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# PROVING THE EFFICIENCY OF PRO-2-GROUPS OF FIXED CO-CLASSES

A. ARJOMANDFAR* AND H. DOOSTIE

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ABSTRACT. Among the six classes of pro-2-groups of finite and fixed co-classes and trivial Schur Multiplicator which studied by Abdolzadeh and Eick in 2009, there are two classes

$$S_5 = \langle a, b \mid [b, a^2] = 1, a^2 = [b, a]^2, (b^2)^{[b,a]} b^2 = 1 \rangle$$

and

$$S_6 = \langle a, t, b \mid a^2 = b^2, [b, a]^2 = 1, t^a = t^{-1}[b, a], b^t = aba \rangle$$

that have been conjectured to have deficiency zero presentations. In this paper we prove these conjectures. This completes the efficiency of all six classes of pro-2-groups of fixed co-classes.

## 1. Introduction

For detailed information on pro-$p$-groups one may see [6, 7, 8]. The pro-2-groups of fixed co-classes were first investigated in [1] and the Schur Multiplicator is used to get the appropriate presentations for such classes of groups. For a useful and prolific information on the Schur Multiplicator of a group one may consult [10]. Briefly, for a finitely presented finite group $G = \langle X \mid R \rangle$ the Schur Multiplicator of $G$ is defined to be the group $M(G) = \frac{F' \cap \overline{R}}{[F, \overline{R}]}$, where $F = F(X)$ is the free group

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*Corresponding author

of rank $|X|$ and $\overline{R}$ is the normal closure of $R$ ($R \subseteq F$). More detailed consideration about the relationship between the Schur Multiplicator and the deficiency of a presentation of a finitely presented group may be found in [9]. It is a classical fact that the groups with a deficiency zero presentation will have the trivial Schur Multiplicators. However, looking for a deficiency zero presentation of a group which has the trivial Schur Multiplicator, is a long-standing question and many attempts have been made during the years on finite $p$-groups and even in infinite groups. Our notations are merely standard, we use $|G : H|$ for the index of a subgroup $H$ in a group $G$, $[a, b]$ is used for the commutator $a^{-1}b^{-1}ab$ and we will use the Modified Todd-Coxeter coset enumeration algorithm in the form as given in [2] to get a presentation for subgroups. More application of this algorithm may be found in [3, 4, 5]. Following [1] and consider the groups:

$$S_5 = \langle a, b \mid [b, a^2] = 1, a^2 = [b, a]^2, (b^2)^{[b,a]}b^2 = 1\rangle$$

and

$$S_6 = \langle a, t, b \mid a^2 = b^2, [a, b]^2 = 1, t^a = t^{-1}[b, a], b^t = aba\rangle.$$

Just proved in [1], these groups have trivial Schur Multiplicators and the above presentations are the simplified presentations for them (see the Lemmas 13 and 14 of [1]). We now recall the conjectures 14 and 16 of [1] as the following propositions:

**Proposition 1.1.** *The group $S_5$ has a deficiency zero presentation isomorphic to*

$$\langle a, b \mid a^2 = [b, a]^2, (b^2)^{[b,a]}b^2 = 1\rangle.$$

**Proposition 1.2.** *The group $S_6$ has a deficiency zero presentation isomorphic to*

$$\langle a, t, b \mid a^2 = b^2, t^a = t^{-1}[b, a], b^t = aba\rangle.$$

## 2. The proofs

We give a suitable generating set for the derived subgroups of $S_5$ and $S_6$. Then, by using the Modified Todd-Coxeter coset enumeration algorithm we get a presentation for the derived subgroups. Note that, using GAP [11] we are able to get a presentation for the derived subgroup in each case and then finding the image of the word $[b, a^2]$ in the derived subgroup of the group $S_5$ and the image of the word $[a,b]^2$ in the derived subgroup of $S_6$ are possible, however, checking that this image is the identity element of the group (in each case) is not possible every time, i.e; we checked it for $S_5$ and it was not possible for $S_6$. For this reason we are interested in to use the combinatorial method of Modified Todd-Coxeter coset enumeration algorithm to give a clear and exact proof.

**Lemma 2.1.** *Let $G = \langle a, b \mid a^2 = [b,a]^2, (b^2)^{[b,a]}b^2 = 1\rangle$. Then, $G'$, the derived subgroup of $G$, has a presentation isomorphic to*

$$G' = \langle a_1, \ldots, a_6 \mid r_1, ..., r_{12}\rangle$$

*where, the relations $r_1, \ldots, r_{12}$ are as follows:*

$$\begin{aligned}
&r_1 : a_1a_3^{-2} = 1, && r_2 : (a_1a_6^{-1})^2a_1^{-1}a_4 = 1,\\
&r_3 : a_1a_5^{-2} = 1, && r_4 : (a_6a_2)^2a_5^{-1}a_1a_3^{-1} = 1,\\
&r_5 : a_2a_3a_2a_3^{-1} = 1, && r_6 : a_2a_6a_2a_6^{-1} = 1,\\
&r_7 : a_3^2a_6a_3^2a_6^{-1} = 1, && r_8 : (a_1a_6^{-1})^2a_2a_3^{-1}a_4^{-1}a_1a_2a_3 = 1,
\end{aligned}$$

$$\begin{aligned}
&r_9 : a_4a_3^2a_2^{-1}a_6^{-1}a_3a_5^{-1}a_6^{-1}a_1^{-1} = 1,\\
&r_{10} : (a_6^{-1}a_3a_1^{-1})a_5(a_2^{-1}a_3)^2a_2^{-1}(a_6^{-1}a_3a_1^{-1})a_4^{-1}(a_1a_3a_2^{-1}) = 1,\\
&r_{11} : a_1^{-1}a_4a_1a_6^{-1}a_2a_3^{-1}a_2a_5^{-1}a_1a_6^{-1} = 1,\\
&r_{12} : (a_4^{-1}a_1)a_3a_2^{-1}a_6^{-1}(a_3a_1^{-1}a_5)a_1a_6^{-1}a_2a_3^{-1} = 1.
\end{aligned}$$

*Proof.* It is easy to check that $\frac{G}{G'} \cong Z_2 \times Z_4$. We now consider the subgroup

$$K = \langle a^2, b^4, [b,a], [a^{-2}, b^{-1}], [a^{-1}, b], [b^{-1}, a]\rangle$$

of G and define eight cosets as

$$1 = K,\ ib = i+1,\ (i = 1,2,3),\ 1a = 5,\ 4a^{-1} = 6,\ 2a^{-1} = 7,\ 3a^{-1} = 8$$

to see that $K$ is of index 8 in $G$. Since $a^2, b^4 \in G'$ then $K \subseteq G'$ , which together with $|G : H| = 8$ proves that $K = G'$.
We let

$$a_1 = a^2,\ a_2 = b^4,\ a_3 = [b,a],\ a_4 = [a^{-2}, b^{-1}],\ a_5 = [a^{-1}, b],\ a_6 = [b^{-1}, a]$$

and we use the Modified Todd-Coxeter coset enumeration algorithm to get a presentation for $K$. In this way we have to adopt the name of a coset and its representative to get the following table of coset representatives:

$$\begin{array}{ll}
1a = 5, & 1b = 2, \\
2a = a_4^{-1}a_1.7, & 2b = 3, \\
3a = a_3a_2^{-1}a_3a_2^{-1}.8, & 3b = 4, \\
4a = a_2a_5^{-1}a_1a_3^{-1}a_2^{-1}.6, & 4b = a_2.1, \\
5a = a_1.1, & 5b = a_1a_6^{-1}.7, \\
6a = 4, & 6b = a_2a_3a_1^{-1}.5, \\
7a = 2, & 7b = a_3a_2^{-1}.8, \\
8a = 3, & 8b = a_6^{-1}a_2^{-1}.6\ .
\end{array}$$

These coset representatives have been obtained by using the eight defined cosets as $1a = 5, 1b = 2, 2b = 3, 3b = 4, 4a^{-1} = 6, 2a^{-1} = 7, 3a^{-1} = 8$, and by using the subgroup tables of the coset enumeration algorithm. Indeed,

$$\begin{array}{ll}
1a^2 = a_1.1 & \Rightarrow 5a = a_1.1, \\
1b^4 = a_2.1 & \Rightarrow 4b = a_2.1, \\
1[b,a] = a_3.1 & \Rightarrow 6b = a_2a_3a_1^{-1}.5, \\
1[a^{-1},b] = a_5.1 & \Rightarrow 4a = a_2a_5^{-1}a_1a_3^{-1}a_2^{-1}.6, \\
1[a^{-2},b^{-1}] = a_4.1 & \Rightarrow 2a = a_4^{-1}a_1.7, \\
1[b^{-1},a] = a_6.1 & \Rightarrow 5b = a_1a_6^{-1}.7, \\
1R_2 = 1 & \Rightarrow 7b = a_3a_2^{-1}.8, \\
2R_2 = 2 & \Rightarrow 8b = a_6^{-1}a_2^{-1}.6, \\
3R_1 = 3 & \Rightarrow 3a = (a_3a_2^{-1})^2.8.
\end{array}$$

where, $R_1 = a^2[a,b]^2$ and $R_2 = (b^2)^{[b,a]}b^2$.

To get a presentation for $K$ we examine all of the equations:

$$iR_1 = i,\ \ (i = 1,2,4,5,6,7,8),$$

and

$$iR_2 = i,\ \ (i = 3,4,5,6,7,8).$$

Many of them give us complicated relations, however, by using the new results in each step we will simplify the relations to get the desired

presentation for $K$. A comprehensive and detailed computation may be given as follows:
The equation $1R_1 = 1$ gives us $r_1 : a_1a_3^{-2} = 1$, and the equation $2R_2 = 2$ yields the trivial relation. The equation $7R_1 = 7$ yields the relation $r_2 : (a_1a_6^{-1})^2a_1^{-1}a_4 = 1$.
The equations $5R_1 = 5$, $4R_1 = 4$, $3R_2 = 3$ and $4R_2 = 4$ yield the relations:

$$r_3 : a_1a_5^{-2} = 1,$$

$$r_4 : (a_6a_2)^2a_5^{-1}a_1a_3^{-1} = 1,$$

$$r_5 : a_2a_3a_2a_3^{-1} = 1,$$

$$r_6 : a_2a_6a_2a_6^{-1} = 1,$$

respectively. The preliminary relation $[(a_2^{-1}a_3)^2a_2^{-1}a_6^{-1}]^2a_3a_1^{-1}a_5 = 1$ is a result of the equation $6R_1 = 6$, and using the relations $r_4$ and $r_5$ this becomes to:

$$r_7 : a_3^2a_6a_3^2a_6^{-1} = 1.$$

The equation $8R_1 = 8$ gives us the relation $a_4^{-1}a_1a_2a_3^{-1}a_4^{-1}a_1a_3a_2^{-1} = 1$ that becomes to the simpler relation

$$r_8 : (a_1a_6^{-1})^2a_2a_3^{-1}a_4^{-1}a_1a_2a_3 = 1$$

by using the relations $r_2$ and $r_5$. The equation $5R_2 = 5$ gives the relation

$$a_4(a_3a_2^{-1})^2a_6^{-1}a_3a_1^{-1}a_5^{-1}a_1a_6^{-1}a_1^{-1} = 1$$

that becomes to $r_9 : a_4a_3^2a_2^{-1}a_6^{-1}a_3a_5^{-1}a_6^{-1}a_1^{-1} = 1$ by using the relations $r_3$ and $r_5$. Finally, the equations $6R_2 = 6, 7R_2 = 7$ and $8R_2 = 8$ will give us the relations $r_{10}$, $r_{11}$ and $r_{12}$, respectively. This completes the proof. $\square$

**Proof of Proposition 1.1.** By considering the presentation of $K$, we prove that $a_4 = 1$ holds in $K$. This will tends to establishing the relation $[a^2, b] = 1$ in $G$ and so, the group $S_5$ has a deficiency zero presentation.

Proving $a_4 = 1$ is by using the relation $r_{10}$ of $K$ and we proceed as follows. First, we use the relations $r_5$ to get $[a_3^2, a_2] = 1$ and then the relations $r_3, r_4$ and $r_7$ to simplify the relations $r_{10}$, i.e;

$$\begin{aligned}
&(a_6^{-1}a_3a_1^{-1})a_5(a_2^{-1}a_3a_2^{-1}a_3)a_2^{-1}(a_6^{-1}a_3a_1^{-1})a_4^{-1}(a_1a_3a_2^{-1}) = 1\\
\Rightarrow\ &(a_6^{-1}a_3a_5a_1^{-1}).a_3^2a_2^{-1}.(a_6^{-1}a_3a_1^{-1}).a_4^{-1}a_1.a_3a_2^{-1} = 1,\ (by\ r_3),\\
\Rightarrow\ &(a_6^{-1}a_6a_2a_6a_2).a_3^2a_2^{-1}.a_6^{-1}a_3a_1^{-1}.(a_4^{-1}a_1).a_3a_2^{-1} = 1,\ (by\ r_4d),
\end{aligned}$$

$$\Rightarrow a_6(a_2a_3^2a_2^{-1}).a_6^{-1}a_3a_1^{-1}.(a_4^{-1}a_1)a_3 = 1,$$
$$(a_6a_3^2a_6^{-1}a_3)a_1^{-1}.(a_4^{-1}a_1)a_3 = 1,\ (by\ [a_3^2, a_2] = 1),$$
$$\Rightarrow a_3^{-1}a_1^{-1}a_4^{-1}a_1a_3 = 1,\ (by\ r_7),$$
$$\Rightarrow a_4 = 1.$$

□

**Lemma 2.2.** *Let $G = \langle a, t, b \mid a^2 = b^2, t^a = t^{-1}[b, a], b^t = aba\rangle$. Then, $G' = \langle A, B, C\rangle$, where $A = a^2, B = t^2$ and $C = [a, b]$. Moreover, $C^2 = 1$ holds in $G'$.*

*Proof.* Obviously, $\frac{G}{G'} \cong Z_2 \times Z_2 \times Z_2$ and letting $K = \langle A, B, C\rangle$ and defining eight cosets as

$$K = 1,\ 1a = 2,\ 1t = 3,\ 1b = 4,\ 5b = 2,\ 3b = 6,\ 3a = 7,\ 6a = 8$$

shows that $|G : H| = 8$. Since $a^2, t^2 \in G'$ then $K \subseteq G'$, so, $G' \cong K$. To prove the equation $C^2 = 1$ we use the Modified Todd-Coxeter coset enumeration algorithm as well as in Lemma 2.1 and get the coset representatives as,

| | | |
|---|---|---|
| $1a = 2,$ | $1b = 4,$ | $1t = 3,$ |
| $2a = A.1,$ | $2b = A.5,$ | $2t = CB^{-1}.7,$ |
| $3a = 7,$ | $3b = 6,$ | $3t = B.1,$ |
| $4a = AC^{-1}.5,$ | $4b = A.1,$ | $4t = BC^{-1}AB^{-1}.6,$ |
| $5a = C.4,$ | $5b = 2,$ | $5t = C^{-1}A^{-6}B^{-1}.8,$ |
| $6a = 8,$ | $6b = BA^3B^{-1}.3,$ | $6t = BAC.4,$ |
| $7a = BA^3B^{-1}.3,$ | $7b = BC^{-1}A^{-2}B^{-1}.8,$ | $7t = BA^3B^{-1}A^{-1}C^{-1}.2,$ |
| $8a = BA^3B^{-1}.6,$ | $8b = BA^5CB^{-1}.7,$ | $8t = BA^5B^{-1}AC^{-2}.5.$ |

To get a presentation for $G'$ we have to consider all of the equations

$$iR_j = i,\ (i = 1, 2, ..., 8,\ j = 1, 2, 3)$$

where, $R_1 = a^2b^{-2}$, $R_2 = t^a[a, b]t$ and $R_3 = bta^{-1}b^{-1}a^{-1}t^{-1}$. To prove the relation $C^2 = 1$ we do not need to calculate all of the relations of $G'$ we just examine the suitable equation $5R_3 = 5$ to get the result. Indeed, by using the above coset representatives we get:

$$5bta^{-1}b^{-1}a^{-1}t^{-1} = 5 \quad \Rightarrow CB^{-1}.BA^{-3}B^{-1}.BA^{-3}B^{-1}.BA^6C = 1$$
$$\Rightarrow C^2 = 1.$$

This completes the proof. □

**Proof of Proposition 1.2.** Using the result of Lemma 2.2 and substituting for $C$ will conclude the validity of the relation $[a,b]^2 = 1$ in the group $G$.

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**A. Arjomandfar**
Department of Mathematics, Science and Research Branch, Islamic Azad University
P.O. Box 14515/ 1775, Tehran, Iran
Email: ab.arj44@gmail.com

**H. Doostie**
Mathematics Department, Tarbiat Moallem University, 49 Mofateh Ave., Tehran 15614, Iran
Email: doostih@saba.tmu.ac.ir