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RANKS OF MODULES RELATIVE TO A TORSION THEORY

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Abstract. Relative to a hereditary torsion theory τ we introduce a dimension for a module M, called *τ-rank of M*, which coincides with the reduced rank of M whenever τ is the Goldie torsion theory. It is shown that the τ-rank of M is measured by the length of certain decompositions of the τ-injective hull of M. Moreover, some relations between the τ-rank of M and complements to τ-torsionfree submodules of M are obtained.

1. **Introduction**

Throughout the paper, rings will have unit elements and modules will be unitary right modules. The category of all right R-modules is denoted by Mod-R, and the notation $\leq_e$ will denote an essential submodule. In the paper $\tau = (\mathcal{T}, \mathcal{F})$ will denote a fixed hereditary torsion theory on Mod-R. Then, $\tau(M) = \sum\{N : N \leq M,\ N \in \mathcal{T}\}$ is the τ-torsion submodule of $M \in$ Mod-R. The module M is called τ-torsion, if $M \in \mathcal{T}$, and τ-torsionfree, if $M \in \mathcal{F}$. In fact, M is τ-torsion, if $\tau(M) = M$, and τ-torsionfree, if $\tau(M) = 0$. A submodule A of M is called τ-dense, if

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M/A is τ-torsion and we denote this by $A \leq_{\tau-d} M$. It is clear that

$$\tau(M) = \{m \in M : \mathrm{ann}(m) \leq_{\tau-d} R_R\}.$$

More information on torsion-theoretic concepts can be found in [5]. In this paper we prove that whenever the uniform dimension of a complement to $\tau(M)$ is a finite number n then the uniform dimension of every complement to $\tau(M)$ is n. We call this integer n, the τ-rank of M, and if no such integer exists we say that M is not of finite τ-rank. We shall prove that the hereditary torsion τ is stable, if and only if the τ-rank of M coincides with the uniform dimension of $M/\tau(M)$ for every $M \in$ Mod-R.

In Goldie's theory of uniform dimension, within a module M, one seeks for a submodule $A = A_1 \oplus \cdots \oplus A_l$ with the largest possible l such that each A_i is non-zero, and if such an A exists then A is essential in M. In fact, M is measured by the largest possible direct sum of non-zero submodules which it can contain. In section 2, we introduce the notion of *pseudo τ-essential submodule* which has a role in the subject of τ-rank similar to that of an essential submodule in Goldie's theory of uniform dimension. We say that a submodule A of M is pseudo τ-essential, if for every submodule B of M, $A \cap B \leq \tau(M)$ implies that $B \leq \tau(M)$. This concept is a generalization of the notion of *τ-essential submodule*. A submodule A of M is called τ-essential, if A is τ-dense and essential in M. Such submodules appear in many concepts such as (s-)τ-CS modules and the τ-injective hulls of modules which are torsion-theoretic analogues of CS modules and the injective hulls of modules; see [1] and [2]. Some properties of τ-essential submodules can be found in [1, Proposition 3.1], and in Propositions 2.3 and 2.6 we show that most of these properties hold more generally for pseudo τ-essential submodules.

In section 3 we deal with the theory of τ-rank. Then, we will prove that to find the τ-rank of a module M, one should look for a submodule $A = A_1 \oplus \cdots \oplus A_l \oplus B$ with the largest possible l such that each A_i is non-zero τ-torsionfree and B is quasi-τ-torsion (that is $\tau(B) \leq_e B$), and if such an A exists then A is pseudo τ-essential in M. As the uniform dimension of a module M relates to the decomposition length of the injective hull of M, we show that the τ-rank of M is measured by the length of certain decompositions of the τ-injective hull of M. Indeed, in Proposition 3.11 we show that the τ-rank of M is a finite number n, if and only if the τ-injective hull of M is a direct sum of n pseudo τ-uniform modules and a quasi τ-torsion module. Finally section 4 is devoted to some relations between τ-ranks and certain complements analogous to the well known relations between uniform dimensions and complements. In fact, the τ-rank of M is the supremum of the set of nonnegative integers k for which M contains a chain of length k of complements to τ-torsionfree submodules of M.

2. **Pseudo τ-essential submodules**

In this section we introduce the notion of a pseudo τ-essential submodule and give some properties of such submodules for later use. We say that a submodule A of a module M is *pseudo τ-essential in M* and write $A \leq_{p.\tau.e} M$, if for every submodule B of M, $A \cap B \leq \tau(M)$ implies that $B \leq \tau(M)$. Clearly every τ-torsionfree essential submodule of M is pseudo τ-essential. Moreover, if A is a submodule of a τ-torsionfree module M, then A is pseudo τ-essential in M if and only if A is essential in M.

The first result shows that the notion of pseudo τ-essential is a generalization of the notion of τ-dense. In particular, every τ-essential submodule is pseudo τ-essential.

Proposition 2.1. *Every τ-dense submodule of M is pseudo τ-essential in M.*

Proof. Assume that A is a τ-dense submodule of M. Let $A \cap B \leq \tau(M)$ for some submodule B of M, and $b \in B$. As $\tau(M/A) = M/A$, there exists a τ-dense right ideal I of R such that $bI \leq A$. Then, $bI \leq A \cap B \leq \tau(M)$, hence $b + \tau(M) \in \tau(M/\tau(M)) = 0$ and so $b \in \tau(M)$. □

Proposition 2.2. *Let A be a τ-torsionfree submodule of M and B a submodule of M which is maximal with respect to the property $A \cap B = 0$. Then, $A \oplus B \leq_{p.\tau.e} M$ and $(A \oplus B)/B \leq_{p.\tau.e} M/B$.*

Proof. Assume that C is a submodule of M such that $(A \oplus B) \cap C \leq \tau(M)$. Let $a \in A$; if $a = b + c$, for some $b \in B$ and $c \in C$, then $a - b \in (A \oplus B) \cap C$ and so $(a - b)I = 0$ for some dense right ideal I of R. Thus, $aI = bI \leq A \cap B = 0$, hence $a \in \tau(M)$ and so $a = 0$. This implies that $A \cap (B+C) = 0$, hence by hypothesis $C \leq B$ and so $C \leq \tau(M)$. This shows that $A \oplus B \leq_{p.\tau.e} M$. The pseudo τ-essentiality of $(A \oplus B)/B$ is clear as it is a τ-torsionfree essential submodule of M/B. □

A hereditary torsion theory is called *stable* if the torsion class is closed under injective envelopes; equivalently, $\tau(M)$ is (essentially) closed, for every module M.

Proposition 2.3. *The following statements are equivalent for a submodule A of M.*

(1) $A \leq_{p.\tau.e} M$.
(2) $(A+\tau(M))/\tau(M) \leq_e M/\tau(M)$.
(3) *For all* $m \in M \backslash \tau(M)$, *there exists* $r \in R$ *such that* $mr \in A \backslash \tau(A)$.

If the hereditary torsion theory τ is stable, then the above statements are equivalent to

(4) $A+\tau(M) \leq_e M$.
(5) $A \oplus B \leq_e M$, *for some τ-torsion submodule B of M.*
(6) $A \cap B \neq 0$, *for every non-zero τ-torsionfree submodule B of M.*

Proof. Clearly (1) ⇒ (3) and (3) ⇒ (2).

(2) ⇒ (1). Let B be a submodule of M, for which $A \cap B \leq \tau(M)$. If $a \in A$, $b \in B$ with $a+\tau(M) = b+\tau(M)$ then there exists a τ-dense right ideal I such that $(a-b)I = 0$. Then, $aI = bI \leq A \cap B \leq \tau(M)$, hence $a+\tau(M) \in \tau(M/\tau(M)) = 0$. This implies that $(A+\tau(M))/\tau(M) \cap (B+\tau(M))/\tau(M) = 0$, thus by hypothesis $B \leq \tau(M)$.

Now, assume that the hereditary torsion theory τ is stable.

(1) ⇒ (6). Let B be a non-zero τ-torsionfree submodule of M. Then, by (1), $A \cap B$ is non-τ-torsion, hence $A \cap B \neq 0$.

(6) ⇒ (5). There exists a submodule B of M such that $A \oplus B \leq_e M$. Also, there exists a submodule B' of B such that $B' \oplus \tau(B) \leq_e B$. Then, (6) implies that $B' = 0$, hence $\tau(B) \leq_e B$. Since $\tau(B)$ is closed in B we conclude that $B = \tau(B)$.

(5) ⇒ (4). This is obvious.

(4) ⇒ (2). This is clear as $A+\tau(M) \leq_e M$ and $\tau(M)$ is a closed submodule of M. □

Corollary 2.4. *Every essential submodule of M is pseudo τ-essential in M if and only if $\tau(M)$ is (essentially) closed. Consequently, for every* *module M the set of essential submodules in M is a subset of pseudo τ-essential submodules in M if and only if the hereditary torsion theory τ is stable.*

Proof. This follows by Proposition 2.3-(2) and [4, Proposition 1.27-((1) $\Leftrightarrow$ (3))]. $\square$

Corollary 2.5. *The following statements are equivalent for a module M.*

(1) *Every submodule of M is pseudo τ-essential.*
(2) *Every submodule of M is τ-dense.*
(3) *Every submodule of M is τ-torsion.*
(4) *M has a τ-torsion pseudo τ-essential submodule.*
(5) *$\tau(M)$ is pseudo τ-essential in M.*
(6) *M is τ-torsion.*

Proposition 2.6. (1) *Suppose $A \leq B \leq C$ are modules. Then, $A \leq_{p.\tau.e} C$ if and only if $A \leq_{p.\tau.e} B$ and $B \leq_{p.\tau.e} C$.*
(2) *Let A_1, A_2, B_1 and B_2 be modules such that $A_1 \leq_{p.\tau.e} B_1$ and $A_2 \leq_{p.\tau.e} B_2$. Then, $A_1 \cap A_2 \leq_{p.\tau.e} B_1 \cap B_2$.*
(3) *Assume that $f : B \to C$ is a homomorphism of modules, and $A \leq_{p.\tau.e} C$. Then, $f^{-1}(A) \leq_{p.\tau.e} B$.*
(4) *Let A_λ be a submodule of B_λ, for all λ in a set Λ. Then, $\bigoplus_\Lambda A_\lambda \leq_{p.\tau.e} \bigoplus_\Lambda B_\lambda$ if and only if $A_\lambda \leq_{p.\tau.e} B_\lambda$, for all $\lambda \in \Lambda$.*

Proof. (1) and (2) follow easily from the definition.

(3). Let $b \in B \setminus \tau(B)$. By Proposition 2.3-(3) it suffices to show that there exists $r \in R$ such that $br \in f^{-1}(A) \setminus \tau(f^{-1}(A))$. If $f(b) \notin \tau(C)$, as $A \leq_{p.\tau.e} C$, there exists $r \in R$ such that $f(b)r \in A \setminus \tau(A)$. Hence, $br \in f^{-1}(A) \setminus \tau(f^{-1}(A))$. Now, assume that $f(b) \in \tau(C)$. There exists a τ-dense right ideal I of R such that $f(b)I = 0$. If $bI \leq \tau(B)$ then $b + \tau(B) \in \tau(B/\tau(B)) = 0$ which is a contradiction. Thus, $bI \nleq \tau(B)$ and so $bI \nleq \tau(f^{-1}(A))$. Hence, there exists $x \in I$ such that $bx \in f^{-1}(A) \setminus \tau(f^{-1}(A))$.

(4). The implication ($\Rightarrow$) follows from Proposition 2.3-(3). For the converse implication ($\Leftarrow$), by Proposition 2.3-(3) it is enough to check the case of a finite direct sum, and by induction it suffices to check the case $\Lambda = \{1, 2\}$. The latter follows from (2) and (3) for projections $B_1 \oplus B_2 \to B_\lambda$ ($\lambda = 1, 2$). □

Let us call a module M *quasi-τ-torsion* if $\tau(M) \leq_e M$. Clearly every submodule of a quasi-τ-torsion module is quasi-τ-torsion, and every direct sum of quasi-τ-torsion modules is quasi-τ-torsion. Every τ-torsion module is quasi-τ-torsion, however the next result shows that the class of τ-torsion modules coincides with the class of quasi-τ-torsion modules precisely when the hereditary torsion theory τ is stable.

Proposition 2.7. *The class of τ-torsion modules is equal to the class of quasi-τ-torsion modules if and only if the hereditary torsion theory τ is stable.*

Proof. For ($\Rightarrow$), it suffices to show that $\tau(M)$ is (essentially) closed in M, for any module M. Then, assume that $\tau(M) \leq_e K \leq M$. Clearly $\tau(K) = \tau(M)$, hence K is quasi-τ-torsion and so it is τ-torsion by hypothesis. Thus, $K = \tau(M)$. The converse implication ($\Leftarrow$) is clear. □

Proposition 2.8. *A module M is quasi-τ-torsion if and only if M has a pseudo τ-essential submodule which is quasi-τ-torsion.*

Proof. The implication ($\Rightarrow$) is clear. For ($\Leftarrow$), let M have a pseudo τ-essential submodule A which is quasi-τ-torsion. If $\tau(M)$ is not essential in M then there exists a non-zero submodule K of M such that $\tau(M) \cap K = 0$. Since $A \leq_{p.\tau.e} M$ we conclude that $A \cap K \neq 0$. Thus, $\tau(A) \leq_e A$ implies that $\tau(A) \cap K \neq 0$ which is a contradiction. □

3. τ-Ranks

Let M be a non-quasi-τ-torsion module. We say that M is *pseudo τ-uniform* if every non-quasi-τ-torsion submodule of M is pseudo τ-essential. Equivalently, a *pseudo τ-uniform module* is a non-quasi-τ-torsion module M such that for every $A, B \leq M$ if $A \cap B$ is τ-torsion then A is quasi-τ-torsion or B is τ-torsion. For a τ-torsionfree module the properties of uniform and pseudo τ-uniform are equivalent. By

Proposition 2.3, if M is a non-quasi-τ-torsion module for which $M/\tau(M)$ is uniform then M is pseudo τ-uniform. Moreover, by Proposition 2.1, if M is a non-quasi-τ-torsion module which is τ-uniform (see [1, Definition 3.18]) then M is pseudo τ-uniform.

Theorem 3.1. *Let $A_1 \oplus \cdots \oplus A_m \oplus K$ and $B_1 \oplus \cdots \oplus B_n \oplus L$ be pseudo τ-essential submodules of a module M such that each A_i and each B_j is pseudo τ-uniform and K and L are quasi-τ-torsion. Then, $m = n$.*

Proof. Clearly a complement to $\tau(A_i)$ in A_i is a non-zero τ-torsionfree submodule and so it is pseudo τ-essential in A_i. Thus, by Proposition 2.6-(4), we can assume that each A_i is τ-torsionfree and uniform. Similarly we can assume that each B_j is τ-torsionfree and uniform. Now, let $m \leq n$ and set $A = A_2 \oplus \cdots \oplus A_m$. If $A \cap B_j \neq 0$, for all j, then $A \cap B_j \leq_{p.\tau.e} B_j$ and so

$$(A \cap B_1) \oplus \cdots \oplus (A \cap B_n) \oplus L \leq_{p.\tau.e} B_1 \oplus \cdots \oplus B_n \oplus L,$$

hence $(A \cap (B_1 \oplus \cdots \oplus B_n)) \oplus L \leq_{p.\tau.e} B_1 \oplus \cdots \oplus B_n \oplus L \leq_{p.\tau.e} M$. Thus, by Proposition 2.6-(1), $A \oplus L \leq_{p.\tau.e} M$; note that $A \cap L$ is a τ-torsionfree submodule of the quasi-τ-torsion module L and so it is zero. However $(A_1 \oplus \cdots \oplus A_m) \cap L = 0$ and so $(A \oplus L) \cap A_1 = 0$ which is impossible since A_1 is non-τ-torsion. Hence, $A \cap B_j = 0$, for some j, say $j = 1$ and set $B = A \oplus B_1$. If $A_1 \cap B = 0$ then $A_1 + A + B_1$ would be a direct sum and so it is τ-torsionfree, hence $(A_1 \oplus A \oplus B_1) \cap K = 0$. Thus, $(A_1 \oplus A \oplus K) \cap B_1 = 0$ which is impossible as $A_1 \oplus A \oplus K \leq_{p.\tau.e} M$. Therefore, $A_1 \cap B \neq 0$ and so

$$(A_1 \cap B) \oplus A_2 \oplus \cdots \oplus A_m \oplus K \leq_{p.\tau.e} A_1 \oplus \cdots \oplus A_m \oplus K \leq_{p.\tau.e} M,$$

hence by Proposition 2.6-(1), $B \oplus K \leq_{p.\tau.e} M$. This shows that we can replace the summand A_1 of $A_1 \oplus \cdots \oplus A_m \oplus K$ by B_1. By repeating this process we obtain

$$B_1 \oplus B_2 \oplus A_3 \oplus \cdots \oplus A_m \oplus K \leq_{p.\tau.e} M,$$

and after m steps we will arrive at $B_1 \oplus \cdots \oplus B_m \oplus K \leq_{p.\tau.e} M$ which is impossible if $m < n$, since $(B_1 \oplus \cdots \oplus B_m \oplus K) \cap B_{m+1} = 0$ and B_{m+1} is non-τ-torsion. Thus, $m = n$ as desired. □

Proposition 3.2. *Let $M_1 \oplus \cdots \oplus M_n \oplus N$ be a pseudo τ-essential submodule of a module M such that each M_i is pseudo τ-uniform and N*

is quasi-τ-torsion. Then, M does not contain any direct sum of $n+1$ non-quasi-τ-torsion submodules.

Proof. If $n=0$ then M is quasi-τ-torsion by Proposition 2.8 and so the conclusion is clear. Now, let $n>0$ and assume that the statement holds for $n-1$. Let M contain a direct sum $A_1 \oplus \cdots \oplus A_{n+1}$ of $n+1$ non-quasi-τ-torsion submodules. As every non-quasi-τ-torsion module has a nonzero τ-torsionfree submodule, we can assume that $A_1, \ldots, A_{n+1}$ are non-zero τ-torsionfree. Moreover, $B_i = (M_1 \oplus \cdots \oplus M_n \oplus N) \cap A_i$ is non-quasi-τ-torsion since $M_1 \oplus \cdots \oplus M_n \oplus N$ is pseudo τ-essential, and clearly $B_1 \oplus \cdots \oplus B_{n+1} \leq M_1 \oplus \cdots \oplus M_n \oplus N$. Hence, we may assume that $M = M_1 \oplus \cdots \oplus M_n \oplus N$. Now, set $A = A_1 \oplus \cdots \oplus A_n$. If $A \cap M_1$ is quasi-τ-torsion then $A \cap M_1 = 0$ since A is τ-torsionfree. Then, we can embed A in $M_2 \oplus \cdots \oplus M_n \oplus N$ by using the natural projection $M \to M_2 \oplus \cdots \oplus M_n \oplus N$. Thus, $M_2 \oplus \cdots \oplus M_n \oplus N$ contains a direct sum of n non-quasi-τ-torsion submodules, contradicting the induction hypothesis. Therefore, $A \cap M_1$ is non-quasi-τ-torsion and similarly so is $A \cap M_i$, for all i. Thus, $A \cap M_i \leq_{p.\tau.e} M_i$ and so

$$(A \cap M_1) \oplus \cdots \oplus (A \cap M_n) \oplus N \leq_{p.\tau.e} M_1 \oplus \cdots \oplus M_n \oplus N \leq_{p.\tau.e} M.$$

Consequently $A \oplus N \leq_{p.\tau.e} M$. However $(A_1 \oplus \cdots \oplus A_{n+1}) \cap N$ is a τ-torsionfree submodule of the quasi-τ-torsion module N, hence it is zero and so $(A \oplus N) \cap A_{n+1} = 0$ which is impossible as A_{n+1} is non-τ-torsion. Hence, M does not contain a direct sum of $n+1$ non-quasi-τ-torsion submodules. □

Corollary 3.3. *For any module M, the uniform dimensions of all complements to $\tau(M)$ (in M) are equal.*

Proof. Assume that there exists a complement C to $\tau(M)$ of finite uniform dimension n. Then, C contains an essential submodule $C_1 \oplus \cdots \oplus C_n$ such that each C_i is uniform. By Proposition 2.2, there exists a submodule D such that $C \oplus D \leq_{p.\tau.e} M$, hence by Proposition 2.6-(4), (1), $C_1 \oplus \cdots \oplus C_n \oplus D \leq_{p.\tau.e} M$. If D is non-quasi-τ-torsion then it contains a non-zero τ-torsionfree submodule B. Thus, $(B \oplus C) \cap \tau(M) = 0$ which is impossible, hence D is quasi-τ-torsion. Therefore, by Proposition 3.2, if a complement to $\tau(M)$ is of finite uniform dimension then every complement to $\tau(M)$ is of finite uniform dimension and by Theorem 3.1, the uniform dimensions of all complements to $\tau(M)$ are equal. □

As Corollary 3.3 shows, for any module M either all complements to $\tau(M)$ are not of finite uniform dimension or all complements to $\tau(M)$ are of finite uniform dimension n. Let us call this integer n, *the τ-rank of* M and denote this by $\mathbf{r}_\tau(M)$. Note that $\mathbf{r}_\tau(M) = 0$ if and only if M is quasi-τ-torsion. If a complement (hence, every complement) to $\tau(M)$ is not of finite uniform dimension, we say that M is not of finite τ-rank and write $\mathbf{r}_\tau(M) = \infty$. Let u.dim$(M)$ denote the uniform dimension of M. Clearly $\text{u.dim}(M) = \mathbf{r}_\tau(M) + \text{u.dim}(\tau(M))$, hence $\mathbf{r}_\tau(M) = \text{u.dim}(M)$ if M is τ-torsionfree and the converse holds if M is of finite uniform dimension.

Proposition 3.4. *The following statements are equivalent for a module M.*

(1) *M has finite τ-rank n.*
(2) *M has a pseudo τ-essential submodule which is a finite direct sum of n τ-torsionfree uniform submodules and a quasi-τ-torsion submodule.*
(3) *M has a pseudo τ-essential submodule which is a finite direct sum of n pseudo τ-uniform submodules and a quasi-τ-torsion submodule.*
(4) *M contains a direct sum of n non-quasi-τ-torsion submodules, but no direct sum of $n+1$ non-quasi-τ-torsion submodules.*
(5) *M contains a direct sum of n non-zero τ-torsionfree submodules, but no direct sum of $n+1$ non-zero τ-torsionfree submodules.*

Proof. (1) $\Rightarrow$ (2). Assume that C is a complement to $\tau(M)$. As the proof of Corollary 3.3 shows there exist some τ-torsionfree uniform submodules $C_1, \ldots, C_n$ of C and a quasi-τ-torsion submodule D of M such that $C_1 \oplus \cdots \oplus C_n \oplus D \leq_{p.\tau.e} M$.

(2) $\Rightarrow$ (3). This implication is clear.

(3) $\Rightarrow$ (4). This follows by Proposition 3.2.

(4) $\Rightarrow$ (5). This implication is clear as every non-quasi-τ-torsion submodule has a non-zero τ-torsionfree submodule and every non-zero τ-torsionfree submodule is non-quasi-τ-torsion.

(5) $\Rightarrow$ (1). By hypothesis there exists a direct sum of n non-zero τ-torsionfree submodules $K_1 \oplus \cdots \oplus K_n$. This direct sum can be enlarged into a complement C of $\tau(M)$. Then, C contains a direct sum of n

non-zero τ-torsionfree submodules, but no direct sum of $n+1$ non-zero τ-torsionfree submodules and so $\mathrm{u.dim}(C) = n$. □

Corollary 3.5. *The following statements are equivalent for a module M.*

(1) *M is of finite τ-rank.*
(2) *M contains no infinite direct sum of non-quasi-τ-torsion submodules.*
(3) *M contains no infinite direct sum of non-zero τ-torsionfree submodules.*

Proof. The implication (1) ⇒ (2) follows by Proposition 3.4, and (2) ⇒ (3) is clear.

(3) ⇒ (1). Let C be a complement to $\tau(M)$. By hypothesis C contains no infinite direct sum of non-zero submodules, hence C is of finite uniform dimension. □

Corollary 3.6. *For any module M,*
$\mathbf{r}_\tau(M) = \sup\{k$ *: M contains a direct sum of k non-quasi-τ-torsion submodules*$\} = \sup\{k$ *: M contains a direct sum of k non-zero τ-torsionfree submodules*$\}$.

Corollary 3.7. $\mathbf{r}_\tau(M) \leq \mathbf{r}_\tau(M/N) \leq \mathrm{u.dim}(M/N)$, *for every τ-torsion submodule N of M. In particular,* $\mathbf{r}_\tau(M) \leq \mathrm{u.dim}(M/\tau(M))$. *Moreover,* $\mathbf{r}_\tau(M) = \mathrm{u.dim}(M/\tau(M))$ *for every module M, if and only if the hereditary torsion theory τ is stable.*

Proof. Assume that M contains a direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ of non-zero τ-torsionfree submodules. If N is a τ-torsion submodule of M then M/N has a direct sum of non-zero τ-torsionfree submodules $(A_1+N)/N \oplus (A_2+N)/N \oplus \cdots \oplus (A_k+N)/N$. Thus, $\mathbf{r}_\tau(M) \leq \mathbf{r}_\tau(M/N)$ by Corollary 3.6. If τ is stable and $A_1/\tau(M) \oplus \cdots \oplus A_k/\tau(M)$ is a direct sum of non-zero submodules of $M/\tau(M)$, then $B_1 \oplus \cdots \oplus B_k$ is a direct sum of non-zero τ-torsionfree submodules of M, where B_i is a complement to $\tau(M)$ in A_i. Thus, $\mathrm{u.dim}(M/\tau(M)) \leq \mathbf{r}_\tau(M)$ and so $\mathbf{r}_\tau(M) = \mathrm{u.dim}(M/\tau(M))$. Now, let $\mathbf{r}_\tau(M) = \mathrm{u.dim}(M/\tau(M))$, for every module M. Then, $\text{u.dim}(M/\tau(M)) = 0$, for every quasi-$\tau$-torsion module M. Hence, every quasi-τ-torsion module is τ-torsion and so the hereditary torsion theory is stable by Proposition 2.7. □

Corollary 3.8. *Let A be a submodule of M.*

(1) $\mathbf{r}_\tau(A) \leq \mathbf{r}_\tau(M)$.
(2) $\mathbf{r}_\tau(A) = \mathbf{r}_\tau(M)$ *if and only if* $\mathbf{r}_\tau(A) = \infty$ *or if* $\mathbf{r}_\tau(A) = k < \infty$ *then every complement in M of each direct sum of k non-zero τ-torsionfree submodules of A is quasi-τ-torsion.*
(3) $\mathbf{r}_\tau(A) = \mathbf{r}_\tau(M)$ *if* $\mathbf{r}_\tau(A) = \infty$ *or* $A \leq_{p.\tau.e} M$. *The converse holds if the hereditary torsion theory τ is stable.*

Proof. Clearly (1) follows by Corollary 3.6.

(2). ($\Leftarrow$). By Corollary 3.5, if $\mathbf{r}_\tau(A) = \infty$ then $\mathbf{r}_\tau(M) = \infty$. Now, assume that $\mathbf{r}_\tau(A) = k < \infty$. By Proposition 3.4, A contains a direct sum of k non-zero τ-torsionfree submodules $A_1 \oplus \cdots \oplus A_k$ and so by Proposition 2.2, $A_1 \oplus \cdots \oplus A_k \oplus B \leq_{p.\tau.e} M$ for a submodule B of M which is maximal with respect to the property $(A_1 \oplus \cdots \oplus A_k) \cap B = 0$. By hypothesis B is quasi-τ-torsion, hence $\mathbf{r}_\tau(M) = k$.

($\Rightarrow$). Let $\mathbf{r}_\tau(A) = k < \infty$ and $A_1 \oplus \cdots \oplus A_k$ be a direct sum of k non-zero τ-torsionfree submodules of A. If a complement B in M of $A_1 \oplus \cdots \oplus A_k$ is non-quasi-τ-torsion, then there exists a non-zero τ-torsionfree submodule C of B and so M contains the direct sum $A_1 \oplus \cdots \oplus A_k \oplus C$ of $k+1$ non-zero τ-torsionfree submodules which is impossible as $\mathbf{r}_\tau(M) = k$.

(3). The first statement is clear by Proposition 3.4 and Corollary 3.5. Now, assume that τ is stable and $\mathbf{r}_\tau(A) = k < \infty$, moreover A is not pseudo τ-essential in M. Then, A contains a direct sum $A_1 \oplus \cdots \oplus A_k$ of k non-zero τ-torsionfree submodules. Since A is not pseudo τ-essential in M, there exists a non-τ-torsion submodule B such that $A \oplus B \leq_e M$ by Proposition 2.3-(5). Thus, M contains the direct sum $A_1 \oplus \cdots \oplus A_k \oplus B$ of non-τ-torsion submodules. Hence, $\mathbf{r}_\tau(M) \geq k+1$ as the notions of non-τ-torsion and non-quasi-τ-torsion are the same whenever a hereditary torsion theory is stable. □

Note that by Corollaries 3.7 and 3.8, $\mathbf{r}_\tau(M) \leq \mathbf{r}_\tau(A) + \mathbf{r}_\tau(M/A)$ if A is a τ-torsion submodule or a pseudo τ-essential submodule of M. The next corollary shows that the inequality holds for some other submodules of M. Recall that a submodule A of M is called τ-pure (or τ-closed) if M/A is τ-torsionfree.

Corollary 3.9. *Let A be a τ-torsionfree submodule or a τ-pure submodule of M. Then,*

$$\mathbf{r}_\tau(M) \leq \mathbf{r}_\tau(A) + \mathbf{r}_\tau(M/A).$$

Proof. Let A be a τ-torsionfree submodule. There exists a submodule B such that $A \oplus B \leq_{p.\tau.e} M$. Then, $\mathbf{r}_\tau(M) = \mathbf{r}_\tau(A \oplus B) = \mathbf{r}_\tau(A) + \mathbf{r}_\tau(B)$. But, $B \cong (A \oplus B)/A \leq M/A$, hence $\mathbf{r}_\tau(B) \leq \mathbf{r}_\tau(M/A)$. Now, assume that A is τ-pure, moreover C is a complement to $\tau(M)$ in M. Clearly $C \cap A$ can be enlarged to a complement D to $\tau(A)$ in A. Then,

$$\begin{aligned} \text{u.dim}(C) &\leq \text{u.dim}(C \cap A) + \text{u.dim}(C/(C \cap A)) \\ &\leq \text{u.dim}(D) + \text{u.dim}(M/A). \end{aligned}$$

Since A is τ-pure, $\mathbf{r}_\tau(M/A) = \text{u.dim}(M/A)$ and so $\mathbf{r}_\tau(M) \leq \mathbf{r}_\tau(A) + \mathbf{r}_\tau(M/A)$. □

A module M is called *τ-injective* if for any τ-dense submodule A of B, any homomorphism $A \to M$ extends to a homomorphism $B \to M$. If $E_\tau(M)$ is a τ-injective τ-essential extension of M, then $E_\tau(M)$ is the smallest τ-injective module containing M. Moreover, it is unique up to isomorphism. $E_\tau(M)$ is called the *τ-injective hull* of M. More properties of the τ-injective hull of a module can be found in [2, § 3]. Note that by Proposition 2.1, $M \leq_{p.\tau.e} E_\tau(M)$. Proposition 3.11 below interprets the finiteness of the τ-rank of M via a certain decomposition length of $E_\tau(M)$. The following lemma is helpful.

Lemma 3.10. *A module M is pseudo τ-uniform if and only if $E_\tau(M)$ is pseudo τ-uniform.*

Proof. Clearly if M is non-quasi-τ-torsion then $E_\tau(M)$ is non-quasi-τ-torsion and the converse holds by Proposition 2.8. For ($\Rightarrow$), assume that $A \cap B \leq \tau(E_\tau(M))$. Then, $(A \cap M) \cap (B \cap M) \leq \tau(M)$, hence by hypothesis $A \cap M$ is quasi-τ-torsion or $B \cap M$ is τ-torsion. However $M \leq_{p.\tau.e} E_\tau(M)$ and so $A \cap M \leq_{p.\tau.e} A$, therefore A is quasi-τ-torsion by Proposition 2.8 or B is τ-torsion. The converse implication ($\Leftarrow$) is clear. □

Proposition 3.11. $\mathbf{r}_\tau(M) = n < \infty$ *if and only if* $E_\tau(M)$ *is a direct sum of* n *pseudo* τ*-uniform modules and a quasi-*τ*-torsion module.*

Proof. $(\Rightarrow)$. By hypothesis M contains pseudo τ-uniform submodules $A_1, \ldots, A_n$ and a quasi-τ-torsion submodule B such that

$$A_1 \oplus \cdots \oplus A_n \oplus B \leq_{p.\tau.e} M.$$

There exists a submodule C of M for which $A_1 \oplus \cdots \oplus A_n \oplus B \oplus C \leq_e M$. Then, C is τ-torsion and by Proposition 2.6-(1),

$$A_1 \oplus \cdots \oplus A_n \oplus B \oplus C \leq_{p.\tau.e} M.$$

Thus, $A_1 \oplus \cdots \oplus A_n \oplus B \oplus C$ is essential and pseudo τ-essential in $E_\tau(M)$. Thus,

$$\begin{aligned} E_\tau(M) &= E_\tau(A_1 \oplus \cdots \oplus A_n \oplus B \oplus C) \\ &= E_\tau(A_1) \oplus \cdots \oplus E_\tau(A_n) \oplus E_\tau(B) \oplus E_\tau(C), \end{aligned}$$

where, each $E_\tau(A_i)$ is pseudo τ-uniform by Lemma 3.10 and $E_\tau(B)$ and $E_\tau(C)$ are quasi-τ-torsion by Proposition 2.8.

$(\Leftarrow)$. By Proposition 3.4, $\mathbf{r}_\tau(E_\tau(M)) = n$ and so by Corollary 3.8-(3), $\mathbf{r}_\tau(M) = \mathbf{r}_\tau(E_\tau(M)) = n$. □

Corollary 3.12. $\mathbf{r}_\tau(\bigoplus_{i=1}^{k} M_i) = \sum_{i=1}^{k} \mathbf{r}_\tau(M_i)$.

4. Complements and τ-ranks

Recall that for a module M, if $\text{u.dim}(M) = n < \infty$, then any chain of complements has length $\leq n$. In addition, $\text{u.dim}(M) = \infty$ if and only if there exists an infinite strictly ascending chain of complements in M if and only if there exists an infinite strictly descending chain of complements (See [3, Propositions (6.29) and (6.30)]). In this section we obtain similar relations for τ-rank of a module M in terms of certain complement submodules.

Proposition 4.1. *Let* M *be a module and* $\mathbf{r}_\tau(M) = n < \infty$*. Then, in* M *any chain of complements to* τ*-torsionfree submodules has length* $\leq n$*.*

Proof. Let $C_0 < C_1 < \cdots < C_k$, where each C_{i-1} is a complement to some τ-torsionfree submodule T_i of M. Then, each C_{i-1} is a complement to the τ-torsionfree submodule $T_i \cap C_i$ of C_i. Set $S_i = T_i \cap C_i$, for all $i = 1, \dots, k$. Since $C_{i-1} \neq C_i$, we have $S_i \neq 0$. Then, $S_1 \oplus \cdots \oplus S_k$ is a direct sum of k non-zero τ-torsionfree submodules of M, hence $k \leq n$ by Corollary 3.6. □

Theorem 4.2. *The following statements are equivalent for a module M.*

(1) $\mathbf{r}_\tau(M) = \infty$.
(2) *There exists an infinite strictly ascending chain of complements to τ-torsionfree submodules in M.*
(3) *There exists an infinite strictly descending chain of complements to τ-torsionfree submodules in M.*

Proof. (1) ⇒ (2). By Corollary 3.5, M contains an infinite direct sum $T_1 \oplus T_2 \oplus \cdots$, where T_i is a non-zero τ-torsionfree submodule. Enlarge T_1 into a complement to $T_2 \oplus T_3 \oplus \cdots$, say C_1. Then, enlarge $C_1 \oplus T_2$ into a complement to $T_3 \oplus T_4 \oplus \cdots$, say C_2. In this way, we get an ascending chain $C_1 \leq C_2 \leq \cdots$, where each C_i is a complement to a τ-torsionfree submodule in M. Since $T_i \leq C_i$ and $T_i \cap C_{i-1} = 0$, we have $C_{i-1} \neq C_i$, for all i.

(2) ⇒ (3). Assume that $C_0 < C_1 < \cdots$, where each C_i is a complement to a τ-torsionfree submodule in M. If C_k is τ-torsion then $C_k = \tau(M)$ and so only C_0 can be τ-torsion. Moreover, similar to the proof of Proposition 4.1, C_{i-1} is a complement to some non-zero τ-torsionfree submodule S_i in C_i. Enlarge $S_2 \oplus S_3 \oplus \cdots$ into a complement to S_1, let L_1 be this complement. Then, enlarge $S_3 \oplus S_4 \oplus \cdots$ into a complement to S_2 in L_1, say L_2. Clearly L_2 is a complement to the non-zero τ-torsionfree submodule $S_1 \oplus S_2$ in M. Moreover, $L_2 < L_1$ since $S_2 \leq L_1$ and $L_2 \cap S_2 = 0$. By this process we get a strictly descending chain of complements to τ-torsionfree submodules in M, i.e., $L_1 > L_2 > \cdots$.

(3) ⇒ (1) is clear by Proposition 4.1. □

Recall that u.dim $(M) = \sup\{k : M$ contains a chain of complements of length $k\}$. A similar result holds for the τ-rank of M.

Corollary 4.3. *For any module M,*

$\mathbf{r}_\tau(M) = \sup\{k : M$ *contains a chain of length* k *of complements to* τ*-torsionfree submodules*$\}$.

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