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ON THE NILPOTENCY CLASS OF THE AUTOMORPHISM GROUP OF SOME FINITE p-GROUPS

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ABSTRACT. Let G be a p-group of order p^n and $\Phi = \Phi(G)$ be the Frattini subgroup of G. It is shown that the nilpotency class of $\operatorname{Aut}^{\Phi}(G)$, the group of all automorphisms of G centralizing $G/\Phi(G)$, takes the maximum value n-2 if and only if G is of maximal class. We also determine the nilpotency class of $\operatorname{Aut}^{\Phi}(G)$ when G is a finite abelian p-group.

1. Introduction

It is well known [3, III, Satz 3.17] that if G is a finite p-group with the Frattini subgroup $\Phi = \Phi(G)$, then $\operatorname{Aut}^{\Phi}(G)$ is a finite p-group. Liebeck [5] found an upper bound for the nilpotency class of $\operatorname{Aut}^{\Phi}(G)$.

Here, we find the nilpotency class of $\operatorname{Aut}^{\Phi}(G)$ in some cases. A straightforward consequence of the result in [5] shows that the nilpotency class of $\operatorname{Aut}^{\Phi}(G)$ is less than or equal to n-2, for all non-cyclic p-groups of order p^n . Here, we show that the nilpotency class of $\operatorname{Aut}^{\Phi}(G)$ takes the maximum value n-2 if and only if G is of maximal class. Moreover, we find the nilpotency class of $\operatorname{Aut}^{\Phi}(G)$ for a finite abelian p-group G in terms of its invariants, where p is an odd prime.

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Throughout, the following notation is used. The terms of the lower central series of G are denoted by $\Gamma_i = \Gamma_i(G)$. The center of G is denoted by Z = Z(G). The nilpotency class of a group G is denoted by cl(G). If α is an automorphism of G and x is an element of G, we write x^{α} for the image of x under α . The inner automorphism induced by the element g is denoted by σ_q . For a normal subgroup N of G, we let $\operatorname{Aut}^{N}(G)$ denote the group of all automorphisms of G centralizing G/N. We write d(G) for the minimal number of generators of G. An extra-special p-group is a p-group G with $\Phi(G) = Z(G) = G' \cong \mathbb{Z}_p$. A non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian. Also, \mathbb{Z}_n is the cyclic group of order n. All unexplained notation is standard and follows that of [4].

2. Maximum Value of $cl(Aut^{\Phi}(G))$

Let G be a non-cyclic p-group of order p^n . In this section, we prove that $cl(Aut^{\Phi}(G))$ takes the maximum value n-2 if and only if G is of maximal class. First, we give some basic results that are needed for the main results of this section. In [5], Liebeck proved the following theorems which play important roles in our proofs.

Theorem 2.1. [5, Theorem 2] Let G be a finite d-generator p-group with lower central series $G = \Gamma_1 > \cdots > \Gamma_c > \Gamma_{c+1} = 1$ and $\Phi(G) \neq 1$. Let Γ_c have exponent p^m . If $N = \Gamma_c^{p^{m-1}}$, the group generated by all p^{m-1} th powers of elements of Γ_c , then

- (i) N is elementwise fixed by all automorphisms in $\operatorname{Aut}^{\Phi}(G)$,
- (ii) $\operatorname{Aut}^{N}(G) \leq Z(\operatorname{Aut}^{\Phi}(G)),$
- (iii) Aut^N(G) has order p^{rd} , where p^r is the order of N, (iv) Aut^Φ(G)/Aut^N(G) \hookrightarrow Aut^{Φ/N}(G/N).

Theorem 2.2. [5, Theorem 3] Let G be as in Theorem 2.1, with $\Phi(G) \neq \Phi(G)$ 1, and let $\Gamma_i(G)/\Gamma_{i+1}(G)$ have exponent p^{m_i} , for $1 \leq i \leq c$. Then, Aut^{Φ}(G) is nilpotent and cl(Aut^{Φ}(G)) $\leq (\sum_{i=1}^c m_i) - 1$.

Now, we begin by stating a number of lemmas that will be used in the sequel.

Lemma 2.3. Let G be a non-cyclic p-group of order p^n . Then, $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \le n-2.$

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Proof. Suppose that cl(G) = c and $(n_{i1}, n_{i2}, \ldots, n_{ir_i})$ are the invariants of $\Gamma_i(G)/\Gamma_{i+1}(G)$, with $n_{i1} \ge n_{i2} \ge \cdots \ge n_{ir_i}$, for $1 \le i \le c$. Hence,

$$\sum_{i=1}^{c} (n_{i1} + n_{i2} + \dots + n_{ir_i}) = n,$$

and so $\sum_{i=1}^{c} n_{i1} \leq n-1$, since $\Gamma_1(G)/\Gamma_2(G)$ is not cyclic. Now, by Theorem 2.2, we deduce that $cl(Aut^{\Phi}(G)) \leq n-2$.

Lemma 2.4. Let G be a non-abelian p-group of order p^n and $cl(Aut^{\Phi}(G)) = t$, where $1 \leq t \leq n-2$. If $d(G/\Gamma_2(G)) \geq n-t$, then $d(G/\Gamma_2(G)) = n-t$, $G/\Gamma_2(G) \cong \mathbb{Z}_{p^{t-c+2}} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ and $\Gamma_i(G)/\Gamma_{i+1}(G) \cong \mathbb{Z}_p$, for $2 \leq i \leq c$, where c is the nilpotency class of G.

Proof. Suppose that $(n_{i1}, n_{i2}, \ldots, n_{ir_i})$ are the invariants of $\Gamma_i(G)/\Gamma_{i+1}(G)$, with $n_{i1} \ge n_{i2} \ge \cdots \ge n_{ir_i}$ for $1 \le i \le c$. We have

$$\sum_{i=1}^{c} (n_{i1} + n_{i2} + \dots + n_{ir_i}) = n.$$

If $n_{1j} > 1$, for some $2 \leq j \leq n-t$, then $\sum_{i=1}^{c} n_{i1} \leq t$. Hence, $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \leq t-1$, by Theorem 2.2, which is impossible. Therefore, $n_{1j} = 1$, for $2 \leq j \leq n-t$. Now, we have $r_1 = n-t$; for otherwise, $r_1 > n-t$ and by the same argument as above we deduce that $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \leq t-1$, which is a contradiction. So,

$$G/\Gamma_2(G) \cong \mathbb{Z}_{p^{n+1}} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$$

with $d(G/\Gamma_2(G)) = n - t$. We see that $n_{i1} = 1$, for $2 \le i \le c$, by [2, Theorem 1.5]. Moreover if $|\Gamma_i(G)/\Gamma_{i+1}(G)| > p$, for some $2 \le i \le c$, then again $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \le t - 1$. Thus, $\Gamma_i(G)/\Gamma_{i+1}(G) \cong \mathbb{Z}_p$, for $2 \le i \le c$, completing the proof.

Lemma 2.5. Let G be a non-abelian group of order p^n $(n \ge 4)$ and $\operatorname{cl}(G) = n - 2$. If $G/\Gamma_2(G) \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, then $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \le n - 3$.

Proof. We use induction on n. For n = 4, we claim that G is purely non-abelian. Otherwise, we may write $G = A \times B$, where $A \neq 1$ is abelian and B is purely non-abelian. Hence, B is extra-special of order p^3 and so $G/G' \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, which is a contradiction. By [6, Lemma 0.4], we have $\exp(G/Z(G)) = \exp(G') = p$, which implies that G/Z(G) is elementary abelian. Therefore, $\Phi(G) \leq Z(G)$. Now, we prove that $\operatorname{Aut}^{\Phi}(G)$ is abelian. To see this, we consider two cases for Z(G). First, we suppose that $Z(G) \cong \mathbb{Z}_{p^2}$. Then by [1, Theorem 1], we have $|\operatorname{Aut}^Z(G)| = p^3$, and so $|\operatorname{Aut}^{\Phi}(G)| \leq p^3$. Also, $|\operatorname{Aut}^{G'}(G)| \geq p^2$ and $\operatorname{Aut}^{G'}(G) \leq Z(\operatorname{Aut}^{\Phi}(G))$, by Theorem 2.1 (iii) and (ii). This yields that $\operatorname{Aut}^{\Phi}(G)$ is abelian. Next, suppose that Z(G) is elementary abelian. Therefore, $\operatorname{Aut}^{\Phi}(G)$ fixes G^p elementwise, since $\Phi(G) \leq Z(G)$. Hence, by Theorem 2.1(i), $\operatorname{Aut}^{\Phi}(G)$ fixes $\Phi(G)$ elementwise and consequently $\operatorname{Aut}^{\Phi}(G)$ is abelian. Now, suppose that $n \geq 5$ and the result holds for any group of order less than p^n . On setting $N = \Gamma_{n-2}(G)$, we may see that $\operatorname{cl}(G/N) = n - 3$, $|G/N| = p^{n-1}$ and G/N satisfies the conditions of the Lemma. Thus, $\operatorname{cl}(\operatorname{Aut}^{\Phi/N}(G/N)) \leq n - 4$, by the induction hypothesis. Now, the parts (ii) and (iv) of Theorem 2.1 imply that $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \leq n - 3$, as desired. \Box

We have the following theorem due to Müller [7]

Theorem 2.6. [7, Theorem] If G is a finite p-group which is neither elementary abelian nor extra-special, then $\operatorname{Aut}^{\Phi}(G)/\operatorname{Inn}(G)$ is a nontrivial normal p-subgroup of the group of outer automorphisms of G.

Corollary 2.7. Let G be an extra-special p-group of order p^n . Then, $\operatorname{Aut}^{\Phi}(G)$ is elementary abelian of order p^{n-1} .

Theorem 2.8. Let G be a non-abelian p-group of order p^n $(n \ge 3)$. Then, G is of maximal class if and only if $cl(Aut^{\Phi}(G)) = n - 2$.

Proof. If G is of maximal class, then cl(Aut^Φ(G)) ≥ n-2, since Inn(G) ≤ Aut^Φ(G). So, cl(Aut^Φ(G)) = n - 2, by Lemma 2.3. Now, suppose that cl(Aut^Φ(G)) = n - 2. By induction on n, we prove that G is of maximal class. If n = 3, then obviously G is of maximal class. Assume that $|G| = p^n$, $n \ge 4$ and the result holds for any group of order less than p^n . If cl(G) = c, then $G/\Gamma_2(G) \cong \mathbb{Z}_{p^{n-c}} \times \mathbb{Z}_p$ and $\Gamma_i(G)/\Gamma_{i+1}(G) \cong \mathbb{Z}_p$, for $2 \le i \le c$, by Lemma 2.4. On setting $N = \Gamma_c(G)$, we see that cl(Aut^{Φ/N}(G/N)) ≥ n - 3, by Theorem 2.1 (ii) and (iv). Consequently, cl(Aut^{Φ/N}(G/N)) = n - 3, by Theorem 2.2. Hence by the induction hypothesis, G/N is of maximal class, which implies that cl(G) = cl(G/N) + 1 = n - 1, as desired.

Lemma 2.9. If G is a p-group of order p^n $(n \ge 4)$ and cl(G) = n - 2, then $Aut^{\Phi}(G)$ is of class n - 3. Proof. We have $p^2 \leq |G/\Gamma_2(G)| \leq p^3$. If $G/\Gamma_2(G)$ is elementary abelian, then $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \leq n-3$, by Theorem 2.2. Therefore, $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) = n-3$, since $\operatorname{Inn}(G) \leq \operatorname{Aut}^{\Phi}(G)$. Now, if $G/\Gamma_2(G) \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, then $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \leq n-3$, by Lemma 2.5, which completes the proof. \Box

Remark 2.10. The converse of Lemma 2.9 does not hold. To see this we consider a family of groups of order p^5 for any prime p as follows:

$$G = \langle a_1, a_2, a_3, a_4, a_5 | a_1^p = a_3, a_2^p = a_4, a_3^p = a_5, a_4^p = a_5^p = 1,$$
$$[a_1, a_2] = a_5, [a_i, a_j] = 1 \rangle,$$

where, $1 \le i < j \le 5$ and $(i, j) \ne (1, 2)$.

Obviously, we have |G'| = p, which implies that $\operatorname{cl}(G) = 2$. Now, since $\exp(G/G') = p^2$, we see that $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \leq 2$, by Theorem 2.2. Furthermore, we define the maps α and β by $a_1^{\alpha} = a_1 a_4$, $a_i^{\alpha} = a_i$ for $2 \leq i \leq 5$ and $a_2^{\beta} = a_2 a_3$, $a_4^{\beta} = a_4 a_5$, $a_i^{\beta} = a_i$, for $i \in \{1, 3, 5\}$. Now, it is easy to show that $\alpha, \beta \in \operatorname{Aut}^{\Phi}(G)$ since $\Phi(G) = \langle a_3, a_4, a_5 \rangle$. Also, $\alpha\beta \neq \beta\alpha$, which yields $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) = 2$.

Theorem 2.11. Let G be a non-abelian p-group of order p^n and $G' = \Phi(G)$. Then, cl(G) = c if and only if $cl(Aut^{\Phi}(G)) = c - 1$, where, $2 \le c \le n - 1$.

Proof. First suppose that cl(G) = c. Then, $exp(\Gamma_i(G)/\Gamma_{i+1}(G)) = p$, for $1 \leq i \leq c$, by [2, Theorem 1.5]. Hence, $cl(Aut^{\Phi}(G)) \leq c - 1$, by Theorem 2.2. Moreover, cl(Inn(G)) = c - 1, completing the proof. Now if $cl(Aut^{\Phi}(G)) = c - 1$ and cl(G) = d, then by the same argument as above we have $cl(Aut^{\Phi}(G)) = d - 1$. This implies that d = c. \Box

3. $cl(Aut^{\Phi}(G))$ When G Is Abelian

Let G be an abelian p-group, where p is an odd prime. In this section, we find the nilpotency class of $\operatorname{Aut}^{\Phi}(G)$ according to the invariants of G. First, we find $\operatorname{Aut}^{\Phi}(G)$ for cyclic and elementary abelian p-groups G.

Lemma 3.1. If G is a cyclic group of order p^n , then $\operatorname{Aut}^{\Phi}(G) \cong \mathbb{Z}_{p^{n-1}}$.

Proof. Let $G = \langle x \rangle$. Then, obviously the automorphism α , defined by $x^{\alpha} = x^{1+p}$, is of order p^{n-1} lying in $\operatorname{Aut}^{\Phi}(G)$. Therefore, we can complete the proof by the fact that $\operatorname{Aut}(G) \cong \mathbb{Z}_{p^{n-1}(p-1)}$ and $|\alpha| = p^{n-1}$.

Lemma 3.2. Let G be a finite p-group. Then, $Aut^{\Phi}(G) = 1$ if and only if G is elementary abelian.

Proof. Let G be an elementary abelian p-group, then $\operatorname{Aut}^{\Phi}(G) = 1$, by Theorem 2.6. Now, if $\operatorname{Aut}^{\Phi}(G) = 1$. Then $\operatorname{Inn}(G) = 1$, or equivalently, G is abelian. Assume that $G = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle$, where $|x_i| = p^{m_i}$, for $1 \leq i \leq r$ and $m_1 \geq m_2 \geq \cdots \geq m_r$. We claim that $m_1 = 1$; for otherwise, $1 \neq x_1^p \in \Phi(G)$ and so the map α defined by $x_1^{\alpha} = x_1^{1+p}$, $x_i^{\alpha} = x_i \ (2 \leq i \leq r)$ is a non-trivial automorphism of $\operatorname{Aut}^{\Phi}(G)$, which is a contradiction.

Lemma 3.3. Let G be a finite p-group. Then, $\operatorname{Aut}^{\Phi}(G)$ is a non-trivial cyclic group if and only if G is cyclic of order greater than p.

Proof. If $G \cong \mathbb{Z}_{p^n}$, where n > 1, then Lemma 3.1 completes the proof. Now, assume that $\operatorname{Aut}^{\Phi}(G)$ is non-trivial and cyclic. Therefore, G is abelian, since $\operatorname{Inn}(G) \leq \operatorname{Aut}^{\Phi}(G)$. If G is not cyclic, then we may write $G = \langle x \rangle \times \langle y \rangle \times H$, where, $|x| = p^m$, $|y| = p^n$, $m \geq n \geq 1$ and $\exp(H) \leq p^n$. Hence, m > 1, by Lemma 3.2, and so $1 \neq x^p \in \Phi(G)$. Therefore, we may define the automorphisms σ and τ by $x^{\sigma} = x^{1+p}$, $y^{\sigma} = y, h^{\sigma} = h$, for all $h \in H$ and $x^{\tau} = x, y^{\tau} = yx^{p^{m-1}}, h^{\tau} = h$, for all $h \in H$. Obviously, $\sigma, \tau \in \operatorname{Aut}^{\Phi}(G)$ and $\langle \sigma \rangle \cap \langle \tau \rangle = 1$. This implies that $\langle \sigma \rangle \times \langle \tau \rangle \leq \operatorname{Aut}^{\Phi}(G)$, which is a contradiction.

Now, let G be an abelian p-group and let (m_1, m_2, \ldots, m_r) be the invariants of G with $m_1 \ge m_2 \ge \cdots \ge m_r$. For the rest of the paper, we assume that G is neither cyclic nor elementary abelian. Therefore, we may write $m_1 > 1$ and r > 1. In Lemma 3.4, we find, $\operatorname{Aut}^{\Phi}(G)$ for the case $m_1 > 1$ and $m_2 = 1$. Then, in Theorem 3.5, we consider the case $m_2 > 1$.

Lemma 3.4. Let G be a non-cyclic abelian p-group and let (m_1, m_2, \ldots, m_r) be the invariants of G with $m_1 \ge m_2 \ge \cdots \ge m_r$ and $m_1 > 1$, $m_2 = 1$. Then,

- (i) $\operatorname{Aut}^{\Phi}(G)$ is abelian of order p^{m_1+r-2} .
- (ii) Aut^{Φ}(G) has the invariants $(m_1 1, m_2, \ldots, m_r)$.

Proof. (i) Let $G = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle$, where, $|x_1| = p^{m_1}$ and $|x_2| = \cdots = |x_r| = p$. Then, we may easily see that any automorphism α of G, which fixes $G/\Phi(G)$ elementwise, has the form: $x_1^{\alpha} = x_1^{1+\ell_1 p}$, $x_i^{\alpha} = x_i x_1^{\ell_i p^{m_1-1}}$, where, $0 \leq \ell_1 < p^{m_1-1}$ and $0 \leq \ell_i < p$, for $2 \leq i \leq r$. This completes the proof.

(ii) For $2 \leq i \leq r$, we define the automorphism α_i by $x_i^{\alpha_i} = x_i x_1^{p^{m_1-1}}$, $x_j^{\alpha_i} = x_j$, where, $1 \leq j \leq r$ and $j \neq i$. Also, we define α_1 by $x_1^{\alpha_1} = x_1^{1+p}$, $x_j^{\alpha_1} = x_j$, where, $2 \leq j \leq r$. Obviously, $|\alpha_1| = p^{m_1-1}$, $|\alpha_2| = |\alpha_3| = \cdots = |\alpha_r| = p$. Therefore, by (i), we deduce that

$$\operatorname{Aut}^{\Phi}(G) = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \cdots \times \langle \alpha_r \rangle$$

as desired.

Theorem 3.5. Let G be a non-cyclic abelian p-group and let (m_1, m_2, \ldots, m_r) be the invariants of G with $m_1 \ge m_2 \ge \cdots \ge m_r$ and $m_2 > 1$. Then,

(i)
$$m_2 - 1 \le cl(Aut^{\Phi}(G)) \le m_1 - 1.$$

(ii) If $m_1 > m_2$ then $m_2 \le cl(Aut^{\Phi}(G)) \le m_1 - 1.$

Proof. (i) Let $G \cong \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_r \rangle$, where, $|x_i| = p^{m_i}$, $1 \leq i \leq r$. For $1 \leq i \leq m_2 - 1$, we define the automorphisms α_i by $x_1^{\alpha_i} = x_1 x_2^{p^i + p^{i+1} + \cdots + p^{m_2 - 1}}$ and $x_j^{\alpha_i} = x_j$, for $2 \leq j \leq r$. Also, we define the automorphism β by $x_1^{\beta} = x_1$, $x_2^{\beta} = x_2^{1 + p + p^2 + \cdots + p^{m_2 - 1}}$ and $x_j^{\beta} = x_j$, for $3 \leq j \leq r$. Obviously, α_i $(1 \leq i \leq m_2 - 1)$ and β are in Aut^{Φ}(G). By an easy calculation, we may see that $[\beta^{-1}, \alpha_i^{-1}] = \alpha_{i+1}^{-1}$, for $1 \leq i \leq m_2 - 2$. This implies that $\alpha_{i+1} \in \Gamma_{i+1}(\operatorname{Aut}^{\Phi}(G))$, for $1 \leq i \leq m_2 - 2$. Since $\alpha_{m_2-1} \neq 1$, we have $\Gamma_{m_2-1}(\operatorname{Aut}^{\Phi}(G)) \neq 1$, and so $m_2 - 1 \leq \operatorname{cl}(\operatorname{Aut}^{\Phi}(G))$. Furthermore, $\operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \leq m_1 - 1$, by Theorem 2.2.

(ii) According to (i), we define the automorphism γ by $x_1^{\gamma} = x_1, x_2^{\gamma} = x_2 x_1^{p^{m_1-m_2}}$ and $x_j^{\gamma} = x_j$, for $3 \leq j \leq r$. We have $[\gamma^{-1}, \alpha_{m_2-1}^{-1}] \neq 1$ and $\alpha_{m_2-1} \in \Gamma_{m_2-1}(\operatorname{Aut}^{\Phi}(G))$. Hence, $\Gamma_{m_2}(\operatorname{Aut}^{\Phi}(G)) \neq 1$, and so $m_2 \leq \operatorname{cl}(\operatorname{Aut}^{\Phi}(G)) \leq m_1 - 1$. \Box

The following corollaries are immediate consequences of Theorem 3.5.

Corollary 3.6. Let G be a non-cyclic abelian p-group and let (m_1, m_2, \ldots, m_r) be the invariants of G with $m_1 \ge m_2 \ge \cdots \ge m_r$ and $m_1 = m_2 > 1$. Then, $cl(Aut^{\Phi}(G)) = m_1 - 1$.

Corollary 3.7. Let G be a non-cyclic abelian p-group and let (m_1, m_2, \ldots, m_r) be the invariants of G with $m_1 \ge m_2 \ge \cdots \ge m_r$ and $m_2 = m_1 - 1, m_1 > 1$. Then, $cl(Aut^{\Phi}(G)) = m_2$.

Theorem 3.8. Let G be an abelian p-group and let (m_1, m_2, \ldots, m_r) be the invariants of G with $m_1 \ge m_2 \ge \cdots \ge m_r$. Then, $\operatorname{Aut}^{\Phi}(G)$ is non-trivial abelian if and only if either $m_1 > 1$, $m_2 \le 1$ or $m_1 = m_2 = 2$.

Proof. If either $m_1 > 1$, $m_2 \leq 1$ or $m_1 = m_2 = 2$, then $\operatorname{Aut}^{\Phi}(G)$ is abelian by considering lemmas 3.4 and 3.1 and Corollary 3.6. Conversely, if $\operatorname{Aut}^{\Phi}(G)$ is non-trivial abelian, then $m_2 \leq 2$ by Theorem 3.5 (i). If $m_2 = 2$ then $m_1 = 2$ by Theorem 3.5 (ii). Now, if $m_2 \leq 1$, then by using lemma 3.1 and 3.2, we may see that $m_1 > 1$.

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