RANKS OF MODULES RELATIVE TO A TORSION THEORY

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Abstract. Relative to a hereditary torsion theory \( \tau \) we introduce a dimension for a module \( M \), called \( \tau \)-rank of \( M \), which coincides with the reduced rank of \( M \) whenever \( \tau \) is the Goldie torsion theory. It is shown that the \( \tau \)-rank of \( M \) is measured by the length of certain decompositions of the \( \tau \)-injective hull of \( M \). Moreover, some relations between the \( \tau \)-rank of \( M \) and complements to \( \tau \)-torsionfree submodules of \( M \) are obtained.

1. Introduction

Throughout the paper, rings will have unit elements and modules will be unitary right modules. The category of all right \( R \)-modules is denoted by \( \text{Mod-}R \), and the notation \( \leq_e \) will denote an essential submodule. In the paper \( \tau = (T,F) \) will denote a fixed hereditary torsion theory on \( \text{Mod-}R \). Then, \( \tau(M) = \sum \{ N : N \leq M, N \in T \} \) is the \( \tau \)-torsion submodule of \( M \in \text{Mod-}R \). The module \( M \) is called \( \tau \)-torsion, if \( M \in T \), and \( \tau \)-torsionfree, if \( M \in F \). In fact, \( M \) is \( \tau \)-torsion, if \( \tau(M) = M \), and \( \tau \)-torsionfree, if \( \tau(M) = 0 \). A submodule \( A \) of \( M \) is called \( \tau \)-dense, if

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$M/A$ is $\tau$-torsion and we denote this by $A \leq_{\tau-d} M$. It is clear that
\[
\tau(M) = \{m \in M : \text{ann}(m) \leq_{\tau-d} \tau R\}.
\]
More information on torsion-theoretic concepts can be found in [5]. In this paper we prove that whenever the uniform dimension of a complement to $\tau(M)$ is a finite number $n$ then the uniform dimension of every complement to $\tau(M)$ is $n$. We call this integer $n$, the $\tau$-rank of $M$, and if no such integer exists we say that $M$ is not of finite $\tau$-rank. We shall prove that the hereditary torsion $\tau$ is stable, if and only if the $\tau$-rank of $M$ coincides with the uniform dimension of $M/\tau(M)$ for every $M \in \text{Mod-}R$.

In Goldie’s theory of uniform dimension, within a module $M$, one seeks for a submodule $A = A_1 \oplus \cdots \oplus A_l$ with the largest possible $l$ such that each $A_i$ is non-zero, and if such an $A$ exists then $A$ is essential in $M$. In fact, $M$ is measured by the largest possible direct sum of non-zero submodules which it can contain. In section 2, we introduce the notion of pseudo $\tau$-essential submodule which has a role in the subject of $\tau$-rank similar to that of an essential submodule in Goldie’s theory of uniform dimension. We say that a submodule $A$ of $M$ is pseudo $\tau$-essential, if for every submodule $B$ of $M$, $A \cap B \leq \tau(M)$ implies that $B \leq \tau(M)$. This concept is a generalization of the notion of $\tau$-essential submodule. A submodule $A$ of $M$ is called $\tau$-essential, if $A$ is $\tau$-dense and essential in $M$. Such submodules appear in many concepts such as (s-)$\tau$-CS modules and the $\tau$-injective hulls of modules which are torsion-theoretic analogues of CS modules and the injective hulls of modules; see [1] and [2]. Some properties of $\tau$-essential submodules can be found in [1, Proposition 3.1], and in Propositions 2.3 and 2.6 we show that most of these properties hold more generally for pseudo $\tau$-essential submodules.

In section 3 we deal with the theory of $\tau$-rank. Then, we will prove that to find the $\tau$-rank of a module $M$, one should look for a submodule $A = A_1 \oplus \cdots \oplus A_l \oplus B$ with the largest possible $l$ such that each $A_i$ is non-zero $\tau$-torsionfree and $B$ is quasi-$\tau$-torsion (that is $\tau(B) \leq_e B$), and if such an $A$ exists then $A$ is pseudo $\tau$-essential in $M$. As the uniform dimension of a module $M$ relates to the decomposition length of the $\tau$-injective hull of $M$, we show that the $\tau$-rank of $M$ is measured by the length of certain decompositions of the $\tau$-injective hull of $M$. Indeed, in Proposition 3.11 we show that the $\tau$-rank of $M$ is a finite number $n$, if and only if the $\tau$-injective hull of $M$ is a direct sum of $n$ pseudo $\tau$-uniform
modules and a quasi $\tau$-torsion module. Finally section 4 is devoted to some relations between $\tau$-ranks and certain complements analogous to the well known relations between uniform dimensions and complements. In fact, the $\tau$-rank of $M$ is the supremum of the set of nonnegative integers $k$ for which $M$ contains a chain of length $k$ of complements to $\tau$-torsionfree submodules of $M$.

2. Pseudo $\tau$-essential submodules

In this section we introduce the notion of a pseudo $\tau$-essential submodule and give some properties of such submodules for later use. We say that a submodule $A$ of a module $M$ is pseudo $\tau$-essential in $M$ and write $A \leq p.\tau.e M$, if for every submodule $B$ of $M$, $A \cap B \leq \tau(M)$ implies that $B \leq \tau(M)$. Clearly every $\tau$-torsionfree essential submodule of $M$ is pseudo $\tau$-essential. Moreover, if $A$ is a submodule of a $\tau$-torsionfree module $M$, then $A$ is pseudo $\tau$-essential in $M$ if and only if $A$ is essential in $M$.

The first result shows that the notion of pseudo $\tau$-essential is a generalization of the notion of $\tau$-dense. In particular, every $\tau$-essential submodule is pseudo $\tau$-essential.

Proposition 2.1. Every $\tau$-dense submodule of $M$ is pseudo $\tau$-essential in $M$.

Proof. Assume that $A$ is a $\tau$-dense submodule of $M$. Let $A \cap B \leq \tau(M)$ for some submodule $B$ of $M$, and $b \in B$. As $\tau(M/A) = M/A$, there exists a $\tau$-dense right ideal $I$ of $R$ such that $bI \leq A$. Then, $bI \leq A \cap B \leq \tau(M)$, hence $b + \tau(M) \in \tau(M/\tau(M)) = 0$ and so $b \in \tau(M)$. $\square$

Proposition 2.2. Let $A$ be a $\tau$-torsionfree submodule of $M$ and $B$ a submodule of $M$ which is maximal with respect to the property $A \cap B = 0$. Then, $A \oplus B \leq p.\tau.e M$ and $(A \oplus B)/B \leq p.\tau.e M/B$.

Proof. Assume that $C$ is a submodule of $M$ such that $(A \oplus B) \cap C \leq \tau(M)$. Let $a \in A$; if $a = b + c$, for some $b \in B$ and $c \in C$, then $a - b \in (A \oplus B) \cap C$ and so $(a - b)I = 0$ for some dense right ideal $I$ of $R$. Thus, $aI = bI \leq A \cap B = 0$, hence $a \in \tau(M)$ and so $a = 0$. This implies
that \( A \cap (B + C) = 0 \), hence by hypothesis \( C \leq B \) and so \( C \leq \tau(M) \). This shows that \( A \oplus B \leq_{\text{p.\tau.e}} M \). The pseudo \( \tau \)-essentiality of \((A \oplus B)/B\) is clear as it is a \( \tau \)-torsionfree essential submodule of \( M/B \). \( \square \)

A hereditary torsion theory is called \textit{stable} if the torsion class is closed under injective envelopes; equivalently, \( \tau(M) \) is (essentially) closed, for every module \( M \).

\textbf{Proposition 2.3.} The following statements are equivalent for a submodule \( A \) of \( M \).

\begin{enumerate}
\item \( A \leq_{\text{p.\tau.e}} M \).
\item \( (A + \tau(M))/\tau(M) \leq_{e} M/\tau(M) \).
\item For all \( m \in M/\tau(M) \), there exists \( r \in R \) such that \( mr \in A \setminus \tau(A) \).
\end{enumerate}

If the hereditary torsion theory \( \tau \) is stable, then the above statements are equivalent to

\begin{enumerate}
\item \( A + \tau(M) \leq_{e} M \).
\item \( A \oplus B \leq_{e} M \), for some \( \tau \)-torsion submodule \( B \) of \( M \).
\item \( A \cap B \neq 0 \), for every non-zero \( \tau \)-torsionfree submodule \( B \) of \( M \).
\end{enumerate}

\textbf{Proof.} Clearly (1) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (2).

(2) \( \Rightarrow \) (1). Let \( B \) be a submodule of \( M \), for which \( A \cap B \leq \tau(M) \). If \( a \in A, b \in B \) with \( a + \tau(M) = b + \tau(M) \) then there exists a \( \tau \)-dense right ideal \( I \) such that \( (a - b)I = 0 \). Then, \( aI = bI \leq A \cap B \leq \tau(M) \), hence \( a + \tau(M) \in \tau(M/\tau(M)) = 0 \). This implies that \( (A + \tau(M))/\tau(M) \cap (B + \tau(M))/\tau(M) = 0 \), thus by hypothesis \( B \leq \tau(M) \).

Now, assume that the hereditary torsion theory \( \tau \) is stable.

(1) \( \Rightarrow \) (6). Let \( B \) be a non-zero \( \tau \)-torsionfree submodule of \( M \). Then, by (1), \( A \cap B \) is non-\( \tau \)-torsion, hence \( A \cap B \neq 0 \).

(6) \( \Rightarrow \) (5). There exists a submodule \( B \) of \( M \) such that \( A \oplus B \leq_{e} M \). Also, there exists a submodule \( B' \) of \( B \) such that \( B' \oplus \tau(B) \leq_{e} B \). Then, (6) implies that \( B' = 0 \), hence \( \tau(B) \leq_{e} B \). Since \( \tau(B) \) is closed in \( B \) we conclude that \( B = \tau(B) \).

(5) \( \Rightarrow \) (4). This is obvious.

(4) \( \Rightarrow \) (2). This is clear as \( A + \tau(M) \leq_{e} M \) and \( \tau(M) \) is a closed submodule of \( M \). \( \square \)

\textbf{Corollary 2.4.} Every essential submodule of \( M \) is pseudo \( \tau \)-essential in \( M \) if and only if \( \tau(M) \) is (essentially) closed. Consequently, for every
module $M$ the set of essential submodules in $M$ is a subset of pseudo $\tau$-essential submodules in $M$ if and only if the hereditary torsion theory $\tau$ is stable.

Proof. This follows by Proposition 2.3-(2) and [4, Proposition 1.27-((1) $\iff$ (3))]. $\Box$

Corollary 2.5. The following statements are equivalent for a module $M$.

1. Every submodule of $M$ is pseudo $\tau$-essential.
2. Every submodule of $M$ is $\tau$-dense.
3. Every submodule of $M$ is $\tau$-torsion.
4. $M$ has a $\tau$-torsion pseudo $\tau$-essential submodule.
5. $\tau(M)$ is pseudo $\tau$-essential in $M$.
6. $M$ is $\tau$-torsion.

Proposition 2.6. (1) Suppose $A \leq B \leq C$ are modules. Then, $A \leq p.\tau.e C$ if and only if $A \leq p.\tau.e B$ and $B \leq p.\tau.e C$.

(2) Let $A_1$, $A_2$, $B_1$ and $B_2$ be modules such that $A_1 \leq p.\tau.e B_1$ and $A_2 \leq p.\tau.e B_2$. Then, $A_1 \cap A_2 \leq p.\tau.e B_1 \cap B_2$.

(3) Assume that $f : B \rightarrow C$ is a homomorphism of modules, and $A \leq p.\tau.e C$. Then, $f^{-1}(A) \leq p.\tau.e B$.

(4) Let $A_\lambda$ be a submodule of $B_\lambda$, for all $\lambda$ in a set $\Lambda$. Then, $\bigoplus_\Lambda A_\lambda \leq p.\tau.e \bigoplus_\Lambda B_\lambda$ if and only if $A_\lambda \leq p.\tau.e B_\lambda$, for all $\lambda \in \Lambda$.

Proof. (1) and (2) follow easily from the definition.

(3). Let $b \in B \setminus \tau(B)$. By Proposition 2.3-(3) it suffices to show that there exists $r \in R$ such that $br \in f^{-1}(A) \setminus \tau(f^{-1}(A))$. If $f(b) \notin \tau(C)$, as $A \leq p.\tau.e C$, there exists $r \in R$ such that $f(b)r \in A \setminus \tau(A)$. Hence, $br \in f^{-1}(A) \setminus \tau(f^{-1}(A))$. Now, assume that $f(b) \in \tau(C)$. There exists a $\tau$-dense right ideal $I$ of $R$ such that $f(b)I = 0$. If $bI \leq \tau(B)$ then $b + \tau(B) \in \tau(B/\tau(B)) = 0$ which is a contradiction. Thus, $bI \not\leq \tau(B)$ and so $bI \not\leq \tau(f^{-1}(A))$. Hence, there exists $x \in I$ such that $bx \in f^{-1}(A) \setminus \tau(f^{-1}(A))$.

(4). The implication ($\Rightarrow$) follows from Proposition 2.3-(3). For the converse implication ($\Leftarrow$), by Proposition 2.3-(3) it is enough to check the case of a finite direct sum, and by induction it suffices to check the
case $\Lambda = \{1, 2\}$. The latter follows from (2) and (3) for projections $B_1 \oplus B_2 \to B_\lambda (\lambda = 1, 2)$. □

Let us call a module $M$ quasi-$\tau$-torsion if $\tau(M) \leq_e M$. Clearly every submodule of a quasi-$\tau$-torsion module is quasi-$\tau$-torsion, and every direct sum of quasi-$\tau$-torsion modules is quasi-$\tau$-torsion. Every $\tau$-torsion module is quasi-$\tau$-torsion, however the next result shows that the class of $\tau$-torsion modules coincides with the class of quasi-$\tau$-torsion modules precisely when the hereditary torsion theory $\tau$ is stable.

**Proposition 2.7.** The class of $\tau$-torsion modules is equal to the class of quasi-$\tau$-torsion modules if and only if the hereditary torsion theory $\tau$ is stable.

**Proof.** For ($\Rightarrow$), it suffices to show that $\tau(M)$ is (essentially) closed in $M$, for any module $M$. Then, assume that $\tau(M) \leq_e K \leq M$. Clearly $\tau(K) = \tau(M)$, hence $K$ is quasi-$\tau$-torsion and so it is $\tau$-torsion by hypothesis. Thus, $K = \tau(M)$. The converse implication ($\Leftarrow$) is clear. □

**Proposition 2.8.** A module $M$ is quasi-$\tau$-torsion if and only if $M$ has a pseudo $\tau$-essential submodule which is quasi-$\tau$-torsion.

**Proof.** The implication ($\Rightarrow$) is clear. For ($\Leftarrow$), let $M$ have a pseudo $\tau$-essential submodule $A$ which is quasi-$\tau$-torsion. If $\tau(M)$ is not essential in $M$ then there exists a non-zero submodule $K$ of $M$ such that $\tau(M) \cap K = 0$. Since $A \leq_{p, e} M$ we conclude that $A \cap K \neq 0$. Thus, $\tau(A) \leq_e A$ implies that $\tau(A) \cap K \neq 0$ which is a contradiction. □

3. $\tau$-Ranks

Let $M$ be a non-quasi-$\tau$-torsion module. We say that $M$ is pseudo $\tau$-uniform if every non-quasi-$\tau$-torsion submodule of $M$ is pseudo $\tau$-essential. Equivalently, a pseudo $\tau$-uniform module is a non-quasi-$\tau$-torsion module $M$ such that for every $A, B \leq M$ if $A \cap B$ is $\tau$-torsion then $A$ is quasi-$\tau$-torsion or $B$ is $\tau$-torsion. For a $\tau$-torsionfree module the properties of uniform and pseudo $\tau$-uniform are equivalent. By
Proposition 2.3, if $M$ is a non-quasi-$\tau$-torsion module for which $M/\tau(M)$ is uniform then $M$ is pseudo $\tau$-uniform. Moreover, by Proposition 2.1, if $M$ is a non-quasi-$\tau$-torsion module which is $\tau$-uniform (see [1, Definition 3.18]) then $M$ is pseudo $\tau$-uniform.

**Theorem 3.1.** Let $A_1 \oplus \cdots \oplus A_m \oplus K$ and $B_1 \oplus \cdots \oplus B_n \oplus L$ be pseudo $\tau$-essential submodules of a module $M$ such that each $A_i$ and each $B_j$ is pseudo $\tau$-uniform and $K$ and $L$ are quasi-$\tau$-torsion. Then, $m = n$.

**Proof.** Clearly a complement to $\tau(A_i)$ in $A_i$ is a non-zero $\tau$-torsionfree submodule and so it is pseudo $\tau$-essential in $A_i$. Thus, by Proposition 2.6-(4), we can assume that each $A_i$ is $\tau$-torsionfree and uniform. Similarly we can assume that each $B_j$ is $\tau$-torsionfree and uniform. Now, let $m \leq n$ and set $A = A_2 \oplus \cdots \oplus A_m$. If $A \cap B_j \neq 0$, for all $j$, then $A \cap B_j \leq_{p.\tau.e} B_j$ and so

$$(A \cap B_1) \oplus \cdots \oplus (A \cap B_n) \oplus L \leq_{p.\tau.e} B_1 \oplus \cdots \oplus B_n \oplus L,$$

hence $(A \cap (B_1 \oplus \cdots \oplus B_n)) \oplus L \leq_{p.\tau.e} B_1 \oplus \cdots \oplus B_n \oplus L \leq_{p.\tau.e} M$. Thus, by Proposition 2.6-(1), $A \oplus L \leq_{p.\tau.e} M$; note that $A \cap L$ is a $\tau$-torsionfree submodule of the quasi-$\tau$-torsion module $L$ and so it is zero. However $(A_1 \oplus \cdots \oplus A_m) \cap L = 0$ and so $(A \oplus L) \cap A_1 = 0$ which is impossible since $A_1$ is non-$\tau$-torsion. Hence, $A \cap B_j = 0$, for some $j$, say $j = 1$ and set $B = A \oplus B_1$. If $A_1 \cap B = 0$ then $A_1 + A + B_1$ would be a direct sum and so it is $\tau$-torsionfree, hence $(A_1 + A \oplus B_1) \cap K = 0$. Thus, $(A_1 \oplus A \oplus K) \cap B_1 = 0$ which is impossible as $A_1 \oplus A \oplus K \leq_{p.\tau.e} M$. Therefore, $A_1 \cap B \neq 0$ and so

$$(A_1 \cap B) \oplus A_2 \oplus \cdots \oplus A_m \oplus K \leq_{p.\tau.e} A_1 \oplus A_2 \oplus \cdots \oplus A_m \oplus K \leq_{p.\tau.e} M,$$

hence by Proposition 2.6-(1), $B \oplus K \leq_{p.\tau.e} M$. This shows that we can replace the summand $A_1$ of $A_1 \oplus \cdots \oplus A_m \oplus K$ by $B_1$. By repeating this process we obtain

$$B_1 \oplus B_2 \oplus A_3 \oplus \cdots \oplus A_m \oplus K \leq_{p.\tau.e} M,$$

and after $m$ steps we will arrive at $B_1 \oplus \cdots \oplus B_m \oplus K \leq_{p.\tau.e} M$ which is impossible if $m < n$, since $(B_1 \oplus \cdots \oplus B_m \oplus K) \cap B_{m+1} = 0$ and $B_{m+1}$ is non-$\tau$-torsion. Thus, $m = n$ as desired. $\square$

**Proposition 3.2.** Let $M_1 \oplus \cdots \oplus M_n \oplus N$ be a pseudo $\tau$-essential submodule of a module $M$ such that each $M_i$ is pseudo $\tau$-uniform and $N$
is quasi-\(\tau\)-torsion. Then, \(M\) does not contain any direct sum of \(n + 1\) non-quasi-\(\tau\)-torsion submodules.

**Proof.** If \(n = 0\) then \(M\) is quasi-\(\tau\)-torsion by Proposition 2.8 and so the conclusion is clear. Now, let \(n > 0\) and assume that the statement holds for \(n - 1\). Let \(M\) contain a direct sum \(A_1 \oplus \cdots \oplus A_{n+1}\) of \(n + 1\) non-quasi-\(\tau\)-torsion submodules. As every non-quasi-\(\tau\)-torsion module has a nonzero \(\tau\)-torsionfree submodule, we can assume that \(A_1, \ldots, A_{n+1}\) are non-zero \(\tau\)-torsionfree. Moreover, \(B_i = (M_1 \oplus \cdots \oplus M_n \oplus N) \cap A_i\) is non-quasi-\(\tau\)-torsion since \(M_1 \oplus \cdots \oplus M_n \oplus N\) is pseudo-\(\tau\)-essential, and clearly \(B_1 \oplus \cdots \oplus B_{n+1} \leq M_1 \oplus \cdots \oplus M_n \oplus N\). Hence, we may assume that \(M = M_1 \oplus \cdots \oplus M_n \oplus N\). Now, set \(A = A_1 \oplus \cdots \oplus A_n\). If \(A \cap M_1\) is quasi-\(\tau\)-torsion then \(A \cap M_1 = 0\) since \(A\) is \(\tau\)-torsionfree. Then, we can embed \(A\) in \(M_2 \oplus \cdots \oplus M_n \oplus N\) by using the natural projection \(M \to M_2 \oplus \cdots \oplus M_n \oplus N\). Thus, \(M_2 \oplus \cdots \oplus M_n \oplus N\) contains a direct sum of \(n\) non-quasi-\(\tau\)-torsion submodules, contradicting the induction hypothesis. Therefore, \(A \cap M_1\) is non-quasi-\(\tau\)-torsion and similarly so is \(A \cap M_i\) for all \(i\). Thus, \(A \cap M_i \leq_{p.\tau,e} M_i\) and so

\[
(A \cap M_1) \oplus \cdots \oplus (A \cap M_n) \oplus N \leq_{p.\tau,e} M_1 \oplus \cdots \oplus M_n \oplus N \leq_{p.\tau,e} M.
\]

Consequently \(A \oplus N \leq_{p.\tau,e} M\). However \((A_1 \oplus \cdots \oplus A_{n+1}) \cap N\) is a \(\tau\)-torsionfree submodule of the quasi-\(\tau\)-torsion module \(N\), hence it is zero and so \((A \oplus N) \cap A_{n+1} = 0\) which is impossible as \(A_{n+1}\) is non-\(\tau\)-torsion. Hence, \(M\) does not contain a direct sum of \(n + 1\) non-quasi-\(\tau\)-torsion submodules. \(\square\)

**Corollary 3.3.** For any module \(M\), the uniform dimensions of all complements to \(\tau(M)\) (in \(M\)) are equal.

**Proof.** Assume that there exists a complement \(C\) to \(\tau(M)\) of finite uniform dimension \(n\). Then, \(C\) contains an essential submodule \(C_1 \oplus \cdots \oplus C_n\) such that each \(C_i\) is uniform. By Proposition 2.2, there exists a submodule \(D\) such that \(C \oplus D \leq_{p.\tau,e} M\), hence by Proposition 2.6-(4), (1), \(C_1 \oplus \cdots \oplus C_n \oplus D \leq_{p.\tau,e} M\). If \(D\) is non-quasi-\(\tau\)-torsion then it contains a non-zero \(\tau\)-torsionfree submodule \(B\). Thus, \((B \oplus C) \cap \tau(M) = 0\) which is impossible, hence \(D\) is quasi-\(\tau\)-torsion. Therefore, by Proposition 3.2, if a complement to \(\tau(M)\) is of finite uniform dimension then every complement to \(\tau(M)\) is of finite uniform dimension and by Theorem 3.1, the uniform dimensions of all complements to \(\tau(M)\) are equal. \(\square\)
As Corollary 3.3 shows, for any module $M$ either all complements to $	au(M)$ are not of finite uniform dimension or all complements to $	au(M)$ are of finite uniform dimension $n$. Let us call this integer $n$, the $\tau$-rank of $M$ and denote this by $r_\tau(M)$. Note that $r_\tau(M) = 0$ if and only if $M$ is quasi-$\tau$-torsion. If a complement (hence, every complement) to $\tau(M)$ is not of finite uniform dimension, we say that $M$ is not of finite $\tau$-rank and write $r_\tau(M) = \infty$. Let $u\dim(M)$ denote the uniform dimension of $M$. Clearly $u\dim(M) = r_\tau(M) + u\dim(\tau(M))$, hence $r_\tau(M) = u\dim(M)$ if $M$ is $\tau$-torsionfree and the converse holds if $M$ is of finite uniform dimension.

**Proposition 3.4.** The following statements are equivalent for a module $M$.

1. $M$ has finite $\tau$-rank $n$.
2. $M$ has a pseudo $\tau$-essential submodule which is a finite direct sum of $n$ $\tau$-torsionfree uniform submodules and a quasi-$\tau$-torsion submodule.
3. $M$ has a pseudo $\tau$-essential submodule which is a finite direct sum of $n$ pseudo $\tau$-uniform submodules and a quasi-$\tau$-torsion submodule.
4. $M$ contains a direct sum of $n$ non-quasi-$\tau$-torsion submodules, but no direct sum of $n + 1$ non-quasi-$\tau$-torsion submodules.
5. $M$ contains a direct sum of $n$ non-zero $\tau$-torsionfree submodules, but no direct sum of $n + 1$ non-zero $\tau$-torsionfree submodules.

**Proof.** (1) $\Rightarrow$ (2). Assume that $C$ is a complement to $\tau(M)$. As the proof of Corollary 3.3 shows there exist some $\tau$-torsionfree uniform submodules $C_1, \ldots, C_n$ of $C$ and a quasi-$\tau$-torsion submodule $D$ of $M$ such that $C_1 \oplus \cdots \oplus C_n \oplus D \leq p.\tau.e M$.

(2) $\Rightarrow$ (3). This implication is clear.

(3) $\Rightarrow$ (4). This follows by Proposition 3.2.

(4) $\Rightarrow$ (5). This implication is clear as every non-quasi-$\tau$-torsion submodule has a non-zero $\tau$-torsionfree submodule and every non-zero $\tau$-torsionfree submodule is non-quasi-$\tau$-torsion.

(5) $\Rightarrow$ (1). By hypothesis there exists a direct sum of $n$ non-zero $\tau$-torsionfree submodules $K_1 \oplus \cdots \oplus K_n$. This direct sum can be enlarged into a complement $C$ of $\tau(M)$. Then, $C$ contains a direct sum of $n$
non-zero $\tau$-torsionfree submodules, but no direct sum of $n+1$ non-zero $\tau$-torsionfree submodules and so $u\dim(C) = n$. \hfill \square

**Corollary 3.5.** The following statements are equivalent for a module $M$.

1. $M$ is of finite $\tau$-rank.
2. $M$ contains no infinite direct sum of non-quasi-$\tau$-torsion submodules.
3. $M$ contains no infinite direct sum of non-zero $\tau$-torsionfree submodules.

**Proof.** The implication (1) $\Rightarrow$ (2) follows by Proposition 3.4, and (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Let $C$ be a complement to $\tau(M)$. By hypothesis $C$ contains no infinite direct sum of non-zero submodules, hence $C$ is of finite uniform dimension. \hfill \square

**Corollary 3.6.** For any module $M$, 
$$r_\tau(M) = \sup\{k : M \text{ contains a direct sum of } k \text{ non-quasi-}\tau\text{-torsion submodules}\} = \sup\{k : M \text{ contains a direct sum of } k \text{ non-zero } \tau\text{-torsionfree submodules}\}.$$

**Corollary 3.7.** $r_\tau(M) \leq r_\tau(M/N) \leq u\dim(M/N)$, for every $\tau$-torsion submodule $N$ of $M$. In particular, $r_\tau(M) \leq u\dim(M/\tau(M))$. Moreover, $r_\tau(M) = u\dim(M/\tau(M))$ for every module $M$, if and only if the hereditary torsion theory $\tau$ is stable.

**Proof.** Assume that $M$ contains a direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ of non-zero $\tau$-torsionfree submodules. If $N$ is a $\tau$-torsion submodule of $M$ then $M/N$ has a direct sum of non-zero $\tau$-torsionfree submodules $(A_1+N)/N \oplus (A_2+N)/N \oplus \cdots \oplus (A_k+N)/N$. Thus, $r_\tau(M) \leq r_\tau(M/N)$ by Corollary 3.6. If $\tau$ is stable and $A_1/\tau(M) \oplus \cdots \oplus A_k/\tau(M)$ is a direct sum of non-zero submodules of $M/\tau(M)$, then $B_1 \oplus \cdots \oplus B_k$ is a direct sum of non-zero $\tau$-torsionfree submodules of $M$, where $B_i$ is a complement to $\tau(M)$ in $A_i$. Thus, $u\dim(M/\tau(M)) \leq r_\tau(M)$ and so $r_\tau(M) = u\dim(M/\tau(M))$. Now, let $r_\tau(M) = u\dim(M/\tau(M))$, for
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Let \( M \) be any module. Then, \( \text{u.dim}(M/\tau(M)) = 0 \), for every quasi-\( \tau \)-torsion module \( M \). Hence, every quasi-\( \tau \)-torsion module is \( \tau \)-torsion and so the hereditary torsion theory is stable by Proposition 2.7. □

**Corollary 3.8.** Let \( A \) be a submodule of \( M \).

1. \( r_\tau(A) \leq r_\tau(M) \).
2. \( r_\tau(A) = r_\tau(M) \) if and only if \( r_\tau(A) = \infty \) or if \( r_\tau(A) = k < \infty \) then every complement in \( M \) of each direct sum of \( k \) non-zero \( \tau \)-torsionfree submodules of \( A \) is quasi-\( \tau \)-torsion.
3. \( r_\tau(A) = r_\tau(M) \) if \( r_\tau(A) = \infty \) or if \( A \leq \text{p.\tau.e} \leq \tau \) and \( r_\tau(A) = k < \infty \). The converse holds if the hereditary torsion theory \( \tau \) is stable.

**Proof.** Clearly (1) follows by Corollary 3.6.

(2). (\( \Leftarrow \)). By Corollary 3.5, if \( r_\tau(A) = \infty \) then \( r_\tau(M) = \infty \). Now, assume that \( r_\tau(A) = k < \infty \). By Proposition 3.4, \( A \) contains a direct sum of \( k \) non-zero \( \tau \)-torsionfree submodules \( A_1 \oplus \cdots \oplus A_k \) and so by Proposition 2.2, \( A_1 \oplus \cdots \oplus A_k \oplus B \leq \text{p.\tau.e} \leq \tau \) for a submodule \( B \) of \( M \) which is maximal with respect to the property \( (A_1 \oplus \cdots \oplus A_k) \cap B = 0 \). By hypothesis \( B \) is quasi-\( \tau \)-torsion, hence \( r_\tau(M) = k \).

(\( \Rightarrow \)). Let \( r_\tau(A) = k < \infty \) and \( A_1 \oplus \cdots \oplus A_k \) be a direct sum of \( k \) non-zero \( \tau \)-torsionfree submodules of \( A \). If a complement \( B \) in \( M \) of \( A_1 \oplus \cdots \oplus A_k \) is non-quasi-\( \tau \)-torsion, then there exists a non-zero \( \tau \)-torsionfree submodule \( C \) of \( B \) and so \( M \) contains the direct sum \( A_1 \oplus \cdots \oplus A_k \oplus C \) of \( k + 1 \) non-zero \( \tau \)-torsionfree submodules which is impossible as \( r_\tau(M) = k \).

(3). The first statement is clear by Proposition 3.4 and Corollary 3.5. Now, assume that \( \tau \) is stable and \( r_\tau(A) = k < \infty \), moreover \( A \) is not \( \text{p.\tau.e} \)-essential in \( M \). Then, \( A \) contains a direct sum \( A_1 \oplus \cdots \oplus A_k \) of \( k \) non-zero \( \tau \)-torsionfree submodules. Since \( A \) is not \( \text{p.\tau.e} \)-essential in \( M \), there exists a non-\( \tau \)-torsion submodule \( B \) such that \( A \oplus B \leq \tau \) by Proposition 2.3-(5). Thus, \( M \) contains the direct sum \( A_1 \oplus \cdots \oplus A_k \oplus B \) of non-\( \tau \)-torsion submodules. Hence, \( r_\tau(M) \geq k + 1 \) as the notions of non-\( \tau \)-torsion and non-quasi-\( \tau \)-torsion are the same whenever a hereditary torsion theory is stable. □

Note that by Corollaries 3.7 and 3.8, \( r_\tau(M) \leq r_\tau(A) + r_\tau(M/A) \) if \( A \) is a \( \tau \)-torsion submodule or a \( \text{p.\tau.e} \)-essential submodule of \( M \). The next corollary shows that the inequality holds for some other submodules of \( M \). Recall that a submodule \( A \) of \( M \) is called \( \tau \)-pure (or \( \tau \)-closed)
if $M/A$ is $\tau$-torsionfree.

**Corollary 3.9.** Let $A$ be a $\tau$-torsionfree submodule or a $\tau$-pure submodule of $M$. Then,

$$r_\tau(M) \leq r_\tau(A) + r_\tau(M/A).$$

**Proof.** Let $A$ be a $\tau$-torsionfree submodule. There exists a submodule $B$ such that $A \oplus B \leq p.\tau.e M$. Then, $r_\tau(M) = r_\tau(A \oplus B) = r_\tau(A) + r_\tau(B)$. But, $B \cong (A \oplus B)/A \leq M/A$, hence $r_\tau(B) \leq r_\tau(M/A)$. Now, assume that $A$ is $\tau$-pure, moreover $C$ is a complement to $\tau(M)$ in $M$. Clearly $C \cap A$ can be enlarged to a complement $D$ to $\tau(A)$ in $A$. Then,

$$u.\dim(C) \leq u.\dim(C \cap A) + u.\dim((C \cap A)/(C \cap A)) \leq u.\dim(D) + u.\dim(M/A).$$

Since $A$ is $\tau$-pure, $r_\tau(M/A) = u.\dim(M/A)$ and so $r_\tau(M) \leq r_\tau(A) + r_\tau(M/A)$. $\square$

A module $M$ is called $\tau$-**injective** if for any $\tau$-dense submodule $A$ of $B$, any homomorphism $A \rightarrow M$ extends to a homomorphism $B \rightarrow M$. If $E_\tau(M)$ is a $\tau$-injective $\tau$-essential extension of $M$, then $E_\tau(M)$ is the smallest $\tau$-injective module containing $M$. Moreover, it is unique up to isomorphism. $E_\tau(M)$ is called the $\tau$-**injective hull** of $M$. More properties of the $\tau$-injective hull of a module can be found in [2, § 3]. Note that by Proposition 2.1, $M \leq p.\tau.e E_\tau(M)$. Proposition 3.11 below interprets the finiteness of the $\tau$-rank of $M$ via a certain decomposition length of $E_\tau(M)$. The following lemma is helpful.

**Lemma 3.10.** A module $M$ is pseudo $\tau$-uniform if and only if $E_\tau(M)$ is pseudo $\tau$-uniform.

**Proof.** Clearly if $M$ is non-quasi-$\tau$-torsion then $E_\tau(M)$ is non-quasi-$\tau$-torsion and the converse holds by Proposition 2.8. For ($\Rightarrow$), assume that $A \cap B \leq \tau(E_\tau(M))$. Then, $(A \cap M) \cap (B \cap M) \leq \tau(M)$, hence by hypothesis $A \cap M$ is quasi-$\tau$-torsion or $B \cap M$ is $\tau$-torsion. However $M \leq p.\tau.e E_\tau(M)$ and so $A \cap M \leq p.\tau.e A$, therefore $A$ is quasi-$\tau$-torsion by Proposition 2.8 or $B$ is $\tau$-torsion. The converse implication ($\Leftarrow$) is clear. $\square$
Proposition 3.11. \( r_\tau(M) = n < \infty \) if and only if \( E_\tau(M) \) is a direct sum of \( n \) pseudo \( \tau \)-uniform modules and a quasi-\( \tau \)-torsion module.

Proof. \((\Rightarrow)\). By hypothesis \( M \) contains pseudo \( \tau \)-uniform submodules \( A_1, \ldots, A_n \) and a quasi-\( \tau \)-torsion submodule \( B \) such that
\[
A_1 \oplus \cdots \oplus A_n \oplus B \leq p.\tau.e \ M.
\]
There exists a submodule \( C \) of \( M \) for which
\[
A_1 \oplus \cdots \oplus A_n \oplus B \oplus C \leq e \ M.
\]
Thus, \( A_1 \oplus \cdots \oplus A_n \oplus B \oplus C \) is essential and pseudo \( \tau \)-essential in \( E_\tau(M) \).

\[
E_\tau(M) = E_\tau(A_1 \oplus \cdots \oplus A_n \oplus B \oplus C) = E_\tau(A_1) \oplus \cdots \oplus E_\tau(A_n) \oplus E_\tau(B) \oplus E_\tau(C),
\]
where, each \( E_\tau(A_i) \) is pseudo \( \tau \)-uniform by Lemma 3.10 and \( E_\tau(B) \) and \( E_\tau(C) \) are quasi-\( \tau \)-torsion by Proposition 2.8.

\((\Leftarrow)\). By Proposition 3.4, \( r_\tau(E_\tau(M)) = n \) and so by Corollary 3.8-(3),
\[
r_\tau(M) = r_\tau(E_\tau(M)) = n. \qed
\]

Corollary 3.12. \( r_\tau(\bigoplus_{i=1}^k M_i) = \sum_{i=1}^k r_\tau(M_i). \)

4. Complements and \( \tau \)-ranks

Recall that for a module \( M \), if \( u.d(M) = n < \infty \), then any chain of complements has length \( \leq n \). In addition, \( u.d(M) = \infty \) if and only if there exists an infinite strictly ascending chain of complements in \( M \) if and only if there exists an infinite strictly descending chain of complements (See [3, Propositions (6.29) and (6.30)]). In this section we obtain similar relations for \( \tau \)-rank of a module \( M \) in terms of certain complement submodules.

Proposition 4.1. Let \( M \) be a module and \( r_\tau(M) = n < \infty \). Then, in \( M \) any chain of complements to \( \tau \)-torsionfree submodules has length \( \leq n \).
Proof. Let $C_0 < C_1 < \cdots < C_k$, where each $C_{i-1}$ is a complement to some $\tau$-torsionfree submodule $T_i$ of $M$. Then, each $C_{i-1}$ is a complement to the $\tau$-torsionfree submodule $T_i \cap C_i$ of $C_i$. Set $S_i = T_i \cap C_i$, for all $i = 1, \ldots, k$. Since $C_{i-1} \neq C_i$, we have $S_i \neq 0$. Then, $S_1 \oplus \cdots \oplus S_k$ is a direct sum of $k$ non-zero $\tau$-torsionfree submodules of $M$, hence $k \leq n$ by Corollary 3.6. \hfill \square

Theorem 4.2. The following statements are equivalent for a module $M$.

1. $r_\tau(M) = \infty$.
2. There exists an infinite strictly ascending chain of complements to $\tau$-torsionfree submodules in $M$.
3. There exists an infinite strictly descending chain of complements to $\tau$-torsionfree submodules in $M$.

Proof. (1) $\Rightarrow$ (2). By Corollary 3.5, $M$ contains an infinite direct sum $T_1 \oplus T_2 \oplus \cdots$, where $T_i$ is a non-zero $\tau$-torsionfree submodule. Enlarge $T_1$ into a complement to $T_2 \oplus T_3 \oplus \cdots$, say $C_1$. Then, enlarge $C_1 \oplus T_2$ into a complement to $T_3 \oplus T_4 \oplus \cdots$, say $C_2$. In this way, we get an ascending chain $C_1 \leq C_2 \leq \cdots$, where each $C_i$ is a complement to a $\tau$-torsionfree submodule in $M$. Since $T_i \leq C_i$ and $T_i \cap C_i = 0$, we have $C_{i-1} \neq C_i$, for all $i$.

(2) $\Rightarrow$ (3). Assume that $C_0 < C_1 < \cdots$, where each $C_i$ is a complement to a $\tau$-torsionfree submodule in $M$. If $C_k$ is $\tau$-torsion then $C_k = \tau(M)$ and so only $C_0$ can be $\tau$-torsion. Moreover, similar to the proof of Proposition 4.1, $C_{i-1}$ is a complement to some non-zero $\tau$-torsionfree submodule $S_i$ in $C_i$. Enlarge $S_2 \oplus S_3 \oplus \cdots$ into a complement to $S_1$, let $L_1$ be this complement. Then, enlarge $S_3 \oplus S_4 \oplus \cdots$ into a complement to $S_2$ in $L_1$, say $L_2$. Clearly $L_2$ is a complement to the non-zero $\tau$-torsionfree submodule $S_1 \oplus S_2$ in $M$. Moreover, $L_2 < L_1$ since $S_2 \leq L_1$ and $L_2 \cap S_2 = 0$. By this process we get a strictly descending chain of complements to $\tau$-torsionfree submodules in $M$, i.e., $L_1 > L_2 > \cdots$.

(3) $\Rightarrow$ (1) is clear by Proposition 4.1. \hfill \square

Recall that $u.\dim (M) = \sup \{k : M \text{ contains a chain of complements of length } k\}$. A similar result holds for the $\tau$-rank of $M$. 
Corollary 4.3. For any module $M$,
\[ r_\tau(M) = \sup\{k : M \text{ contains a chain of length } k \text{ of complements to } \tau\text{-torsionfree submodules}\}. \]

References


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