ON THE NILPOTENCY CLASS OF THE AUTOMORPHISM GROUP OF SOME FINITE $p$-GROUPS

S. FOULADI* AND R. ORFI

Communicated by Saeid Azam

Abstract. Let $G$ be a $p$-group of order $p^n$ and $\Phi = \Phi(G)$ be the Frattini subgroup of $G$. It is shown that the nilpotency class of $\text{Aut}^\Phi(G)$, the group of all automorphisms of $G$ centralizing $G/\Phi(G)$, takes the maximum value $n - 2$ if and only if $G$ is of maximal class. We also determine the nilpotency class of $\text{Aut}^\Phi(G)$ when $G$ is a finite abelian $p$-group.

1. Introduction

It is well known [3, III, Satz 3.17] that if $G$ is a finite $p$-group with the Frattini subgroup $\Phi = \Phi(G)$, then $\text{Aut}^\Phi(G)$ is a finite $p$-group. Liebeck [5] found an upper bound for the nilpotency class of $\text{Aut}^\Phi(G)$.

Here, we find the nilpotency class of $\text{Aut}^\Phi(G)$ in some cases. A straightforward consequence of the result in [5] shows that the nilpotency class of $\text{Aut}^\Phi(G)$ is less than or equal to $n - 2$, for all non-cyclic $p$-groups of order $p^n$. Here, we show that the nilpotency class of $\text{Aut}^\Phi(G)$ takes the maximum value $n - 2$ if and only if $G$ is of maximal class. Moreover, we find the nilpotency class of $\text{Aut}^\Phi(G)$ for a finite abelian $p$-group $G$ in terms of its invariants, where $p$ is an odd prime.

Keywords: Finite $p$-group, automorphism group, nilpotency class.
Received: 18 January 2010, Accepted: 4 April 2010.
*Corresponding author
© 2011 Iranian Mathematical Society.
Throughout, the following notation is used. The terms of the lower central series of \( G \) are denoted by \( \Gamma_i = \Gamma_i(G) \). The center of \( G \) is denoted by \( Z = Z(G) \). The nilpotency class of a group \( G \) is denoted by \( \text{cl}(G) \). If \( \alpha \) is an automorphism of \( G \) and \( x \) is an element of \( G \), we write \( x^\alpha \) for the image of \( x \) under \( \alpha \). The inner automorphism induced by the element \( g \) is denoted by \( \sigma_g \). For a normal subgroup \( N \) of \( G \), we let \( \text{Aut}^N(G) \) denote the group of all automorphisms of \( G \) centralizing \( G/N \). We write \( d(G) \) for the minimal number of generators of \( G \). An extra-special \( p \)-group is a \( p \)-group \( G \) with \( \Phi(G) = Z(G) = G' \cong \mathbb{Z}_p \). A non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian. Also, \( \mathbb{Z}_n \) is the cyclic group of order \( n \). All unexplained notation is standard and follows that of [4].

2. Maximum Value of \( \text{cl}(\text{Aut}^\Phi(G)) \)

Let \( G \) be a non-cyclic \( p \)-group of order \( p^n \). In this section, we prove that \( \text{cl}(\text{Aut}^\Phi(G)) \) takes the maximum value \( n - 2 \) if and only if \( G \) is of maximal class. First, we give some basic results that are needed for the main results of this section. In [5], Liebeck proved the following theorems which play important roles in our proofs.

**Theorem 2.1.** [5, Theorem 2] Let \( G \) be a finite \( d \)-generator \( p \)-group with lower central series \( G = \Gamma_1 > \cdots > \Gamma_c \) with \( \Gamma_{c+1} = 1 \) and \( \Phi(G) \neq 1 \). Let \( \Gamma_c \) have exponent \( p^m \). If \( N = \Gamma_c^{p^m-1} \), the group generated by all \( p^m-1 \)th powers of elements of \( \Gamma_c \), then

(i) \( N \) is elementwise fixed by all automorphisms in \( \text{Aut}^\Phi(G) \),

(ii) \( \text{Aut}^N(G) \leq Z(\text{Aut}^\Phi(G)) \),

(iii) \( \text{Aut}^N(G) \) has order \( p^r \), where \( p^r \) is the order of \( N \),

(iv) \( \text{Aut}^\Phi(G)/\text{Aut}^N(G) \hookrightarrow \text{Aut}^\Phi(G/N) \).

**Theorem 2.2.** [5, Theorem 3] Let \( G \) be as in Theorem 2.1, with \( \Phi(G) \neq 1 \), and let \( \Gamma_i(G)/\Gamma_{i+1}(G) \) have exponent \( p^{m_i} \), for \( 1 \leq i \leq c \). Then, \( \text{Aut}^\Phi(G) \) is nilpotent and \( \text{cl}(\text{Aut}^\Phi(G)) \leq (\sum_{i=1}^c m_i) - 1 \).

Now, we begin by stating a number of lemmas that will be used in the sequel.

**Lemma 2.3.** Let \( G \) be a non-cyclic \( p \)-group of order \( p^n \). Then, \( \text{cl}(\text{Aut}^\Phi(G)) \leq n - 2 \).
Proof. Suppose that $\text{cl}(G) = c$ and $(n_{i_1}, n_{i_2}, \ldots, n_{i_r})$ are the invariants of $\Gamma_i(G)/\Gamma_{i+1}(G)$, with $n_{i_1} \geq n_{i_2} \geq \cdots \geq n_{i_r}$, for $1 \leq i \leq c$. Hence,

$$\sum_{i=1}^{c} (n_{i_1} + n_{i_2} + \cdots + n_{i_r}) = n,$$

and so $\sum_{i=1}^{c} n_{i_1} \leq n - 1$, since $\Gamma_1(G)/\Gamma_2(G)$ is not cyclic. Now, by Theorem 2.2, we deduce that $\text{cl}(\text{Aut}^G(G)) \leq n - 2$. \hfill $\square$

**Lemma 2.4.** Let $G$ be a non-abelian $p$-group of order $p^n$ and $\text{cl}(\text{Aut}^G(G)) = t$, where $1 \leq t \leq n - 2$. If $d(G/\Gamma_2(G)) \geq n - t$, then $d(G/\Gamma_2(G)) = n - t$, $G/\Gamma_2(G) \cong \mathbb{Z}_p^{c+2} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ and $\Gamma_i(G)/\Gamma_{i+1}(G) \cong \mathbb{Z}_p$, for $2 \leq i \leq c$, where $c$ is the nilpotency class of $G$.

**Proof.** Suppose that $(n_{i_1}, n_{i_2}, \ldots, n_{i_r})$ are the invariants of $\Gamma_i(G)/\Gamma_{i+1}(G)$, with $n_{i_1} \geq n_{i_2} \geq \cdots \geq n_{i_r}$, for $1 \leq i \leq c$. We have

$$\sum_{i=1}^{c} (n_{i_1} + n_{i_2} + \cdots + n_{i_r}) = n.$$

If $n_{1j} > 1$, for some $2 \leq j \leq n - t$, then $\sum_{i=1}^{c} n_{i_1} \leq t$. Hence, $\text{cl}(\text{Aut}^G(G)) \leq t - 1$, by Theorem 2.2, which is impossible. Therefore, $n_{1j} = 1$, for $2 \leq j \leq n - t$. Now, we have $r_1 = n - t$; for otherwise, $r_1 > n - t$ and by the same argument as above we deduce that $\text{cl}(\text{Aut}^G(G)) \leq t - 1$, which is a contradiction. So,

$$G/\Gamma_2(G) \cong \mathbb{Z}_p^{r_1+1} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$$

with $d(G/\Gamma_2(G)) = n - t$. We see that $n_{i_1} = 1$, for $2 \leq i \leq c$, by [2, Theorem 1.5]. Moreover if $|\Gamma_i(G)/\Gamma_{i+1}(G)| > p$, for some $2 \leq i \leq c$, then again $\text{cl}(\text{Aut}^G(G)) \leq t - 1$. Thus, $\Gamma_i(G)/\Gamma_{i+1}(G) \cong \mathbb{Z}_p$, for $2 \leq i \leq c$, completing the proof. \hfill $\square$

**Lemma 2.5.** Let $G$ be a non-abelian group of order $p^n$ ($n \geq 4$) and $\text{cl}(G) = n - 2$. If $G/\Gamma_2(G) \cong \mathbb{Z}_p^{c} \times \mathbb{Z}_p$, then $\text{cl}(\text{Aut}^G(G)) \leq n - 3$.

**Proof.** We use induction on $n$. For $n = 4$, we claim that $G$ is purely non-abelian. Otherwise, we may write $G = A \times B$, where $A \neq 1$ is abelian and $B$ is purely non-abelian. Hence, $B$ is extra-special of order $p^3$ and so $G/G' \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, which is a contradiction. By [6, Lemma 0.4], we have $\text{exp}(G/Z(G)) = \text{exp}(G') = p$, which implies that $G/Z(G)$ is elementary abelian. Therefore, $\Phi(G) \leq Z(G)$. Now, we prove
that $\text{Aut}^\Phi(G)$ is abelian. To see this, we consider two cases for $Z(G)$. First, we suppose that $Z(G) \cong \mathbb{Z}_{p^2}$. Then by [1, Theorem 1], we have $|\text{Aut}_G^Z(G)| = p^3$, and so $|\text{Aut}^\Phi(G)| \leq p^3$. Also, $|\text{Aut}_G^G(G)| \geq p^2$ and $\text{Aut}_G^G(G) \leq Z(\text{Aut}^\Phi(G))$, by Theorem 2.1 (iii) and (ii). This yields that $\text{Aut}^\Phi(G)$ is abelian. Next, suppose that $Z(G)$ is elementary abelian.

Therefore, $\text{Aut}^\Phi(G)$ fixes $G$ elementwise, since $\Phi(G) \leq Z(G)$. Hence, by Theorem 2.1(i), $\text{Aut}^\Phi(G)$ fixes $\Phi(G)$ elementwise and consequently $\text{Aut}^\Phi(G)$ is abelian. Now, suppose that $n \geq 5$ and the result holds for any group of order less than $p^n$. On setting $N = \Gamma_{n-2}(G)$, we may see that $\text{cl}(G/N) = n - 3$, $|G/N| = p^{n-1}$ and $G/N$ satisfies the conditions of the Lemma. Thus, $\text{cl}(\text{Aut}^\Phi(G/N)(G/N)) \leq n - 4$, by the induction hypothesis. Now, the parts (ii) and (iv) of Theorem 2.1 imply that $\text{cl}(\text{Aut}^\Phi(G)) \leq n - 3$, as desired.

We have the following theorem due to Müller [7]

**Theorem 2.6.** [7, Theorem] If $G$ is a finite $p$-group which is neither elementary abelian nor extra-special, then $\text{Aut}^\Phi(G)/\text{Inn}(G)$ is a non-trivial normal $p$-subgroup of the group of outer automorphisms of $G$.

**Corollary 2.7.** Let $G$ be an extra-special $p$-group of order $p^n$. Then, $\text{Aut}^\Phi(G)$ is elementary abelian of order $p^{n-1}$.

**Theorem 2.8.** Let $G$ be a non-abelian $p$-group of order $p^n$ ($n \geq 3$). Then, $G$ is of maximal class if and only if $\text{cl}(\text{Aut}^\Phi(G)) = n - 2$.

**Proof.** If $G$ is of maximal class, then $\text{cl}(\text{Aut}^\Phi(G)) \geq n - 2$, since $\text{Inn}(G) \leq \text{Aut}^\Phi(G)$. So, $\text{cl}(\text{Aut}^\Phi(G)) = n - 2$, by Lemma 2.3. Now, suppose that $\text{cl}(\text{Aut}^\Phi(G)) = n - 2$. By induction on $n$, we prove that $G$ is of maximal class. If $n = 3$, then obviously $G$ is of maximal class. Assume that $|G| = p^n$, $n \geq 4$ and the result holds for any group of order less than $p^n$. If $c(G) = c$, then $G/\Gamma_2(G) \cong \mathbb{Z}_{p^{n-c}} \times \mathbb{Z}_p$ and $\Gamma_2(G)/\Gamma_{i+1}(G) \cong \mathbb{Z}_p$, for $2 \leq i \leq c$, by Lemma 2.4. On setting $N = \Gamma_c(G)$, we see that $\text{cl}(\text{Aut}^\Phi/G(N)) \geq n - 3$, by Theorem 2.1 (ii) and (iv). Consequently, $\text{cl}(\text{Aut}^\Phi/G(N)) = n - 3$, by Theorem 2.2. Hence by the induction hypothesis, $G/N$ is of maximal class, which implies that $\text{cl}(G) = \text{cl}(G/N) + 1 = n - 1$, as desired.

**Lemma 2.9.** If $G$ is a $p$-group of order $p^n$ ($n \geq 4$) and $\text{cl}(G) = n - 2$, then $\text{Aut}^\Phi(G)$ is of class $n - 3$. 
On the nilpotency class of the automorphism group of some finite $p$-groups

Proof. We have $p^2 \leq |G/\Gamma_2(G)| \leq p^3$. If $G/\Gamma_2(G)$ is elementary abelian, then $\text{cl}(\text{Aut}^\Phi(G)) \leq n - 3$, by Theorem 2.2. Therefore, $\text{cl}(\text{Aut}^\Phi(G)) = n - 3$, since $\text{Inn}(G) \leq \text{Aut}^\Phi(G)$. Now, if $G/\Gamma_2(G) \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, then $\text{cl}(\text{Aut}^\Phi(G)) \leq n - 3$, by Lemma 2.5, which completes the proof. \hfill \Box

Remark 2.10. The converse of Lemma 2.9 does not hold. To see this we consider a family of groups of order $p^5$ for any prime $p$ as follows:

$$G = \langle a_1, a_2, a_3, a_4, a_5 | a_1^p = a_3, a_2^p = a_4, a_3^p = a_5, a_4^p = a_5^p = 1, [a_1, a_2] = a_5, [a_i, a_j] = 1 \rangle,$$

where, $1 \leq i < j \leq 5$ and $(i, j) \neq (1, 2)$.

Obviously, we have $|G'| = p$, which implies that $\text{cl}(G) = 2$. Now, since $\exp(G/G') = p^2$, we see that $\text{cl}(\text{Aut}^\Phi(G)) \leq 2$, by Theorem 2.2. Furthermore, we define the maps $\alpha$ and $\beta$ by $a_1^\alpha = a_1a_4$, $a_i^\alpha = a_i$ for $2 \leq i \leq 5$ and $a_2^\beta = a_2a_3$, $a_4^\beta = a_4a_5$, $a_i^\beta = a_i$, for $i \in \{1, 3, 5\}$. Now, it is easy to show that $\alpha, \beta \in \text{Aut}^\Phi(G)$ since $\Phi(G) = \langle a_3, a_4, a_5 \rangle$. Also, $\alpha\beta \neq \beta\alpha$, which yields $\text{cl}(\text{Aut}^\Phi(G)) = 2$.

Theorem 2.11. Let $G$ be a non-abelian $p$-group of order $p^a$ and $G' = \Phi(G)$. Then, $\text{cl}(G) = c$ if and only if $\text{cl}(\text{Aut}^\Phi(G)) = c - 1$, where, $2 \leq c \leq n - 1$.

Proof. First suppose that $\text{cl}(G) = c$. Then, $\exp(\Gamma_i(G)/\Gamma_{i+1}(G)) = p$, for $1 \leq i \leq c$, by [2, Theorem 1.5]. Hence, $\text{cl}(\text{Aut}^\Phi(G)) \leq c - 1$, by Theorem 2.2. Moreover, $\text{cl}(\text{Inn}(G)) = c - 1$, completing the proof. Now if $\text{cl}(\text{Aut}^\Phi(G)) = c - 1$ and $\text{cl}(G) = d$, then by the same argument as above we have $\text{cl}(\text{Aut}^\Phi(G)) = d - 1$. This implies that $d = c$. \hfill \Box

3. $\text{cl}(\text{Aut}^\Phi(G))$ When $G$ Is Abelian

Let $G$ be an abelian $p$-group, where $p$ is an odd prime. In this section, we find the nilpotency class of $\text{Aut}^\Phi(G)$ according to the invariants of $G$. First, we find $\text{Aut}^\Phi(G)$ for cyclic and elementary abelian $p$-groups $G$.

Lemma 3.1. If $G$ is a cyclic group of order $p^a$, then $\text{Aut}^\Phi(G) \cong \mathbb{Z}_{p^{a-1}}$.

Proof. Let $G = \langle x \rangle$. Then, obviously the automorphism $\alpha$, defined by $x^\alpha = x^{1+p}$, is of order $p^{a-1}$ lying in $\text{Aut}^\Phi(G)$. Therefore, we can
complete the proof by the fact that $\text{Aut}(G) \cong \mathbb{Z}_{p^{n-1}(p-1)}$ and $|\alpha| = p^{n-1}$.

Lemma 3.2. Let $G$ be a finite $p$-group. Then, $\text{Aut}^\Phi(G) = 1$ if and only if $G$ is elementary abelian.

Proof. Let $G$ be an elementary abelian $p$-group, then $\text{Aut}^\Phi(G) = 1$, by Theorem 2.6. Now, if $\text{Aut}^\Phi(G) = 1$. Then $\text{Inn}(G) = 1$, or equivalently, $G$ is abelian. Assume that $G = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle$, where $|x_i| = p^{m_i}$, for $1 \leq i \leq r$ and $m_1 \geq m_2 \geq \cdots \geq m_r$. We claim that $m_1 = 1$; for otherwise, $1 \neq x_1^p \in \Phi(G)$ and so the map $\alpha$ defined by $x_1^\alpha = x_1 + p$, $x_i^\alpha = x_i$ $(2 \leq i \leq r)$ is a non-trivial automorphism of $\text{Aut}^\Phi(G)$, which is a contradiction. □

Lemma 3.3. Let $G$ be a finite $p$-group. Then, $\text{Aut}^\Phi(G)$ is a non-trivial cyclic group if and only if $G$ is cyclic of order greater than $p$.

Proof. If $G \cong \mathbb{Z}_{p^n}$, where $n > 1$, then Lemma 3.1 completes the proof. Now, assume that $\text{Aut}^\Phi(G)$ is non-trivial and cyclic. Therefore, $G$ is abelian, since $\text{Inn}(G) \leq \text{Aut}^\Phi(G)$. If $G$ is not cyclic, then we may write $G = \langle x \rangle \times \langle y \rangle \times H$, where, $|x| = p^m$, $|y| = p^n$, $m \geq n \geq 1$ and $\exp(H) \leq p^n$. Hence, $m > 1$, by Lemma 3.2, and so $1 \neq x^p \in \Phi(G)$. Therefore, we may define the automorphisms $\sigma$ and $\tau$ by $x^\sigma = x^{1+p}$, $y^\sigma = y$, $h^\sigma = h$, for all $h \in H$ and $x^\tau = x$, $y^\tau = yx^{p^{m-1}}$, $h^\tau = h$, for all $h \in H$. Obviously, $\sigma, \tau \in \text{Aut}^\Phi(G)$ and $\langle \sigma \rangle \cap \langle \tau \rangle = 1$. This implies that $\langle \sigma \rangle \times \langle \tau \rangle \leq \text{Aut}^\Phi(G)$, which is a contradiction. □

Now, let $G$ be an abelian $p$-group and let $(m_1, m_2, \ldots, m_r)$ be the invariants of $G$ with $m_1 \geq m_2 \geq \cdots \geq m_r$. For the rest of the paper, we assume that $G$ is neither cyclic nor elementary abelian. Therefore, we may write $m_1 > 1$ and $r > 1$. In Lemma 3.4, we find, $\text{Aut}^\Phi(G)$ for the case $m_1 > 1$ and $m_2 = 1$. Then, in Theorem 3.5, we consider the case $m_2 > 1$.

Lemma 3.4. Let $G$ be a non-cyclic abelian $p$-group and let $(m_1, m_2, \ldots, m_r)$ be the invariants of $G$ with $m_1 \geq m_2 \geq \cdots \geq m_r$ and $m_1 > 1, m_2 = 1$. Then,

(i) $\text{Aut}^\Phi(G)$ is abelian of order $p^{m_1+r-2}$.

(ii) $\text{Aut}^\Phi(G)$ has the invariants $(m_1 - 1, m_2, \ldots, m_r)$. 
Proof. (i) Let $G = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle$, where, $|x_1| = p^{m_1}$ and $|x_2| = \cdots = |x_r| = p$. Then, we may easily see that any automorphism $\alpha$ of $G$, which fixes $G/\Phi(G)$ elementwisely, has the form: $x_1^{\ell_1} = x_1^{1+\ell_1p}$, $x_i^{\alpha} = x_i x_i^{\ell_i p^{m_1-1}}$, where, $0 \leq \ell_1 < p^{m_1-1}$ and $0 \leq \ell_i < p$, for $2 \leq i \leq r$. This completes the proof.

(ii) For $2 \leq i \leq r$, we define the automorphism $\alpha_i$ by $x_i^{\alpha_i} = x_i x_i^{p^{m_1-1}}$, $x_j^{\alpha_i} = x_j$, where, $1 \leq j \leq r$ and $j \neq i$. Also, we define $\alpha_1$ by $x_1^{\alpha_1} = x_1^{1+p}$, $x_j^{\alpha_1} = x_j$, where, $2 \leq j \leq r$. Obviously, $|\alpha_1| = p^{m_1-1}$, $|\alpha_2| = |\alpha_3| = \cdots = |\alpha_r| = p$. Therefore, by (i), we deduce that

$$\text{Aut}^{\Phi}(G) = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \cdots \times \langle \alpha_r \rangle,$$

as desired. \qed

**Theorem 3.5.** Let $G$ be a non-cyclic abelian $p$-group and let $(m_1, m_2, \ldots, m_r)$ be the invariants of $G$ with $m_1 \geq m_2 \geq \cdots \geq m_r$ and $m_2 > 1$. Then,

(i) $m_2 - 1 \leq \text{cl} (\text{Aut}^{\Phi}(G)) \leq m_1 - 1$.

(ii) If $m_1 > m_2$ then $m_2 \leq \text{cl} (\text{Aut}^{\Phi}(G)) \leq m_1 - 1$.

Proof. (i) Let $G \cong \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_r \rangle$, where, $|x_i| = p^{m_i}$, $1 \leq i \leq r$. For $1 \leq i \leq m_2 - 1$, we define the automorphisms $\alpha_i$ by $x_1^{\alpha_i} = x_1 x_1^{p^{i+1}+\cdots+p^{m_2-1}}$ and $x_j^{\alpha_i} = x_j$, for $2 \leq j \leq r$. Also, we define the automorphism $\beta$ by $x_1^{\beta} = x_1$, $x_2^{\beta} = x_2^{1+p^{i}+\cdots+p^{m_2-1}}$ and $x_j^{\beta} = x_j$, for $3 \leq j \leq r$. Obviously, $\alpha_i$ (for $1 \leq i \leq m_2 - 1$) and $\beta$ are in $\text{Aut}^{\Phi}(G)$. By an easy calculation, we may see that $[\beta^{-1}, \alpha_i^{-1}] = \alpha_i^{-1+1}$, for $1 \leq i \leq m_2 - 1$. This implies that $\alpha_{i+1} \in \Gamma_{i+1}(\text{Aut}^{\Phi}(G))$, for $1 \leq i \leq m_2 - 2$. Since $\alpha_{m_2-1} \neq 1$, we have $\Gamma_{m_2-1}(\text{Aut}^{\Phi}(G)) \neq 1$, and so $m_2 - 1 \leq \text{cl} (\text{Aut}^{\Phi}(G))$. Furthermore, $\text{cl} (\text{Aut}^{\Phi}(G)) \leq m_1 - 1$, by Theorem 2.2.

(ii) According to (i), we define the automorphism $\gamma$ by $x_1^{\gamma} = x_1$, $x_2^{\gamma} = x_2 x_1^{p^{m_1-2}}$, and $x_j^{\gamma} = x_j$, for $3 \leq j \leq r$. We have $[\gamma^{-1}, \alpha_{m_2-1}^{-1}] \neq 1$ and $\alpha_{m_2-1} \in \Gamma_{m_2-1}(\text{Aut}^{\Phi}(G))$. Hence, $\Gamma_{m_2}(\text{Aut}^{\Phi}(G)) \neq 1$, and so $m_2 \leq \text{cl} (\text{Aut}^{\Phi}(G)) \leq m_1 - 1$. \qed

The following corollaries are immediate consequences of Theorem 3.5.
Corollary 3.6. Let $G$ be a non-cyclic abelian $p$-group and let $(m_1, m_2, \ldots, m_r)$ be the invariants of $G$ with $m_1 \geq m_2 \geq \cdots \geq m_r$ and $m_1 = m_2 > 1$. Then, $\text{cl} (\text{Aut}^\Phi(G)) = m_1 - 1$.

Corollary 3.7. Let $G$ be a non-cyclic abelian $p$-group and let $(m_1, m_2, \ldots, m_r)$ be the invariants of $G$ with $m_1 \geq m_2 \geq \cdots \geq m_r$ and $m_2 = m_1 - 1$, $m_1 > 1$. Then, $\text{cl} (\text{Aut}^\Phi(G)) = m_2$.

Theorem 3.8. Let $G$ be an abelian $p$-group and let $(m_1, m_2, \ldots, m_r)$ be the invariants of $G$ with $m_1 \geq m_2 \geq \cdots \geq m_r$. Then, $\text{Aut}^\Phi(G)$ is non-trivial abelian if and only if either $m_1 > 1$, $m_2 \leq 1$ or $m_1 = m_2 = 2$.

Proof. If either $m_1 > 1$, $m_2 \leq 1$ or $m_1 = m_2 = 2$, then $\text{Aut}^\Phi(G)$ is abelian by considering lemmas 3.4 and 3.1 and Corollary 3.6. Conversely, if $\text{Aut}^\Phi(G)$ is non-trivial abelian, then $m_2 \leq 2$ by Theorem 3.5 (i). If $m_2 = 2$ then $m_1 = 2$ by Theorem 3.5 (ii). Now, if $m_2 \leq 1$, then by using lemma 3.1 and 3.2, we may see that $m_1 > 1$.

Acknowledgments

The authors are grateful to the referee for his valuable suggestions. The paper was revised according to his/her comments. The work of authors was in part supported by Arak University.

References

S. Fouladi
Department of Mathematics, Arak University, Arak, Iran
Email: s-fouladi@araku.ac.ir

R. Orfi
Department of Mathematics, Arak University, Arak, Iran
Email: r-orfi@araku.ac.ir