aut}^{N}(G) \leq Z\left(\operatorname{Aut}^{\Phi}(G)\right)$,
(iii) $\operatorname{Aut}^{N}(G)$ has order $p^{r d}$, where $p^{r}$ is the order of $N$,
(iv) $\operatorname{Aut}^{\Phi}(G) / \operatorname{Aut}^{N}(G) \hookrightarrow \operatorname{Aut}^{\Phi / N}(G / N)$.

Theorem 2.2. [5, Theorem 3] Let $G$ be as in Theorem 2.1, with $\Phi(G) \neq$ 1 , and let $\Gamma_{i}(G) / \Gamma_{i+1}(G)$ have exponent $p^{m_{i}}$, for $1 \leq i \leq c$. Then, $\operatorname{Aut}^{\Phi}(G)$ is nilpotent and $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq\left(\sum_{i=1}^{c} m_{i}\right)-1$.

Now, we begin by stating a number of lemmas that will be used in the sequel.
Lemma 2.3. Let $G$ be a non-cyclic p-group of order $p^{n}$. Then, $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq n-2$.

Proof. Suppose that $\operatorname{cl}(G)=c$ and $\left(n_{i 1}, n_{i 2}, \ldots, n_{i r_{i}}\right)$ are the invariants of $\Gamma_{i}(G) / \Gamma_{i+1}(G)$, with $n_{i 1} \geq n_{i 2} \geq \cdots \geq n_{i r_{i}}$, for $1 \leq i \leq c$. Hence,

$$
\sum_{i=1}^{c}\left(n_{i 1}+n_{i 2}+\cdots+n_{i r_{i}}\right)=n
$$

and so $\sum_{i=1}^{c} n_{i 1} \leq n-1$, since $\Gamma_{1}(G) / \Gamma_{2}(G)$ is not cyclic. Now, by Theorem 2.2, we deduce that $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq n-2$.

Lemma 2.4. Let $G$ be a non-abelian $p$-group of order $p^{n}$ and $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=t$, where $1 \leq t \leq n-2$. If $d\left(G / \Gamma_{2}(G)\right) \geq n-t$, then $d\left(G / \Gamma_{2}(G)\right)=n-t, G / \Gamma_{2}(G) \cong \mathbb{Z}_{p^{t-c+2}} \times \mathbb{Z}_{p} \times \cdots \times \mathbb{Z}_{p}$ and $\Gamma_{i}(G) / \Gamma_{i+1}(G) \cong \mathbb{Z}_{p}$, for $2 \leq i \leq c$, where $c$ is the nilpotency class of $G$.

Proof. Suppose that $\left(n_{i 1}, n_{i 2}, \ldots, n_{i r_{i}}\right)$ are the invariants of $\Gamma_{i}(G) / \Gamma_{i+1}(G)$, with $n_{i 1} \geq n_{i 2} \geq \cdots \geq n_{i r_{i}}$ for $1 \leq i \leq c$. We have

$$
\sum_{i=1}^{c}\left(n_{i 1}+n_{i 2}+\cdots+n_{i r_{i}}\right)=n
$$

If $n_{1 j}>1$, for some $2 \leq j \leq n-t$, then $\sum_{i=1}^{c} n_{i 1} \leq t$. Hence, $\operatorname{cl}\left(\right.$ Aut $\left.^{\Phi}(G)\right) \leq t-1$, by Theorem 2.2, which is impossible. Therefore, $n_{1 j}=1$, for $2 \leq j \leq n-t$. Now, we have $r_{1}=n-t$; for otherwise, $r_{1}>n-t$ and by the same argument as above we deduce that $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq t-1$, which is a contradiction. So,

$$
G / \Gamma_{2}(G) \cong \mathbb{Z}_{p^{n 11}} \times \mathbb{Z}_{p} \times \cdots \times \mathbb{Z}_{p}
$$

with $d\left(G / \Gamma_{2}(G)\right)=n-t$. We see that $n_{i 1}=1$, for $2 \leq i \leq c$, by [2, Theorem 1.5]. Moreover if $\left|\Gamma_{i}(G) / \Gamma_{i+1}(G)\right|>p$, for some $2 \leq i \leq c$, then again $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq t-1$. Thus, $\Gamma_{i}(G) / \Gamma_{i+1}(G) \cong \mathbb{Z}_{p}$, for $2 \leq i \leq c$, completing the proof.

Lemma 2.5. Let $G$ be a non-abelian group of order $p^{n}(n \geq 4)$ and $\operatorname{cl}(G)=n-2$. If $G / \Gamma_{2}(G) \cong \mathbb{Z}_{p^{2}} \times \mathbb{Z}_{p}$, then $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq n-3$.

Proof. We use induction on $n$. For $n=4$, we claim that $G$ is purely non-abelian. Otherwise, we may write $G=A \times B$, where $A \neq 1$ is abelian and $B$ is purely non-abelian. Hence, $B$ is extra-special of order $p^{3}$ and so $G / G^{\prime} \cong \mathbb{Z}_{p} \times \mathbb{Z}_{p} \times \mathbb{Z}_{p}$, which is a contradiction. By [ 6 , Lemma 0.4], we have $\exp (G / Z(G))=\exp \left(G^{\prime}\right)=p$, which implies that $G / Z(G)$ is elementary abelian. Therefore, $\Phi(G) \leq Z(G)$. Now, we prove
that $\operatorname{Aut}^{\Phi}(G)$ is abelian. To see this, we consider two cases for $Z(G)$. First, we suppose that $Z(G) \cong \mathbb{Z}_{p^{2}}$. Then by [1, Theorem 1], we have $\left|\operatorname{Aut}^{Z}(G)\right|=p^{3}$, and so $\left|\operatorname{Aut}^{\Phi}(G)\right| \leq p^{3}$. Also, $\mid$ Aut $^{G^{\prime}}(G) \mid \geq p^{2}$ and Aut ${ }^{G^{\prime}}(G) \leq Z\left(\right.$ Aut $\left.^{\Phi}(G)\right)$, by Theorem 2.1 (iii) and (ii). This yields that Aut ${ }^{\Phi}(G)$ is abelian. Next, suppose that $Z(G)$ is elementary abelian. Therefore, Aut ${ }^{\Phi}(G)$ fixes $G^{p}$ elementwise, since $\Phi(G) \leq Z(G)$. Hence, by Theorem 2.1(i), Aut ${ }^{\Phi}(G)$ fixes $\Phi(G)$ elementwise and consequently Aut ${ }^{\Phi}(G)$ is abelian. Now, suppose that $n \geq 5$ and the result holds for any group of order less than $p^{n}$. On setting $N=\Gamma_{n-2}(G)$, we may see that $\operatorname{cl}(G / N)=n-3,|G / N|=p^{n-1}$ and $G / N$ satisfies the conditions of the Lemma. Thus, $\operatorname{cl}\left(\mathrm{Aut}^{\Phi / N}(G / N)\right) \leq n-4$, by the induction hypothesis. Now, the parts (ii) and (iv) of Theorem 2.1 imply that $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq n-3$, as desired.

We have the following theorem due to Müller [7]
Theorem 2.6. [7, Theorem] If $G$ is a finite p-group which is neither elementary abelian nor extra-special, then $\operatorname{Aut}^{\Phi}(G) / \operatorname{Inn}(G)$ is a nontrivial normal p-subgroup of the group of outer automorphisms of $G$.

Corollary 2.7. Let $G$ be an extra-special p-group of order $p^{n}$. Then, Aut ${ }^{\Phi}(G)$ is elementary abelian of order $p^{n-1}$.

Theorem 2.8. Let $G$ be a non-abelian p-group of order $p^{n}(n \geq 3)$. Then, $G$ is of maximal class if and only if $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=n-2$.
Proof. If $G$ is of maximal class, then $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \geq n-2$, since $\operatorname{Inn}(G) \leq$ $\operatorname{Aut}^{\Phi}(G)$. So, $\operatorname{cl}\left(\mathrm{Aut}^{\Phi}(G)\right)=n-2$, by Lemma 2.3. Now, suppose that $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=n-2$. By induction on $n$, we prove that $G$ is of maximal class. If $n=3$, then obviously $G$ is of maximal class. Assume that $|G|=p^{n}, n \geq 4$ and the result holds for any group of order less than $p^{n}$. If $\operatorname{cl}(G)=c$, then $G / \Gamma_{2}(G) \cong \mathbb{Z}_{p^{n-c}} \times \mathbb{Z}_{p}$ and $\Gamma_{i}(G) / \Gamma_{i+1}(G) \cong \mathbb{Z}_{p}$, for $2 \leq i \leq c$, by Lemma 2.4. On setting $N=\Gamma_{c}(G)$, we see that $\operatorname{cl}\left(\operatorname{Aut}^{\Phi / N}(G / N)\right) \geq n-3$, by Theorem 2.1 (ii) and (iv). Consequently, $\operatorname{cl}\left(\operatorname{Aut}^{\Phi / N}(G / N)\right)=n-3$, by Theorem 2.2. Hence by the induction hypothesis, $G / N$ is of maximal class, which implies that $\operatorname{cl}(G)=\operatorname{cl}(G / N)+1=n-1$, as desired.

Lemma 2.9. If $G$ is a p-group of order $p^{n}(n \geq 4)$ and $\operatorname{cl}(G)=n-2$, then $\operatorname{Aut}^{\Phi}(G)$ is of class $n-3$.

Proof. We have $p^{2} \leq\left|G / \Gamma_{2}(G)\right| \leq p^{3}$. If $G / \Gamma_{2}(G)$ is elementary abelian, then $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq n-3$, by Theorem 2.2. Therefore, $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=$ $n-3$, since $\operatorname{Inn}(G) \leq \operatorname{Aut}^{\Phi}(G)$. Now, if $G / \Gamma_{2}(G) \cong \mathbb{Z}_{p^{2}} \times \mathbb{Z}_{p}$, then $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq n-3$, by Lemma 2.5, which completes the proof.

Remark 2.10. The converse of Lemma 2.9 does not hold. To see this we consider a family of groups of order $p^{5}$ for any prime $p$ as follows:

$$
\begin{aligned}
G=\left\langle a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right| a_{1}^{p} & =a_{3}, a_{2}^{p}=a_{4}, a_{3}^{p}=a_{5}, a_{4}^{p}=a_{5}^{p}=1, \\
{\left[a_{1}, a_{2}\right] } & \left.=a_{5},\left[a_{i}, a_{j}\right]=1\right\rangle,
\end{aligned}
$$

where, $1 \leq i<j \leq 5$ and $(i, j) \neq(1,2)$.
Obviously, we have $\left|G^{\prime}\right|=p$, which implies that $\operatorname{cl}(G)=2$. Now, since $\exp \left(G / G^{\prime}\right)=p^{2}$, we see that $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq 2$, by Theorem 2.2. Furthermore, we define the maps $\alpha$ and $\beta$ by $a_{1}^{\alpha}=a_{1} a_{4}, a_{i}^{\alpha}=a_{i}$ for $2 \leq i \leq 5$ and $a_{2}^{\beta}=a_{2} a_{3}, a_{4}^{\beta}=a_{4} a_{5}, a_{i}^{\beta}=a_{i}$, for $i \in\{1,3,5\}$. Now, it is easy to show that $\alpha, \beta \in \operatorname{Aut}^{\Phi}(G)$ since $\Phi(G)=\left\langle a_{3}, a_{4}, a_{5}\right\rangle$. Also, $\alpha \beta \neq \beta \alpha$, which yields $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=2$.

Theorem 2.11. Let $G$ be a non-abelian p-group of order $p^{n}$ and $G^{\prime}=\Phi(G)$. Then, $\operatorname{cl}(G)=c$ if and only if $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=c-1$, where, $2 \leq c \leq n-1$.

Proof. First suppose that $\mathrm{cl}(G)=c$. Then, $\exp \left(\Gamma_{i}(G) / \Gamma_{i+1}(G)\right)=p$, for $1 \leq i \leq c$, by $\left[2\right.$, Theorem 1.5]. Hence, $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq c-1$, by Theorem 2.2. Moreover, $\mathrm{cl}(\operatorname{Inn}(G))=c-1$, completing the proof. Now if $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=c-1$ and $\operatorname{cl}(G)=d$, then by the same argument as above we have $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=d-1$. This implies that $d=c$.

## 3. $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)$ When $G$ Is Abelian

Let $G$ be an abelian $p$-group, where $p$ is an odd prime. In this section, we find the nilpotency class of Aut ${ }^{\Phi}(G)$ according to the invariants of $G$. First, we find $\operatorname{Aut}^{\Phi}(G)$ for cyclic and elementary abelian $p$-groups $G$.

Lemma 3.1. If $G$ is a cyclic group of order $p^{n}$, then $\operatorname{Aut}^{\Phi}(G) \cong \mathbb{Z}_{p^{n-1}}$.
Proof. Let $G=\langle x\rangle$. Then, obviously the automorphism $\alpha$, defined by $x^{\alpha}=x^{1+p}$, is of order $p^{n-1}$ lying in Aut ${ }^{\Phi}(G)$. Therefore, we can
complete the proof by the fact that $\operatorname{Aut}(G) \cong \mathbb{Z}_{p^{n-1}(p-1)}$ and $|\alpha|=$ $p^{n-1}$.

Lemma 3.2. Let $G$ be a finite p-group. Then, $\operatorname{Aut}^{\Phi}(G)=1$ if and only if $G$ is elementary abelian.

Proof. Let $G$ be an elementary abelian $p$-group, then $\operatorname{Aut}^{\Phi}(G)=1$, by Theorem 2.6. Now, if $\operatorname{Aut}^{\Phi}(G)=1$. Then $\operatorname{Inn}(G)=1$, or equivalently, $G$ is abelian. Assume that $G=\left\langle x_{1}\right\rangle \times \cdots \times\left\langle x_{r}\right\rangle$, where $\left|x_{i}\right|=p^{m_{i}}$, for $1 \leq i \leq r$ and $m_{1} \geq m_{2} \geq \cdots \geq m_{r}$. We claim that $m_{1}=1$; for otherwise, $1 \neq x_{1}^{p} \in \Phi(G)$ and so the map $\alpha$ defined by $x_{1}^{\alpha}=x_{1}^{1+p}$, $x_{i}^{\alpha}=x_{i}(2 \leq i \leq r)$ is a non-trivial automorphism of Aut $^{\Phi}(G)$, which is a contradiction.

Lemma 3.3. Let $G$ be a finite p-group. Then, Aut ${ }^{\Phi}(G)$ is a non-trivial cyclic group if and only if $G$ is cyclic of order greater than $p$.

Proof. If $G \cong \mathbb{Z}_{p^{n}}$, where $n>1$, then Lemma 3.1 completes the proof. Now, assume that $\mathrm{Aut}^{\Phi}(G)$ is non-trivial and cyclic. Therefore, $G$ is abelian, since $\operatorname{Inn}(G) \leq \operatorname{Aut}^{\Phi}(G)$. If $G$ is not cyclic, then we may write $G=\langle x\rangle \times\langle y\rangle \times H$, where, $|x|=p^{m},|y|=p^{n}, m \geq n \geq 1$ and $\exp (H) \leq p^{n}$. Hence, $m>1$, by Lemma 3.2, and so $1 \neq x^{p} \in \Phi(G)$. Therefore, we may define the automorphisms $\sigma$ and $\tau$ by $x^{\sigma}=x^{1+p}$, $y^{\sigma}=y, h^{\sigma}=h$, for all $h \in H$ and $x^{\tau}=x, y^{\tau}=y x^{p^{m-1}}, h^{\tau}=h$, for all $h \in H$. Obviously, $\sigma, \tau \in$ Aut $^{\Phi}(G)$ and $\langle\sigma\rangle \cap\langle\tau\rangle=1$. This implies that $\langle\sigma\rangle \times\langle\tau\rangle \leq$ Aut $^{\Phi}(G)$, which is a contradiction.

Now, let $G$ be an abelian $p$-group and let $\left(m_{1}, m_{2}, \ldots, m_{r}\right)$ be the invariants of $G$ with $m_{1} \geq m_{2} \geq \cdots \geq m_{r}$. For the rest of the paper, we assume that $G$ is neither cyclic nor elementary abelian. Therefore, we may write $m_{1}>1$ and $r>1$. In Lemma 3.4, we find, $\operatorname{Aut}^{\Phi}(G)$ for the case $m_{1}>1$ and $m_{2}=1$. Then, in Theorem 3.5, we consider the case $m_{2}>1$.

Lemma 3.4. Let $G$ be a non-cyclic abelian p-group and let $\left(m_{1}, m_{2}, \ldots, m_{r}\right)$ be the invariants of $G$ with $m_{1} \geq m_{2} \geq \cdots \geq m_{r}$ and $m_{1}>1, m_{2}=1$. Then,
(i) $\operatorname{Aut}^{\Phi}(G)$ is abelian of order $p^{m_{1}+r-2}$.
(ii) $\mathrm{Aut}^{\Phi}(G)$ has the invariants $\left(m_{1}-1, m_{2}, \ldots, m_{r}\right)$.

Proof. (i) Let $G=\left\langle x_{1}\right\rangle \times \cdots \times\left\langle x_{r}\right\rangle$, where, $\left|x_{1}\right|=p^{m_{1}}$ and $\left|x_{2}\right|=\cdots=$ $\left|x_{r}\right|=p$. Then, we may easily see that any automorphism $\alpha$ of $G$, which fixes $G / \Phi(G)$ elementwise, has the form: $x_{1}^{\alpha}=x_{1}^{1+\ell_{1} p}, x_{i}^{\alpha}=x_{i} x_{1}^{\ell_{i} p^{m_{1}-1}}$, where, $0 \leq \ell_{1}<p^{m_{1}-1}$ and $0 \leq \ell_{i}<p$, for $2 \leq i \leq r$. This completes the proof.
(ii) For $2 \leq i \leq r$, we define the automorphism $\alpha_{i}$ by $x_{i}^{\alpha_{i}}=x_{i} x_{1}^{p^{m_{1}-1}}$, $x_{j}^{\alpha_{i}}=x_{j}$, where, $1 \leq j \leq r$ and $j \neq i$. Also, we define $\alpha_{1}$ by $x_{1}^{\alpha_{1}}=x_{1}^{1+p}$, $x_{j}^{\alpha_{1}}=x_{j}$, where, $2 \leq j \leq r$. Obviously, $\left|\alpha_{1}\right|=p^{m_{1}-1},\left|\alpha_{2}\right|=\left|\alpha_{3}\right|=$ $\cdots=\left|\alpha_{r}\right|=p$. Therefore, by (i), we deduce that

$$
\operatorname{Aut}^{\Phi}(G)=\left\langle\alpha_{1}\right\rangle \times\left\langle\alpha_{2}\right\rangle \times \cdots \times\left\langle\alpha_{r}\right\rangle,
$$

as desired.
Theorem 3.5. Let $G$ be a non-cyclic abelian p-group and let $\left(m_{1}, m_{2}, \ldots, m_{r}\right)$ be the invariants of $G$ with $m_{1} \geq m_{2} \geq \cdots \geq m_{r}$ and $m_{2}>1$. Then,
(i) $m_{2}-1 \leq \operatorname{cl}\left(\right.$ Aut $\left.^{\Phi}(G)\right) \leq m_{1}-1$.
(ii) If $m_{1}>m_{2}$ then $m_{2} \leq \operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq m_{1}-1$.

Proof. (i) Let $G \cong\left\langle x_{1}\right\rangle \times\left\langle x_{2}\right\rangle \times \cdots \times\left\langle x_{r}\right\rangle$, where, $\left|x_{i}\right|=p^{m_{i}}$, $1 \leq i \leq r$. For $1 \leq i \leq m_{2}-1$, we define the automorphisms $\alpha_{i}$ by $x_{1}^{\alpha_{i}}=x_{1} x_{2}^{p^{i}+p^{i+1}+\cdots+p^{m_{2}-1}}$ and $x_{j}^{\alpha_{i}}=x_{j}$, for $2 \leq j \leq r$. Also, we define the automorphism $\beta$ by $x_{1}^{\beta}=x_{1}, x_{2}^{\beta}=x_{2}^{1+p+p^{2}+\cdots+p^{m_{2}-1}}$ and $x_{j}^{\beta}=x_{j}$, for $3 \leq j \leq r$. Obviously, $\alpha_{i}\left(1 \leq i \leq m_{2}-1\right)$ and $\beta$ are in Aut ${ }^{\Phi}(G)$. By an easy calculation, we may see that $\left[\beta^{-1}, \alpha_{i}^{-1}\right]=\alpha_{i+1}^{-1}$, for $1 \leq i \leq m_{2}-2$. This implies that $\alpha_{i+1} \in \Gamma_{i+1}\left(\operatorname{Aut}^{\Phi}(G)\right)$, for $1 \leq i \leq m_{2}-2$. Since $\alpha_{m_{2}-1} \neq 1$, we have $\Gamma_{m_{2}-1}\left(\operatorname{Aut}^{\Phi}(G)\right) \neq 1$, and so $m_{2}-1 \leq \operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)$. Furthermore, $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq m_{1}-1$, by Theorem 2.2.
(ii) According to (i), we define the automorphism $\gamma$ by $x_{1}^{\gamma}=x_{1}, x_{2}^{\gamma}=$ $x_{2} x_{1}^{p^{m_{1}-m_{2}}}$ and $x_{j}^{\gamma}=x_{j}$, for $3 \leq j \leq r$. We have $\left[\gamma^{-1}, \alpha_{m_{2}-1}^{-1}\right] \neq 1$ and $\alpha_{m_{2}-1} \in \Gamma_{m_{2}-1}\left(\operatorname{Aut}^{\Phi}(G)\right)$. Hence, $\Gamma_{m_{2}}\left(\operatorname{Aut}^{\Phi}(G)\right) \neq 1$, and so $m_{2} \leq \operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right) \leq m_{1}-1$.

The following corollaries are immediate consequences of Theorem 3.5.

Corollary 3.6. Let $G$ be a non-cyclic abelian p-group and let $\left(m_{1}, m_{2}, \ldots, m_{r}\right)$ be the invariants of $G$ with $m_{1} \geq m_{2} \geq \cdots \geq m_{r}$ and $m_{1}=m_{2}>1$. Then, $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=m_{1}-1$.

Corollary 3.7. Let $G$ be a non-cyclic abelian p-group and let $\left(m_{1}, m_{2}, \ldots, m_{r}\right)$ be the invariants of $G$ with $m_{1} \geq m_{2} \geq \cdots \geq m_{r}$ and $m_{2}=m_{1}-1, m_{1}>1$. Then, $\operatorname{cl}\left(\operatorname{Aut}^{\Phi}(G)\right)=m_{2}$.

Theorem 3.8. Let $G$ be an abelian p-group and let $\left(m_{1}, m_{2}, \ldots, m_{r}\right)$ be the invariants of $G$ with $m_{1} \geq m_{2} \geq \cdots \geq m_{r}$. Then, Aut ${ }^{\Phi}(G)$ is non-trivial abelian if and only if either $m_{1}>1, m_{2} \leq 1$ or $m_{1}=m_{2}=2$.
Proof. If either $m_{1}>1, m_{2} \leq 1$ or $m_{1}=m_{2}=2$, then $\operatorname{Aut}^{\Phi}(G)$ is abelian by considering lemmas 3.4 and 3.1 and Corollary 3.6. Conversely, if Aut ${ }^{\Phi}(G)$ is non-trivial abelian, then $m_{2} \leq 2$ by Theorem 3.5 (i). If $m_{2}=2$ then $m_{1}=2$ by Theorem 3.5 (ii). Now, if $m_{2} \leq 1$, then by using lemma 3.1 and 3.2 , we may see that $m_{1}>1$.

## Acknowledgments

The authors are grateful to the referee for his valuable suggestions. The paper was revised according to his/her comments. The work of authors was in part supported by Arak University.

## References

[1] J. E. Adney and T. Yen, Automorphisms of a p-group, Illinois J. Math. 9 (1965) 137-143.
[2] N. Blackburn, On a special class of p-groups, Acta Math. 100 (1958) 45-92.
[3] B. Huppert, Endliche Gruppen I, (German), Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967.
[4] C. R. Leedham-Green and S. McKay. The Structure of Groups of Prime Power Order, London Mathematical Society Monographs, New Series, 27, Oxford Science Publications, Oxford University Press, Oxford, 2002.
[5] H. Liebeck, The Automorphism group of finite p-groups, J. Algebra 4 (1966) 426-432.
[6] M. Morigi, On the minimal number of generators of finite non-abelian $p$-groups having an abelian automorphism group, Comm. Algebra 23(6), (1995) 20452065.
[7] O. Müller, On p-automorphisms of finite p-groups, Arch. Math. (Basel) 32 (1) (1979) 533-538.

## S. Fouladi

Department of Mathematics, Arak University, Arak, Iran
Email: s-fouladi@araku.ac.ir

## R. Orfi

Department of Mathematics, Arak University, Arak, Iran
Email: r-orfi@araku.ac.ir


[^0]:    MSC(2000): Primary: 20D45; Secondary: 20D15.
    Keywords: Finite $p$-group, automorphism group, nilpotency class.
    Received: 18 January 2010, Accepted: 4 April 2010.
    *Corresponding author
    (c) 2011 Iranian Mathematical Society.

