Bulletin of the Iranian Mathematical Society Vol. 35 No. 2 (2009), pp 143-162.

CAUCHY-RASSIAS STABILITY OF LINEAR MAPPINGS IN BANACH MODULES ASSOCIATED WITH A GENERALIZED JENSEN TYPE MAPPING

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Communicated by Heydar Radjavi

ABSTRACT. Let X and Y be vector spaces. We show that a mapping $f : X \to Y$ satisfies the functional equation,

$$f\left(x_1 + \sum_{j=2}^{2d}(-1)^j x_j\right) - f\left(x_1 + \sum_{j=2}^{2d}(-1)^{j-1} x_j\right) = 2\sum_{j=2}^{2d}(-1)^j f(x_j)$$

if and only if the mapping $f : X \to Y$ is Cauchy additive, and prove the Cauchy-Rassias stability of the above functional equation in Banach modules over a unital C^*-algebra, and in Poisson Banach modules over a unital Poisson C^*-algebra. Let A and B be unital C^*-algebras, Poisson C^*-algebras or Poisson JC^*-algebras. As an application, we show that every almost homomorphism $h : A \to B$ of A into B is a homomorphism when $h(2^n uy) = h(2^n u)h(y)$ or $h(2^n u \circ y) = h(2^n u) \circ h(y)$, for all unitaries $u \in A$, all $y \in A$, and $n = 0, 1, 2, \cdots$.

Moreover, we prove the Cauchy-Rassias stability of homomorphisms in C^*-algebras, Poisson C^*-algebras or Poisson JC^*-algebras.

MSC(2000): Primary: 39B52, 46L05, 47B48, 17A36.
Keywords: Cauchy-Rassias stability, C^*-algebra homomorphism, Poisson C^*-algebra homomorphism, Poisson Banach module over Poisson C^*-algebra, Poisson JC^*-algebra homomorphism.
Received: 15 July 2008, Accepted: 29 November 2008.
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1. **Introduction**

Ulam [26] raised the following question: Under what conditions does there exist an additive mapping near an approximate additive mapping? Hyers [4] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. Th.M. Rassias [18] extended the theorem of Hyers by considering the *unbounded Cauchy difference*. His result has provided a lot of influence in the development of what is known as *Cauchy-Rassias stability* of functional equations. Găvruta [2] generalized the Rassias' result to a more general unbounded control function. Beginning around the year 1980, the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was taken up by a number of mathematicians (cf. [1], [5]–[8], [10], [14], [17]–[25]).

Throughout this paper, assume that d is a positive integer.

We solve the following functional equation,

$$f\left(x_1+\sum_{j=2}^{2d}(-1)^j x_j\right) - f\left(x_1+\sum_{j=2}^{2d}(-1)^{j-1} x_j\right) = 2\sum_{j=2}^{2d}(-1)^j f(x_j), \tag{1.1}$$

which is called a *generalized Jensen type functional equation*, and whose solution is called a *generalized Jensen type mapping.* Moreover, we prove the Cauchy-Rassias stability of the functional equation (1.1) in Banach modules over a unital C^*-algebra. Our main purpose is to investigate homomorphisms between C^*-algebras, between Poisson C^*-algebras and between Poisson JC^*-algebras, and to prove their Cauchy-Rassias stability.

2. **A generalized Jensen type mapping**

Throughout this section, assume that X and Y are linear spaces.

Lemma 2.1. *A mapping $f : X \to Y$ satisfies* (1.1) *for all $x_1, x_2, \cdots, x_{2d} \in X$ and $f(0) = 0$ if and only if f is Cauchy additive.*

Proof. Assume that $f : X \to Y$ satisfies (1.1) for all $x_1, x_2, \cdots, x_{2d} \in X$. Putting $x_3 = \cdots = x_{2d} = 0$ in (1.1), we get

$$f(x_1 + x_2) - f(x_1 - x_2) = 2f(x_2) \tag{2.1}$$

for all $x_1, x_2 \in X$. Putting $x_2 = x_1$ in (2.1), we get

$$f(2x_1) = 2f(x_1)$$

for all $x_1 \in X$. Putting $x_1 - x_2 = x$ and $2x_2 = y$ in (2.1), we get

$$f(x+y) = f(x_1 + x_2) = f(x_1 - x_2) + f(2x_2) = f(x) + f(y)$$

for all $x, y \in X$. Thus, f is Cauchy additive.

The converse is obviously true. □

3. Cauchy-Rassias stability of the generalized Jensen type mapping in Banach modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with norm $|\cdot|$ and unitary group $U(A)$, and that X and Y are left Banach modules over A with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Given a mapping $f : X \to Y$, we set

$$\begin{aligned} D_u f(x_1, \cdots, x_{2d}) \quad := \quad & f\left(ux_1 + \sum_{j=2}^{2d} (-1)^j ux_j\right) \\ - \quad & f\left(ux_1 + \sum_{j=2}^{2d} (-1)^{j-1} ux_j\right) - 2\sum_{j=2}^{2d} (-1)^j uf(x_j) \end{aligned}$$

for all $u \in U(A)$ and all $x_1, \cdots, x_{2d} \in X$.

Theorem 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^{2d} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{2d}) \quad := \quad \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \cdots, 2^j x_{2d}) < \infty, \tag{3.1}$$

and

$$\|D_u f(x_1, \cdots, x_{2d})\| \quad \leq \quad \varphi(x_1, \cdots, x_{2d}), \tag{3.2}$$

for all $u \in U(A)$ and all $x_1, \cdots, x_{2d} \in X$. Then, there exists a unique A-linear generalized Jensen type mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \text{ times}}), \tag{3.3}$$

for all $x \in X$.

Proof. Let $u = 1 \in U(A)$. Putting $x_1 = \cdots = x_{2d} = x$ in (3.2), we have

$$\|f(2x) - 2f(x)\| \le \varphi(\underbrace{x, \cdots, x}_{2d \text{ times}}), \tag{3.4}$$

for all $x \in X$. So,

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(\underbrace{x, \cdots, x}_{2d \text{ times}}),$$

for all $x \in X$. Hence,

$$\left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^{n+1}}f(2^{n+1}x)\right\| \le \frac{1}{2^{n+1}}\varphi(\underbrace{2^n x, \cdots, 2^n x}_{2d \text{ times}}), \tag{3.5}$$

for all $x \in X$ and all positive integers n. By (3.5), we have

$$\left\|\frac{1}{2^m}f(2^m x) - \frac{1}{2^n}f(2^n x)\right\| \le \sum_{k=m}^{n-1} \frac{1}{2^{k+1}}\varphi(\underbrace{2^k x, \cdots, 2^k x}_{2d \text{ times}}), \tag{3.6}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ is Cauchy, for all $x \in X$. Since Y is complete, then the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ converges for all $x \in X$. So, we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x),$$

for all $x \in X$. We get

$$\begin{aligned}\|D_1 L(x_1, \cdots, x_{2d})\| &= \lim_{n\to\infty} \frac{1}{2^n}\|D_1 f(2^n x_1, \cdots, 2^n x_{2d})\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x_1, \cdots, 2^n x_{2d}) = 0,\end{aligned}$$

for all $x_1, \cdots, x_{2d} \in X$. By Lemma 2.1, L is Cauchy additive. Putting $m = 0$ and letting $n \to \infty$ in (3.6), we get (3.3).

Now, let $L' : X \to Y$ be another generalized Jensen type mapping satisfying (3.3). Then, we have

$$\begin{aligned}\|L(x) - L'(x)\| &= \frac{1}{2^n}\|L(2^n x) - L'(2^n x)\| \\ &\le \frac{1}{2^n}(\|L(2^n x) - f(2^n x)\| + \|L'(2^n x) - f(2^n x)\|) \\ &\le \frac{2}{2^{n+1}}\widetilde{\varphi}(\underbrace{2^n x, \cdots, 2^n x}_{2d \text{ times}}),\end{aligned}$$

which tends to zero as $n \to \infty$, for all $x \in X$. So, we can conclude that $L(x) = L'(x)$, for all $x \in X$. This proves the uniqueness of L.

By the assumption, for each $u \in U(A)$, we get

$$\begin{aligned}\|D_u L(x, x, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}})\| &= \lim_{n\to\infty} \frac{1}{2^n}\|D_u f(2^n x, 2^n x, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}})\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n x, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}}) = 0,\end{aligned}$$

for all $x \in X$. So,

$$L(2ux) = 2uL(x),$$

for all $u \in U(A)$ and all $x \in X$. Since L is additive, then

$$L(ux) = uL(x),$$

for all $u \in U(A)$ and all $x \in X$.

Now, by the same reasoning as in the proofs of [15] and [16],

$$L(ax + by) = L(ax) + L(by) = aL(x) + bL(y),$$

for all $a, b \in A \quad (a, b \neq 0)$ and all $x, y \in X$. And $L(0x) = 0 = 0L(x)$ for all $x \in X$. So, the unique generalized Jensen type mapping $L : A \to B$ is an A-linear mapping, as desired. □

Corollary 3.2. *Let θ and $p < 1$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$\|D_u f(x_1, \cdots, x_{2d})\| \le \theta \sum_{j=1}^{2d} ||x_j||^p,$$

for all $u \in U(A)$ *and all* $x_1, \cdots, x_{2d} \in X$*. Then, there exists a unique* A*-linear generalized Jensen type mapping* $L : X \to Y$ *such that*

$$\|f(x) - L(x)\| \le \frac{2d}{2 - 2^p}\theta||x||^p,$$

for all $x \in X$*.*

Proof. Define $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} ||x_j||^p$, and apply Theorem 3.1 to obtain the desired result. □

Theorem 3.3. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *for which there is a function* $\varphi : X^{2d} \to [0, \infty)$ *such that*

$$\begin{aligned} \widetilde{\varphi}(x_1, \cdots, x_{2d}) &:= \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x_1}{2^j}, \cdots, \frac{x_{2d}}{2^j}\right) < \infty, \\ \|D_u f(x_1, \cdots, x_{2d})\| &\le \varphi(x_1, \cdots, x_{2d}), \end{aligned}$$

for all $u \in U(A)$ *and all* $x_1, \cdots, x_{2d} \in X$*. Then, there exists a unique* A*-linear generalized Jensen type mapping* $L : X \to Y$ *such that*

$$\|f(x) - L(x)\| \le \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \text{ times}}),$$

for all $x \in X$*.*

Proof. Replacing x by $\frac{x}{2}$ in (3.4), we have

$$\left\|f(x) - 2f(\frac{x}{2})\right\| \le \varphi(\underbrace{\frac{x}{2}, \cdots, \frac{x}{2}}_{2d \text{ times}}),$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

4. Isomorphisms between unital C^*-algebras

Throughout this section, assume that A is a unital C^*-algebra with norm $|| \cdot ||$, unit e and unitary group $U(A)$, and that B is a unital C^*-algebra with norm $\| \cdot \|$.

We investigate C^*-algebra isomorphisms between unital C^*-algebras.

Theorem 4.1. *Let $h : A \to B$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$, for all $u \in U(A)$, all $y \in A$, and $n = 0, 1, 2, \cdots$, for which there is a function $\varphi : A^{2d} \to [0, \infty)$ satisfying* (3.1) *such that*

$$\left\| h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^j \mu x_j\right) - h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^{j-1} \mu x_j\right) - 2\sum_{j=2}^{2d}(-1)^j \mu h(x_j) \right\| \leq \varphi(x_1, \cdots, x_{2d}), \tag{4.1}$$

$$\|h(2^n u^*) - h(2^n u)^*\| \leq \varphi(\underbrace{2^n u, \cdots, 2^n u}_{2d \textit{ times}}), \tag{4.2}$$

for all $u \in U(A)$, all $x_1, \cdots, x_{2d} \in A$, all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and $n = 0, 1, 2, \cdots$. Assume that

$$\lim_{n\to\infty} \frac{h(2^n e)}{2^n} \quad \textit{is invertible.} \tag{4.3}$$

Then, the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. We consider a C^*-algebra as a Banach module over a unital C^*-algebra $\mathbb{C}$. So, by Theorem 3.1, there exists a unique $\mathbb{C}$-linear generalized Jensen type mapping $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \text{ times}}). \tag{4.4}$$

for all $x \in A$. The mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x), \tag{4.5}$$

for all $x \in A$.

By (3.1) and (4.2), we get

$$H(u^*) = \lim_{n\to\infty} \frac{h(2^n u^*)}{2^n} = \lim_{n\to\infty} \frac{h(2^n u)^*}{2^n} = \left(\lim_{n\to\infty} \frac{h(2^n u)}{2^n}\right)^* = H(u)^*,$$

for all $u \in U(A)$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [9]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in$

$\mathbb{C}, u_j \in U(A)$), then

$$\begin{aligned} H(x^*) &= H\left(\sum_{j=1}^{m} \overline{\lambda_j} u_j^*\right) = \sum_{j=1}^{m} \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^{m} \overline{\lambda_j} H(u_j)^* \\ &= \left(\sum_{j=1}^{m} \lambda_j H(u_j)\right)^* = H\left(\sum_{j=1}^{m} \lambda_j u_j\right)^* = H(x)^*, \end{aligned}$$

for all $x \in A$.

Since $h(2^n uy) = h(2^n u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \cdots$, then

$$H(uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u)h(y) = H(u)h(y), \tag{4.6}$$

for all $u \in U(A)$ and all $y \in A$. By the additivity of H and (4.6),

$$2^n H(uy) = H(2^n uy) = H(u(2^n y)) = H(u)h(2^n y),$$

for all $u \in U(A)$ and all $y \in A$. Hence,

$$H(uy) = \frac{1}{2^n} H(u)h(2^n y) = H(u)\frac{1}{2^n} h(2^n y), \tag{4.7}$$

for all $u \in U(A)$ and all $y \in A$. Taking the limit in (4.7) as $n \to \infty$, we obtain:

$$H(uy) = H(u)H(y), \tag{4.8}$$

for all $u \in U(A)$ and all $y \in A$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in U(A)$), then it follows from (4.8) that

$$\begin{aligned} H(xy) &= H\left(\sum_{j=1}^{m} \lambda_j u_j y\right) = \sum_{j=1}^{m} \lambda_j H(u_j y) = \sum_{j=1}^{m} \lambda_j H(u_j)H(y) \\ &= H\left(\sum_{j=1}^{m} \lambda_j u_j\right) H(y) = H(x)H(y), \end{aligned}$$

for all $x, y \in A$.

By (4.6) and (4.8),

$$H(e)H(y) = H(ey) = H(e)h(y),$$

for all $y \in A$. Since $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = H(e)$ is invertible, then

$$H(y) = h(y),$$

for all $y \in A$.

Therefore, the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Corollary 4.2. *Let $h : A \to B$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$, for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}\Bigg\| h\left(\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j\right) &- h\left(\mu x_1 + \sum_{j=2}^{2d}(-1)^{j-1}\mu x_j\right) \\ -2\sum_{j=2}^{2d}(-1)^j \mu h(x_j)\Bigg\| &\leq \theta \sum_{j=1}^{2d} ||x_j||^p, \\ \|h(2^n u^*) - h(2^n u)^*\| &\leq 2d2^{np}\theta\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x_1, \cdots, x_{2d} \in A$. Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ is invertible. Then, the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} ||x_j||^p$, and apply Theorem 4.1 to obtain the desired result. □

Theorem 4.3. *Let $h : A \to B$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$, for all $u \in U(A)$, all $y \in A$, and $n = 0, 1, 2, \cdots$, for which there is a function $\varphi : A^{2d} \to [0, \infty)$ satisfying (3.1), (4.2), and (4.3) such that*

$$\begin{aligned}\Bigg\| h\left(\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j\right) &- h\left(\mu x_1 + \sum_{j=2}^{2d}(-1)^{j-1}\mu x_j\right) \\ -2\sum_{j=2}^{2d}(-1)^j \mu h(x_j)\Bigg\| &\leq \varphi(x_1, \cdots, x_{2d}),\end{aligned} \tag{4.9}$$

for all $x_1, \cdots, x_{2d} \in A$ and $\mu = 1, i$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Put $\mu = 1$ in (4.9). By the same reasoning as in the proof of Theorem 4.1, there exists a unique generalized Jensen type mapping $H : A \to B$ satisfying (4.4). By the same reasoning as in the proof of Theorem of [18], the mapping $H : A \to B$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (4.9). By the same method as in the proof of Theorem 4.1, one can obtain:

$$H(ix) = \lim_{n\to\infty} \frac{h(2^n ix)}{2^n} = \lim_{n\to\infty} \frac{ih(2^n x)}{2^n} = iH(x),$$

for all $x \in A$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So,

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) \\ &= (s + it)H(x) = \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in A$. So,

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y),$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in A$. Hence, the additive mapping $H : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 4.1. □

Now, we prove the Cauchy-Rassias stability of C^*-algebra homomorphisms in unital C^*-algebras.

Theorem 4.4. *Let $h :\to B$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : A^{2d} \to [0, \infty)$ satisfying* (3.1), (4.1) *and* (4.2) *such that*

$$(4.10) \|h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)\| \le \varphi(2^n u, 2^n v, \underbrace{0, \cdots, 0}_{2d-2 \ times}),$$

for all $u, v \in U(A)$ and $n = 0, 1, 2, \cdots$. Then, there exists a unique C^*-algebra homomorphism $H : A \to B$ satisfying (4.4).

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive generalized Jensen type mapping $H : A \to B$ satisfying (4.4).

By (4.10),

$$\begin{aligned}\frac{1}{2^{2n}}\|h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)\| &\leq \frac{1}{2^{2n}}\varphi(2^n u, 2^n v, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}}) \\ &\leq \frac{1}{2^{n}}\varphi(2^n u, 2^n v, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}}),\end{aligned}$$

which tends to zero by (3.1) as $n \to \infty$. By (4.5),

$$\begin{aligned}H(uv) &= \lim_{n\to\infty}\frac{h(2^n u \cdot 2^n v)}{2^{2n}} = \lim_{n\to\infty}\frac{h(2^n u)h(2^n v)}{2^{2n}} \\ &= \lim_{n\to\infty}\frac{h(2^n u)}{2^n}\frac{h(2^n v)}{2^n} = H(u)H(v),\end{aligned}$$

for all $u, v \in U(A)$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in U(A)$), then

$$\begin{aligned}H(xv) &= H\left(\sum_{j=1}^m \lambda_j u_j v\right) = \sum_{j=1}^m \lambda_j H(u_j v) = \sum_{j=1}^m \lambda_j H(u_j)H(v) \\ &= H\left(\sum_{j=1}^m \lambda_j u_j\right) H(v) = H(x)H(v),\end{aligned}$$

for all $x \in A$ and all $v \in U(A)$. By the same method as given above, one can obtain:

$$H(xy) = H(x)H(y),$$

for all $x, y \in A$. So, the mapping $H : A \to B$ is a C^*-algebra homomorphism. □

Theorem 4.5. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : A^{2d} \to [0, \infty)$ satisfying* (3.1), (4.2), (4.9) *and* (4.10). *If $h(tx)$ is continuous in $t \in \mathbb{R}$, for each fixed $x \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ satisfying* (4.4).

Proof. The proof is similar to the proofs of theorems 4.3 and 4.4. □

5. Homomorphisms between Poisson C^*-algebras

A *Poisson C^*-algebra* A is a C^*-algebra with a $\mathbb{C}$-bilinear map $\{\cdot,\cdot\}: A \times A \to A$, called a *Poisson bracket*, such that $(A, \{\cdot,\cdot\})$ is a complex Lie algebra and

$$\{ab, c\} = a\{b, c\} + \{a, c\}b,$$

for all $a, b, c \in A$. Poisson algebras have played important roles in many mathematical areas and have been studied to find sympletic leaves of the corresponding Poisson varieties. It is also important to find or construct a Poisson bracket in the theory of Poisson algebra (see [3], [11], [12], [13]).

Throughout this section, let A be a unital Poisson C^*-algebra with norm $||\cdot||$, unit e and unitary group $U(A)$, and B a unital Poisson C^*-algebra with norm $\|\cdot\|$.

Definition 5.1. A C^*-algebra homomorphism $H : A \to B$ is called a *Poisson C^*-algebra homomorphism* if $H : A \to B$ satisfies

$$H(\{z, w\}) = \{H(z), H(w)\},$$

for all $z, w \in A$.

We investigate Poisson C^*-algebra homomorphisms between Poisson C^*-algebras.

Theorem 5.2. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$, for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : A^{2d+2} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{2d}, z, w) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \cdots, 2^j x_{2d}, 2^j z, 2^j w) < \infty, \tag{5.1}$$

$$\begin{aligned} &\|h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^j \mu x_j + \{z, w\}\right) - h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^{j-1} \mu x_j\right) \\ &\quad -2\sum_{j=2}^{2d} (-1)^j \mu h(x_j) - \{h(z), h(w)\}\| \le \varphi(x_1, \cdots, x_{2d}, z, w), \end{aligned} \tag{5.2}$$

$$\|h(2^n u^*) - h(2^n u)^*\| \le \varphi(\underbrace{2^n u, \cdots, 2^n u}_{2d\ times}, 0, 0), \tag{5.3}$$

for all $u \in U(A)$*, all* $x_1, \cdots, x_{2d}, z, w \in A$*, all* $\mu \in \mathbb{T}^1$ *and* $n = 0, 1, 2, \cdots$*. Assume that* $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ *is invertible. Then, the mapping* $h : A \to B$ *is a Poisson* C^**-algebra homomorphism.*

Proof. The proof is similar to the proof of Theorem 2.1 in [15].

Now, we prove the Cauchy-Rassias stability of Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras. □

Theorem 5.3. *Let* $h : A \to B$ *be a mapping satisfying* $h(0) = 0$ *for which there exists a function* $\varphi : A^{2d+2} \to [0, \infty)$ *satisfying* (5.1)*,* (5.2) *and* (5.3) *such that*

$$\|h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)\| \le \varphi(2^n u, 2^n v, \underbrace{0, \cdots, 0}_{2d \text{ times}}),$$

for all $u, v \in U(A)$ *and* $n = 0, 1, 2, \cdots$*. Then, there exists a unique Poisson* C^**-algebra homomorphism* $H : A \to B$ *satisfying*

$$\|h(x) - H(x)\| \le \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \text{ times}}, 0, 0),$$

for all $x \in A$*.*

Proof. The proof is similar to the proofs of theorems 4.4 and 5.2. □

6. Cauchy-Rassias stability of homomorphisms in Poisson Banach modules over a unital Poisson C^*-algebra

A *Poisson Banach module* X over a Poisson C^*-algebra A is a left Banach A-module endowed with a $\mathbb{C}$-bilinear map $\{\cdot, \cdot\} : A \times X \to X$ such that

$$\begin{aligned} \{\{a, b\}, x\} &= \{a, \{b, x\}\} - \{b, \{a, x\}\}, \\ \{a, b\} \cdot x &= a \cdot \{b, x\} - \{b, a \cdot x\}, \end{aligned}$$

for all $a, b \in A$ and all $x \in X$ (see [3], [11], [12]). Here, "$\cdot$" denotes the associative module action.

Throughout this section, assume that A is a unital Poisson C^*-algebra with unitary group $U(A)$, and that X and Y are left Poisson Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Definition 6.1. A $\mathbb{C}$-linear mapping $H : X \to Y$ is called a *Poisson module homomorphism* if $H : X \to Y$ satisfies

$$\begin{aligned} H(\{\{a,b\},x\}) &= \{\{a,b\},H(x)\}, \\ H(\{a,b\}\cdot x) &= \{a,b\}\cdot H(x), \end{aligned}$$

for all $a, b \in A$ and all $x \in X$.

We prove the Cauchy-Rassias stability of homomorphisms in Poisson Banach modules over a unital Poisson C^*-algebra.

Theorem 6.2. *Let $h : X \to Y$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : X^{2d} \to [0,\infty)$ satisfying (3.1) such that*

$$\begin{aligned} (6.1)\qquad & \left\| h\left(\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j\right) - h\left(\mu x_1 + \sum_{j=2}^{2d}(-1)^{j-1}\mu x_j\right) \right. \\ & \left. - 2\sum_{j=2}^{2d}(-1)^j \mu h(x_j)\right\| \le \varphi(x_1,\cdots,x_{2d}), \\ (6.2)\qquad & \|h(\{\{u,v\},x\}) - \{\{u,v\},h(x)\}\| \le \varphi(\underbrace{x,\cdots,x}_{2d\ times}), \\ (6.3)\qquad & \|h(\{u,v\}\cdot x) - \{u,v\}\cdot h(x)\| \le \varphi(\underbrace{x,\cdots,x}_{2d\ times}), \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $x, x_1, \cdots, x_{2d} \in X$ and all $u, v \in U(A)$. Then, there exists a unique Poisson module homomorphism $H : X \to Y$ such that

$$(6.4)\qquad \|h(x) - H(x)\| \le \frac{1}{2}\widetilde{\varphi}(\underbrace{x,\cdots,x}_{2d\ times}),$$

for all $x \in X$.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $H : X \to Y$ satisfying (6.4). The $\mathbb{C}$-linear mapping $H : X \to Y$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x),$$

for all $x \in X$.

By (6.2),

$$\begin{aligned}&\left\|\frac{1}{2^n}h(2^n\{\{u,v\},x\})-\left\{\{u,v\},\frac{1}{2^n}h(2^nx)\right\}\right\|\\&=\frac{1}{2^n}\|h(\{\{u,v\},2^nx\})-\{\{u,v\},h(2^nx)\}\|\\&\le\frac{1}{2^n}\varphi(\underbrace{2^nx,\cdots,2^nx}_{2d\text{ times}}),\end{aligned}$$

which tends to zero for all $x\in X$ by (3.1). So,

$$\begin{aligned}H(\{\{u,v\},x\}) &= \lim_{n\to\infty}\frac{1}{2^n}h(2^n\{\{u,v\},x\})=\lim_{n\to\infty}\{\{u,v\},\frac{1}{2^n}h(2^nx)\},\\ &= \{\{u,v\},H(x)\}\end{aligned}$$

for all $x\in X$ and all $u,v\in U(A)$. Since H is $\mathbb{C}$-linear and $\{\cdot,\cdot\}$ is $\mathbb{C}$-bilinear and since each $a\in A$ is a finite linear combination of unitary elements, i.e., $a=\sum_{j=1}^m\lambda_ju_j$ ($\lambda_j\in\mathbb{C},u_j\in U(A)$), then

$$\begin{aligned}&H(\{\{a,v\},x\})=H\left(\left\{\left\{\sum_{j=1}^m\lambda_ju_j,v\right\},x\right\}\right)\\&=\sum_{j=1}^m\lambda_jH(\{\{u_j,v\},x\})=\sum_{j=1}^m\lambda_j\{\{u_j,v\},H(x)\}\\&=\left\{\left\{\sum_{j=1}^m\lambda_ju_j,v\right\},H(x)\right\}=\{\{a,v\},H(x)\},\end{aligned}$$

for all $x\in X$ and all $v\in U(A)$. Similarly, one can show that

$$H(\{\{a,b\},x\})=\{\{a,b\},H(x)\},$$

for all $x\in X$ and all $a,b\in A$.

By (6.3),

$$\begin{aligned}&\left\|\frac{1}{2^n}h(2^n\{u,v\}\cdot x)-\{u,v\}\cdot\frac{1}{2^n}h(2^nx)\right\|\\&=\frac{1}{2^n}\|h(\{u,v\}\cdot 2^nx)-\{u,v\}\cdot h(2^nx)\|\\&\le\frac{1}{2^n}\varphi(\underbrace{2^nx,\cdots,2^nx}_{2d\text{ times}}),\end{aligned}$$

which by (3.1) tends to zero for all $x \in X$. So,

$$\begin{aligned} H(\{u,v\}\cdot x) &= \lim_{n\to\infty}\frac{1}{2^n}h(2^n\{u,v\}\cdot x) = \lim_{n\to\infty}\{u,v\}\cdot\frac{1}{2^n}h(2^n x) \\ &= \{u,v\}\cdot H(x), \end{aligned}$$

for all $x \in X$ and all $u, v \in U(A)$. Since H is $\mathbb{C}$-linear and $\{\cdot,\cdot\}$ is $\mathbb{C}$-bilinear and since each $a \in A$ is a finite linear combination of unitary elements, i.e., $a = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in U(A)$), then

$$\begin{aligned} H(\{a,v\}\cdot x) = H\left(\left\{\sum_{j=1}^m \lambda_j u_j, v\right\}\cdot x\right) = \sum_{j=1}^m \lambda_j H(\{u_j,v\}\cdot x) \\ = \sum_{j=1}^m \lambda_j\{u_j,v\}\cdot H(x) = \left\{\sum_{j=1}^m \lambda_j u_j, v\right\}\cdot H(x) = \{a,v\}\cdot H(x), \end{aligned}$$

for all $x \in X$ and all $v \in U(A)$. Similarly, one can show that

$$H(\{a,b\}\cdot x) = \{a,b\}\cdot H(x),$$

for all $x \in X$ and all $a, b \in A$. Thus, $H : X \to Y$ is a Poisson module homomorphism.

Therefore, there exists a unique Poisson module homomorphism $H : X \to Y$ satisfying (6.4). □

7. Homomorphisms between Poisson JC^*-algebras

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [27]). Let $L(H)$ be the real vector space of all bounded self-adjoint linear operators on H, interpreted as the *observables* of the system. In 1932, Jordan observed that $L(H)$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra.* A unital Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a *JC^*-algebra.* A Poisson C^*-algebra, endowed with the anticommutator product, is called a *Poisson JC^*-algebra.*

Throughout this section, assume that A is a unital Poisson JC^*-algebra with unit e, norm $||\cdot||$ and unitary group $U(A)$, and that B is a unital Poisson JC^*-algebra with unit e' and norm $\|\cdot\|$.

Definition 7.1. *A $\mathbb{C}$-linear mapping $H : A \to B$ is called a Poisson JC^*-algebra homomorphism if $H : A \to B$ satisfies*

$$\begin{aligned} H(x \circ y) &= H(x) \circ H(y), \\ H(\{x, y\}) &= \{H(x), H(y)\}, \end{aligned}$$

for all $x, y \in A$.

We investigate Poisson JC^*-algebra homomorphisms between Poisson JC^*-algebras.

Theorem 7.2. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$, for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : A^{2d+2} \to [0, \infty)$ satisfying (5.1) such that*

$$\begin{aligned} &\Bigg\| h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^j \mu x_j + \{z, w\}\right) - h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^{j-1} \mu x_j\right) \\ &-2 \sum_{j=2}^{2d} (-1)^j \mu h(x_j) - \{h(z), h(w)\} \Bigg\| \le \varphi(x_1, \cdots, x_{2d}, z, w), \end{aligned} \tag{7.1}$$

for all $x_1, \cdots, x_{2d}, z, w \in A$, and all $\mu \in \mathbb{T}^1$. Assume:

$$\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = e'. \tag{7.2}$$

Then, the mapping $h : A \to B$ is a Poisson JC^*-algebra homomorphism.

Proof. The proof is similar to the proofs of theorems 4.1 and 5.2. □

Corollary 7.3. *Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$, for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} &\Bigg\| h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^j \mu x_j + \{z, w\}\right) - h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^{j-1} \mu x_j\right) \\ &-2 \sum_{j=2}^{2d} (-1)^j \mu h(x_j) - \{h(z), h(w)\} \Bigg\| \le \theta \left(\sum_{j=1}^{2d} ||x_j||^p + ||z||^p + ||w||^p\right) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, $n = 0, 1, \cdots$, *and all* $x_1, \cdots, x_{2d}, z, w \in A$. *Assume that* $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = e'$. *Then, the mapping* $h : A \to B$ *is a Poisson* JC^**-algebra homomorphism.*

Proof. Define $\varphi(x_1, \cdots, x_{2d}, z, w) = \theta\left(\sum_{j=1}^{2d} ||x_j||^p + ||z||^p + ||w||^p\right)$, and apply Theorem 7.2 to obtain the desired result. □

Theorem 7.4. *Let* $h : A \to B$ *be a mapping satisfying* $h(2x) = 2h(x)$, *for all* $x \in A$ *for which there exists a function* $\varphi : A^{2d+2} \to [0, \infty)$ *satisfying* (5.1), (7.1) *and* (7.2) *such that*

$$\|h(2^n u \circ y) - h(2^n u) \circ h(y)\| \leq \varphi(u, y, \underbrace{0, \cdots, 0}_{2d \text{ times}}),$$

for all $y \in A$, *all* $u \in U(A)$ *and* $n = 0, 1, 2, \cdots$. *Then, the mapping* $h : A \to B$ *is a Poisson* JC^**-algebra homomorphism.*

Proof. The proof is similar to the proofs of theorems 4.1 and 5.2. □

Now we prove the Cauchy-Rassias stability of homomorphisms in Poisson JC^*-algebras.

Theorem 7.5. *Let* $h : A \to B$ *be a mapping satisfying* $h(0) = 0$ *for which there exists a function* $\varphi : A^{2d+4} \to [0, \infty)$ *such that*

$$\begin{aligned} \widetilde{\varphi}(x_1, \cdots, x_{2d}, z, w, a, b) &:= \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \cdots, 2^j x_{2d}, 2^j z, 2^j w, 2^j a, 2^j b) \\ &< \infty, \end{aligned}$$

$$\begin{aligned} &\left\| h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^j \mu x_j + \{z, w\} + a \circ b\right) - h\left(\mu x_1 + \sum_{j=2}^{2d} (-1)^{j-1} \mu x_j\right) \right. \\ &\qquad \left. -2\sum_{j=2}^{2d} (-1)^j \mu h(x_j) - \{h(z), h(w)\} - h(a) \circ h(b) \right\| \\ &\leq \varphi(x_1, \cdots, x_{2d}, z, w, a, b), \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_{2d}, z, w, a, b \in A$. Then, there exists a unique Poisson JC^*-algebra homomorphism $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d\ times}, 0, 0, 0, 0), \tag{7.3}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ satisfying (7.3).

The rest of the proof is similar to the proofs of Theorems 4.1 and 5.2. □

Acknowledgments

The second and corresponding author was supported by the research fund of Hanyang University (HY-2007).

References

[1] Z. Gajda, On stability of additive mappings, *Internat. J. Math. Math. Sci.* **14** (1991) 431–434.

[2] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994) 431–436.

[3] K.R. Goodearl and E.S. Letzter, Quantum n-space as a quotient of classical n-space, *Trans. Amer. Math. Soc.* **352** (2000) 5855–5876.

[4] D.H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. USA* **27** (1941) 222–224.

[5] D.H. Hyers and Th.M. Rassias, Approximate homomorphisms, *Aequationes Math.* **44** (1992) 125–153.

[6] K. Jun and H. Kim, Stability problem of Ulam for generalized forms of Cauchy functional equation, *J. Math. Anal. Appl.* **312** (2005) 535–547.

[7] K. Jun and Y. Lee, A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation, *J. Math. Anal. Appl.* **238** (1999) 305–315.

[8] K. Jun and Y. Lee, A generalization of the Hyers-Ulam-Rassias stability of the Pexiderized quadratic equations, *J. Math. Anal. Appl.* **297** (2004) 70–86.

[9] R.V. Kadison and J.R. Ringrose, Fundamentals of the Theory of Operator Algebras: Elementary Theory, Academic Press, New York, 1983.

[10] M.S. Moslehian, On the orthogonal stability of the Pexiderized quadratic equation, *J. Difference Equ. Appl.* **11** (2005) 999–1004.

[11] S.Oh, C. Park and Y. Shin, Quantum n-space and Poisson n-space, *Commun. Algebra* **30** (2002) 4197–4209.

[12] S.Oh, C. Park and Y. Shin, A Poincaré-Birkhoff-Witt theorem for Poisson enveloping algebras, *Commun. Algebra* **30** (2002) 4867–4887.
[13] C. Park, Poisson brackets on Banach algebras, *Kyungpook Math. J.* **44** (2004) 597–606.
[14] C. Park, On the stability of the orthogonally quartic functional equation, *Bull. Iran. Math. Soc.* **31** (2005) 63–70.
[15] C. Park, Homomorphisms between Poisson JC^*-algebras, *Bull. Braz. Math. Soc.* **36** (2005) 79–97.
[16] C. Park, Homomorphisms between Lie JC^*-algebras and Cauchy-Rassias stability of Lie JC^*-algebra derivations, *J. Lie Theory* **15** (2005) 393–414.
[17] C. Park and J. Hou, Homomorphisms between C^*-algebras associated with the Trif functional equation and linear derivations on C^*-algebras, *J. Korean Math. Soc.* **41** (2004) 461–477.
[18] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978) 297–300.
[19] Th.M. Rassias, Problem 16; 2, Report of the 27^{th} International Symp. on Functional Equations, *Aequationes Math.* **39** (1990) 292–293; 309.
[20] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Appl. Math.* **62** (2000) 23–130.
[21] Th.M. Rassias, The problem of S.M. Ulam for approximately multiplicative mappings, *J. Math. Anal. Appl.* **246** (2000) 352–378.
[22] Th.M. Rassias, On the stability of functional equations in Banach spaces, *J. Math. Anal. Appl.* **251** (2000) 264–284.
[23] Th.M. Rassias and P. Šemrl, On the behavior of mappings which do not satisfy Hyers-Ulam stability, *Proc. Amer. Math. Soc.* **114** (1992) 989–993.
[24] Th.M. Rassias and P. Šemrl, On the Hyers-Ulam stability of linear mappings, *J. Math. Anal. Appl.* **173** (1993) 325–338.
[25] T. Trif, On the stability of a functional equation deriving from an inequality of Popoviciu for convex functions, *J. Math. Anal. Appl.* **272** (2002) 604–616.
[26] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1960.
[27] H. Upmeier, Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics, Regional Conference Series in Mathematics No. 67, Amer. Math. Soc., Providence, 1987.

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