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BANACH MODULE VALUED SEPARATING MAPS AND AUTOMATIC CONTINUITY

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Abstract. For two algebras $\mathcal{A}$ and $\mathcal{B}$, a linear map $T : \mathcal{A} \longrightarrow \mathcal{B}$ is called separating, if $x \cdot y = 0$ implies $Tx \cdot Ty = 0$ for all $x, y \in \mathcal{A}$. The general form and the automatic continuity of separating maps between various Banach algebras have been studied extensively. In this paper, we first extend the notion of separating map for module case and then we give a description of a linear separating map $T : \mathcal{B} \longrightarrow \mathcal{X}$, where $\mathcal{B}$ is a unital commutative semisimple regular Banach algebra satisfying the Ditkin's condition and $\mathcal{X}$ is a left Banach module over a unital commutative Banach algebra. We also show that if $\mathcal{X}$ is hyper semisimple and T is bijective, then T is automatically continuous and T^{-1} is separating as well.

1. Introduction

The notion of separating maps between two arbitrary algebras or between two spaces of functions A and B is a well-known notation. For such structures a map $T : A \longrightarrow B$ is *separating* if $x \cdot y = 0$ implies $Tx \cdot Ty = 0$ for all $x, y \in A$. When A and B are spaces of functions, the separating maps are, indeed, those maps that send every pair of functions in A with disjoint cozero sets to a pair of functions in B with the

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same property. Weighted composition operators are standard examples of linear separating maps between spaces of functions. In fact, it is well known that in certain important cases all linear separating maps are weighted composition operators. For instance, if X and Y are compact Hausdorff spaces, then any linear separating map T from the supremum norm Banach algebra $C(X)$ of all continuous complex-valued functions on X into $C(Y)$ is a weighted composition operator (on a subset of Y) which is automatically continuous, if T is bijective [14]. This result has been extended in [15] for the case, where X is locally compact and $C(X)$ is replaced by the Banach algebra $C_0(X)$ consisting of all continuous complex valued functions on X vanishing at infinity and in [12] for separating maps between certain commutative semisimple regular Banach algebras.

Linear maps $T : L_p(\mu) \longrightarrow L_p(\mu)$ with the property that $f \cdot g = 0$, a.e., implies $Tf \cdot Tg = 0$, a.e., were considered by Banach in [6]. Later on J. Lamperti in [18] and W. Arendt in [5] continued the Banach' research. A separating map between two vector lattices is defined by this property that $|Tf| \wedge |Tg| = 0$ whenever $|f| \wedge |g| = 0$. For some related results in the vector lattices case we refer the reader to [1, 5] and for the separating maps between ceratin subalgebras of continuous functions we refer to [2, 3, 4, 7, 13, 14, 16, 17].

In this paper, introducing the notion of cozero set for elements in a Banach module, we extend the notion of separating maps to Banach module case. Then, we generalize the main results of [12] and give a partial description of a separating map from a unital commutative semisimple Banach algebra $\mathcal{B}$ which is regular and satisfies the Ditkin's condition to a certain left Banach module $\mathcal{X}$. In particular, we show that if $\mathcal{X}$ is hyper semisimple (in the sense of Definition 2.5 in [9]), then every bijective linear separating map $T : \mathcal{B} \longrightarrow \mathcal{X}$ is continuous and T^{-1} is separating as well (Theorem 3.11).

2. **Preliminaries**

Let $\mathcal{A}$ be a unital commutative Banach algebra with maximal ideal space $\sigma(\mathcal{A})$ and let $\mathcal{X}$ be a left Banach $\mathcal{A}$-module which is unital; that is, $1_{\mathcal{A}} \cdot x = x$ for all $x \in \mathcal{X}$, where $1_{\mathcal{A}}$ is the unit element of $\mathcal{A}$. Following [9] we say that a linear functional $\xi \in \mathcal{X}^*$ is a *point multiplier* on $\mathcal{X}$ if there exists a point $\varphi \in \sigma(\mathcal{A})$ such that $\langle \xi, ax \rangle = \varphi(a)\langle \xi, x \rangle$ for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$. Submodules with codimension 1 in $\mathcal{X}$ are called *hyper maximal* submodules of $\mathcal{X}$. By [8, Proposition 4.3] a proper closed submodule P of $\mathcal{X}$ is hyper maximal if and only if there exists a non-trivial point multiplier ξ on $\mathcal{X}$ such that $P = \ker(\xi)$. We note that, in general, a left Banach $\mathcal{A}$-module may have no point multiplier, see for example [9, Example 4.8].

For a unital commutative Banach algebra $\mathcal{A}$ and a unital left Banach $\mathcal{A}$- module $\mathcal{X}$ we denote the set of all non-trivial point multipliers in the closed unit ball of $\mathcal{X}^*$ by $\sigma^h_{\mathcal{A}}(\mathcal{X})$. Since any scalar multiple of a point multiplier is itself a point multiplier, each closed hyper maximal submodule of $\mathcal{X}$ is the kernel of an element in $\sigma^h_{\mathcal{A}}(\mathcal{X})$.

In this section, we assume that $\mathcal{A}$ is a unital commutative Banach algebra and $\mathcal{X}$ is a unital left Banach $\mathcal{A}$-module with $\sigma^h_{\mathcal{A}}(\mathcal{X}) \neq \emptyset$ unless otherwise is specified.

Considering $\mathcal{A}$ as a (left) Banach module over itself we see that $\sigma^h_{\mathcal{A}}(\mathcal{A}) = \{\lambda\varphi : \lambda \in \mathbb{C}, 0 < |\lambda| \leq 1, \varphi \in \sigma(\mathcal{A})\}$ and $\sigma(\mathcal{A}) = \{\xi \in \sigma^h_{\mathcal{A}}(\mathcal{A}) : \langle \xi, 1\rangle = 1\}$. We always endow $\sigma(\mathcal{A})$ with the Gelfand topology and $\sigma^h_{\mathcal{A}}(\mathcal{X})$ with the relative weak-star topology. Since for each $\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X})$ the corresponding $\varphi \in \sigma(\mathcal{A})$ satisfying $\langle \xi, ax\rangle = \varphi(a)\langle \xi, x\rangle$ for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$, is uniquely determined we can define a map $\Lambda_{\mathcal{A}} : \sigma^h_{\mathcal{A}}(\mathcal{X}) \longrightarrow \sigma(\mathcal{A})$ such that for each $\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X})$, $\langle \xi, ax\rangle = \Lambda_{\mathcal{A}}(\xi)(a)\langle \xi, x\rangle$, $a \in \mathcal{A}$, $x \in \mathcal{X}$. It is easy to see that $\sigma^h_{\mathcal{A}}(\mathcal{X}) \cup \{0\}$ is compact and $\Lambda_{\mathcal{A}}$ is continuous.

For an ideal I in $\mathcal{A}$, $h(I)$ denotes the hull of I, i.e., the closed subset $\{\varphi \in \sigma(\mathcal{A}) : \varphi|_I = 0\}$ of $\sigma(\mathcal{A})$ and for a subset E of $\sigma(\mathcal{A})$, we set $k(E) = \bigcap_{\varphi\in E} \ker(\varphi)$. For a submodule $\mathcal{M}$ of $\mathcal{X}$ we define $(\mathcal{M} : \mathcal{X}) = \{a \in \mathcal{A} : a\mathcal{X} \subseteq \mathcal{M}\}$. The Gelfand radical $\mathrm{rad}_{\mathcal{A}}(\mathcal{X})$ of $\mathcal{X}$ is defined by $\mathrm{rad}_A(\mathcal{X}) = \cap_{\xi\in\sigma^h_{\mathcal{A}}(X)} \ker(\xi)$ and we say that $\mathcal{X}$ is *hyper semisimple* if $\mathrm{rad}_{\mathcal{A}}(\mathcal{X}) = \{0\}$. For each subset $\mathcal{M}$ of $\mathcal{X}$ we set $\mathrm{ann}_{\mathcal{A}}(\mathcal{M}) = \{a \in \mathcal{A} : a\mathcal{M} = \{0\}\}$. We note that Proposition 3.2 in [9] shows that the above definition of $\mathrm{rad}_{\mathcal{A}}(\mathcal{X})$ is compatible with the same definition in [9].

Clearly, every (unital) commutative semisimple Banach algebra is a hyper semisimple module over itself. The reader can find some examples of hyper semisimple and non-hyper semisimple left Banach $\mathcal{A}$-modules in [9]. As another example of a hyper semisimple left Banach module, let $\mathcal{X}$ be the sup-norm Banach space $C[0, 1]$ and let $\mathcal{A} = C^n[0, 1]$ be the Banach algebra of all n-times continuously differentiable functions on

$[0,1]$ equipped with the following norm

$$\|f\| = \Sigma_{k=0}^{n} \frac{\|f^{(k)}\|_{[0,1]}}{k!} \quad (f \in \mathcal{A}),$$

where $\|.\|_{[0,1]}$ is the supremum norm on $[0,1]$. It is easy to see that $\mathcal{X}$ is a hyper semisimple (left) Banach $\mathcal{A}$-module under the pointwise multiplication as its module action.

We say that $\mathcal{A}$ satisfies the *Ditkin's condition*, if for each $a \in \mathcal{A}$ and $\varphi \in \sigma(\mathcal{A})$ with $\varphi(a) = 0$, there exists a sequence (a_n) in $\mathcal{A}$ such that for each $n \in \mathbb{N}$, the Gelfand transformation $\widehat{a_n}$ of a_n vanishes on a neighborhood of φ and $\lim \|a_n a - a\| = 0$ as $n \longrightarrow \infty$.

It is easy to verify that $(\mathrm{rad}_{\mathcal{A}}(\mathcal{X}) : \mathcal{X}) = \mathrm{rad}_{\mathcal{A}}(\mathcal{A})$ whenever the natural defined map $\Lambda_{\mathcal{A}}$ is surjective. But, in general, $\Lambda_{\mathcal{A}}$ is not surjective (see Example 2.1). Obviously, for each $\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X})$, $\Lambda_{\mathcal{A}}(\xi) = \Lambda_{\mathcal{A}}(\lambda\xi)$, for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, so that $\Lambda_{\mathcal{A}}$ is not injective.

Remark 2.1. *Let $\Delta_{\mathcal{A}}(\mathcal{X})$ be the set of all closed hyper maximal submodules of $\mathcal{X}$. Then, since for two elements $\xi_1, \xi_2 \in \sigma^h_{\mathcal{A}}(\mathcal{X})$ with $\ker(\xi_1) = \ker(\xi_2)$ we have $\Lambda_{\mathcal{A}}(\xi_1) = \Lambda_{\mathcal{A}}(\xi_2)$, so we can define a natural map $\nu_{\mathcal{A}} : \Delta_{\mathcal{A}}(\mathcal{X}) \longrightarrow \sigma(\mathcal{A})$ such that $\Lambda_{\mathcal{A}}(\xi) = \nu_{\mathcal{A}}(\ker(\xi))$ for each $\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X})$. [9,* [9, Proposition 3.6] *when $\mathcal{X}$ is hyper semisimple the natural defined map $\nu_{\mathcal{A}}$ is surjective if and only if $\mathrm{ann}_{\mathcal{A}}(\mathcal{X}) = \mathrm{rad}_{\mathcal{A}}(\mathcal{A})$. But, the following example shows that the equality $\mathrm{ann}_{\mathcal{A}}(\mathcal{X}) = \mathrm{rad}_{\mathcal{A}}(\mathcal{A})$ does not imply, in general, the surjectivity of $\nu_{\mathcal{A}}$. Indeed, in the proof of [9, Proposition 3.6] the author claims that $\nu_{\mathcal{A}}(\Delta_{\mathcal{A}}(X))$ is hull-kernel closed in $\sigma(\mathcal{A})$ (since $\Delta_{\mathcal{A}}(\mathcal{X})$ is hull-kernel closed, in the sense that he has defined earlier), while his previous result, i.e., Proposition 3.5 in [9] states that for every hull-kernel closed subset S of $\Delta_{\mathcal{A}}(\mathcal{X})$, $\nu_{\mathcal{A}}(S)$ is closed in the relative hull-kernel topology on the image of $\nu_{\mathcal{A}}$.*

Example 2.2. *Consider the algebra $\mathcal{A} =lip([0,1],\alpha)$ of all complex valued functions f on $[0,1]$ satisfying the Lipschitz condition of order α, $0 < \alpha < 1$, such that lim $\frac{|f(x)-f(y)|}{|x-y|^\alpha} = 0$ as $|x-y| \to 0$. Then, $\mathcal{A}$ is a unital commutative semisimple Banach algebra under the following norm*

$$\|f\| = \|f\|_{[0,1]} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \quad (f \in \mathcal{A})$$

and furthermore the maximal ideal space of $\mathcal{A}$ *is equal to* $[0,1]$ *via the evaluation homomorphisms. Now, set* $\mathcal{X} = \{f \in \mathcal{A} : f(0) = 0\}$*. Then,* $\mathcal{X}$ *is a (maximal) ideal of* $\mathcal{A}$*, so that it can be considered as a left Banach* $\mathcal{A}$*-module. For each* $t \in [0,1]$ *let* δ_t *be the evaluation homomorphism at* t *defined on* $\mathcal{A}$*. Then, clearly for each* $t \in (0,1]$*,* $\delta_t|_{\mathcal{X}} \in \sigma^h_{\mathcal{A}}(\mathcal{X})$*. Now, let* $\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X})$*, then by the definition, there exists a point* $t \in [0,1]$ *such that* $\langle \xi, fg\rangle = f(t)\langle \xi, g\rangle$*,* $f \in \mathcal{A}$*,* $g \in \mathcal{X}$*. Since* $\mathcal{A}$ *satisfies the Ditkin's condition [11, Theorem 4.4.30], we can see easily that for each* $g \in \ker(\delta_t)\cap\mathcal{X}$*,* $\langle \xi, g\rangle = 0$*, that is,* $\ker(\delta_t)\cap\mathcal{X} \subseteq \ker(\xi)$*. Hence,* $\xi = \lambda\delta_t|_{\mathcal{X}}$ *and so* $\ker(\xi) = \ker(\delta_t|_{\mathcal{X}})$*. This obviously implies that* $t \neq 0$*. It is simple to observe that* $\mathcal{X}$ *is hyper semisimple and* $\mathrm{rad}_{\mathcal{A}}(\mathcal{A}) = \mathrm{ann}_{\mathcal{A}}(\mathcal{X}) = \{0\}$*, but the above argument shows that the evaluation homomorphism* δ_0 *at* 0 *is not in the range of* $\Lambda_{\mathcal{A}}$*.*

We note that, indeed, the proof of [9, Proposition 3.6] concludes that if $\mathcal{X}$ is hyper semisimple and $\mathrm{ann}_{\mathcal{A}}(\mathcal{X}) = \mathrm{rad}_{\mathcal{A}}(\mathcal{A})$, then $\nu_{\mathcal{A}}$ has a dense range in $\sigma(\mathcal{A})$ with respect to the hull-kernel topology.

3. **Main Results**

In this section, we introduce the notion of cozero set for an element of a left Banach module and the notion of separating maps between two left Banach modules. Then, we give a description of a linear separating map $T : \mathcal{B} \longrightarrow \mathcal{X}$, where $\mathcal{B}$ is a unital commutative semisimple regular Banach algebra satisfying the Ditkin's condition and $\mathcal{X}$ is a unital left Banach module over a unital commutative Banach algebra $\mathcal{A}$. We show that for such T there exists a partition $\{\sigma_0, \sigma_c, \sigma_d\}$ for $\sigma^h_{\mathcal{A}}(\mathcal{X})$ such that σ_0 is closed and σ_d is open and there exist continuous functions $\Phi : \sigma_c \cup \sigma_d \longrightarrow \sigma(\mathcal{A})$ and $\omega : \sigma_c \longrightarrow \mathbb{C}$ such that $\langle \xi, Tb\rangle = \omega(\xi)\Phi(\xi)(b)$ for all $\xi \in \sigma_c$ and $b \in \mathcal{B}$.

As before we assume that $\mathcal{A}$ is a unital commutative Banach algebra and $\mathcal{X}$ is a unital left Banach $\mathcal{A}$-module with $\sigma^h_{\mathcal{A}}(\mathcal{X}) \neq \emptyset$ and with the natural map $\Lambda_{\mathcal{A}} : \sigma^h_{\mathcal{A}}(\mathcal{X}) \longrightarrow \sigma(\mathcal{A})$.

Definition 3.1. For each $x \in \mathcal{X}$ we define the *hyper cozero set* $\mathrm{coz}^h(x)$ of $x \in \mathcal{X}$, by $\mathrm{coz}^h(x) = \{\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X}) : \langle \xi, x\rangle \neq 0\}$ and the *cozero set* $\mathrm{coz}(x)$ of x by $\mathrm{coz}(x) = \Lambda_{\mathcal{A}}(\mathrm{coz}^h(x))$

We note that the notion of the support for an element $x \in \mathcal{X}$ is a well known notion. Indeed, the *support* $\mathrm{supp}(x)$ of $x \in \mathcal{X}$ (which is called as Arveson local spectrum by some authors) is defined as the hull of the closed ideal $\mathrm{ann}_{\mathcal{A}}(x)$ in $\mathcal{A}$, see for example [19, Section 4.12] and [9]. The next lemma shows that if $\mathcal{X}$ is hyper semisimple, then for each $x \in \mathcal{X}$, $\mathrm{supp}(x)$ is, indeed, the the closure of the cozero set $\mathrm{coz}(x)$ with respect to the hull-kernel topology on $\sigma(\mathcal{A})$.

If we consider $\mathcal{A}$ as a left Banach module over itself (with multiplication as its module action), then it is easy to see that for each $a \in \mathcal{A}$, the above defined cozero set $\mathrm{coz}(a)$ is, indeed, the cozero set of $\hat{a}$ as a continuous function on $\sigma(\mathcal{A})$.

Lemma 3.2. *If $\mathcal{A}$ is regular and $\mathcal{X}$ is hyper semisimple, then* $\mathrm{supp}(x) = \overline{\mathrm{coz}(x)}$ *for each $x \in \mathcal{X}$.*

Proof. Assume first that $a \in \mathrm{ann}_{\mathcal{A}}(x)$. Then, since $0 = \langle \xi, ax \rangle = \Lambda_{\mathcal{A}}(\xi)(a)\langle \xi, x\rangle$ for all $\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X})$, it follows that $\Lambda_{\mathcal{A}}(\xi)(a) = 0$ for all $\xi \in \mathrm{coz}^h(x)$. Hence, $\mathrm{ann}_{\mathcal{A}}(x) \subseteq \cap_{\xi \in \mathrm{coz}^h(x)} \ker(\Lambda_{\mathcal{A}}(\xi)) = k(\mathrm{coz}(x))$. On the other hand, if $a \in k(\mathrm{coz}(x))$, then $\langle \xi, ax\rangle = \Lambda_{\mathcal{A}}(\xi)(a)\langle \xi, x\rangle = 0$ for all $\xi \in \mathrm{coz}^h(x)$ which clearly implies that $ax \in \mathrm{rad}_{\mathcal{A}}(\mathcal{X}) = \{0\}$. Therefore, $\mathrm{ann}_{\mathcal{A}}(x) = k(\mathrm{coz}(x))$ and so $h(\mathrm{ann}_{\mathcal{A}}(x)) = hk(\mathrm{coz}(x)) = \overline{\mathrm{coz}(x)}$, since $\mathcal{A}$ is assumed to be regular. □

Proposition 3.3. *(a)* $\mathrm{coz}(x_1+x_2) \subset \mathrm{coz}(x_1) \cup \mathrm{coz}(x_2)$*, for each $x_1, x_2 \in \mathcal{X}$.*

(b) $\mathrm{coz}(ax) \subset \mathrm{coz}(a) \cap \mathrm{coz}(x)$*, for each $a \in \mathcal{A}, x \in \mathcal{X}$.*

(c) If $\mathcal{X}$ is hyper semisimple and $x \in \mathcal{X}$ such that $\mathrm{coz}(x) = \emptyset$*, then $x = 0$.*

Proof. It is straightforward. □

Definition 3.4. Let $\mathcal{Y}$ be a left Banach module over a unital commutative Banach algebra $\mathcal{B}$. We say that a linear map $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is *separating* , if $\mathrm{coz}(x) \cap \mathrm{coz}(x') = \emptyset$ implies $\mathrm{coz}(Tx) \cap \mathrm{coz}(Tx') = \emptyset$ for all $x, x' \in \mathcal{X}$.

A linear operator T on $\mathcal{X}$ is *local*, if $\mathrm{supp}(Tx) \subseteq \mathrm{supp}(x)$ for all $x \in \mathcal{X}$ (see for example [9,10,17]). It is easy to see that if $\mathcal{A}$ is regular and $\mathcal{X}$ is hyper semisimple such that for each $x \in \mathcal{X}$, $\mathrm{coz}(x)$ is an open subset of $\sigma(\mathcal{A})$, then any local operator on $\mathcal{X}$ is separating. In particular, if $\mathcal{B}$

is a non-unital commutative semisimple regular Banach algebra with $\mathcal{A}$ as its unitization and $\mathcal{X}$ is a proper dense ideal in $\mathcal{B}$ which is a Banach algebra with respect to a norm $\|.\|_{\mathcal{X}}$ satisfying $\|bx\|_{\mathcal{X}} \leq \|b\| \cdot \|x\|_{\mathcal{X}}$, for all $b \in \mathcal{B}$ and $x \in \mathcal{X}$, then regarding $\mathcal{X}$ as a Banach $\mathcal{A}$-module (with natural module action) it can be easily verified that $\sigma^h_{\mathcal{A}}(\mathcal{X}) = \{\lambda\varphi|_{\mathcal{X}} : 0 < |\lambda| \leq 1, \varphi \in \sigma(\mathcal{B})\}$ whenever $\mathcal{X}$ is essential as a Banach $\mathcal{B}$-module, i.e., $\mathrm{span}\{bx : b \in \mathcal{B}, x \in \mathcal{X}\}$ is dense in $\mathcal{X}$. Hence, in this case, in the Banach $\mathcal{A}$-module $\mathcal{X}$, $\mathrm{coz}(x) = \{\varphi \in \sigma(\mathcal{B}) : \varphi(x) \neq 0\}$, $x \in \mathcal{X}$, which is an open subset of $\sigma(\mathcal{A})$. Thus, in this case all local operators on $\mathcal{X}$ (as a Banach $\mathcal{A}$-module) are separating. As an example of the ideals with the above mentioned properties we can refer to Segal algebras on a locally compact abelian group G as dense ideals in $L_1(G)$ (see [11, Pages 491-492]).

It is also easy to verify that the class of separating maps on $\mathcal{X}$ includes the class of all simple multipliers on $\mathcal{X}$ in the sense of [9]. By [9, Definition 4.2] a *simple multiplier* on $\mathcal{X}$ is a linear map $T : \mathcal{X} \longrightarrow \mathcal{X}$ satisfying $T(a \cdot x) = a \cdot Tx$ for $a \in \mathcal{A}$ and $x \in \mathcal{X}$, leaving each closed hyper maximal submodule of $\mathcal{X}$ invariant.

In the following we adapt the results of [12] concerning separating maps between commutative semisimple regular Banach algebras to get similar results for separating maps from unital commutative semisimple regular Banach algebras into certain left Banach modules.

In the sequel we assume that $\mathcal{A}$ and $\mathcal{B}$ are unital commutative Banach algebras, where $\mathcal{B}$ is semisimple and regular. We assume, furthermore, that $\mathcal{X}$ is a unital left Banach $\mathcal{A}$-module with $\sigma^h_{\mathcal{A}}(\mathcal{X}) \neq \emptyset$ and $T : \mathcal{B} \longrightarrow \mathcal{X}$ is a linear separating map. Considering $\mathcal{B}$ as a left Banach module over itself by $\mathrm{coz}(b)$ we mean the cozero set of $b \in \mathcal{B}$ which is the same as the cozero set of the continuous function $\hat{b}$ on $\sigma(\mathcal{B})$.

We divide the set $\sigma^h_{\mathcal{A}}(\mathcal{X})$ into three parts as follows

$$\begin{aligned}
\sigma_0 &= \{\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X}) : \xi \circ T = 0\},\\
\sigma_d &= \{\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X}) \backslash \sigma_0 : \xi \circ T \text{ is not continuous }\},\\
\sigma_c &= \{\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X}) \backslash \sigma_0 : \xi \circ T \text{ is continuous }\}.
\end{aligned}$$

For each $\xi \in \sigma_c \cup \sigma_d$ the *support* of the linear functional $\xi \circ T$ on $\mathcal{B}$, denoted by $\mathrm{supp}(\xi \circ T)$, is defined as the set of all $\varphi \in \sigma(\mathcal{B})$ such that for each neighborhood U of φ there exists an element $b \in \mathcal{B}$ with $\mathrm{coz}(b) \subseteq U$ and $\langle \xi, T(b) \rangle \neq 0$.

Lemma 3.5. *For each $\xi \in \sigma_c \cup \sigma_d$,* $\mathrm{supp}(\xi \circ T)$ *is a singleton.*

Proof. Let $\xi \in \sigma_c \cup \sigma_d$. We first show that $\mathrm{supp}(\xi \circ T)$ is non-empty. Assume towards a contradiction that $\mathrm{supp}(\xi \circ T) = \emptyset$, then for each $\varphi \in \sigma(\mathcal{B})$ there exists an open neighborhood U_φ of φ such that for each $b \in \mathcal{B}$ with $\mathrm{coz}(b) \subseteq U_\varphi$ we have $\langle \xi, T(b) \rangle = 0$. Since $\sigma(\mathcal{B})$ is compact, there exist $\varphi_i \in \sigma(B)$, $i = 1, \cdots n$, such that $\sigma(\mathcal{B}) = \bigcup_1^n U_{\varphi_i}$. Set $U_i = U_{\varphi_i}$, $i = 1, \cdots n$. Then, by the regularity of B we can find elements $b_i \in \mathcal{B}$, $i = 1, \cdots n$, such that for each i, $\mathrm{coz}(b_i) \subseteq U_i$ and $\sum_{i=1}^n b_i = 1_{\mathcal{B}}$, where $1_{\mathcal{B}}$ is the unit element of $\mathcal{B}$. The inclusion $\mathrm{coz}(b_i b) \subseteq \mathrm{coz}(b_i) \subseteq U_i$ for each $b \in \mathcal{B}$ and $i = 1, \cdots n$, implies that $\langle \xi, T(b_i b) \rangle = 0$ and so $\langle \xi, T(b) \rangle = \langle \xi, T(\sum_{i=1}^n b_i b) \rangle = 0$ for each $b \in \mathcal{B}$. Hence, $\xi \circ T = 0$ which is impossible, since $\xi \in \sigma_c \cup \sigma_d$.

Assume now that there exist two distinct points φ_1 and φ_2 in $\mathrm{supp}(\xi \circ T)$ and let U_1 and U_2 be disjoint neighborhoods of φ_1 and φ_2, respectively. Then, by the definition of $\mathrm{supp}(\xi \circ T)$, there exist elements $b_i \in \mathcal{B}$, $i = 1, 2$, with $\mathrm{coz}(b_i) \subseteq U_i$ and $\langle \xi, T(b_i) \rangle \neq 0$. In particular, $\mathrm{coz}(b_1) \cap \mathrm{coz}(b_2) = \emptyset$ and hence $\mathrm{coz}(Tb_1) \cap \mathrm{coz}(Tb_2) = \emptyset$. But, since $\langle \xi, T(b_i) \rangle \neq 0$ it follows that $\Lambda_{\mathcal{A}}(\xi) \in \mathrm{coz}(Tb_1) \cap \mathrm{coz}(Tb_2)$ which is a contradiction. □

The above lemma allows us to define a function $\Phi : \sigma_c \cup \sigma_d \longrightarrow \sigma(\mathcal{B})$ in such a way that $\mathrm{supp}(\xi \circ T) = \{\Phi(\xi)\}$. We call Φ the *support map* of T.

Lemma 3.6. *The support map Φ is continuous.*

Proof. Let (ξ_α) be a net in $\sigma_c \cup \sigma_d$ converging to a point $\xi_0 \in \sigma_c \cup \sigma_d$. Then, since $\sigma(\mathcal{B})$ is compact we can assume that $(\Phi(\xi_\alpha))$ converges to a point $\varphi \in \sigma(\mathcal{B})$. Assume now that $\Phi(\xi_0) \neq \varphi$. Let U_0 and U_1 be disjoint neighborhoods of $\Phi(\xi_0)$ and φ, respectively. Then, by the definition of Φ, there exists an element $b_0 \in \mathcal{B}$ with $\mathrm{coz}(b_0) \subseteq U_0$ and $\langle \xi_0, T(b_0) \rangle \neq 0$. Since $(\Phi(\xi_\alpha))$ converges to φ and $\langle \xi_0, Tb_0 \rangle \neq 0$ it follows that $\langle \xi_{\alpha_0}, Tb_0 \rangle \neq 0$ and $\Phi(\xi_{\alpha_0}) \in U_1$ for a sufficiently large α_0. We can now choose an element $b_1 \in \mathcal{B}$ such that $\mathrm{coz}(b_1) \subseteq U_1$ and $\langle \xi_{\alpha_0}, T(b_1) \rangle \neq 0$. Then, $\xi_{\alpha_0} \in \mathrm{coz}^h(Tb_0) \cap \mathrm{coz}^h(Tb_1)$ and so $\Lambda_{\mathcal{A}}(\xi_{\alpha_0}) \in \mathrm{coz}(Tb_0) \cap \mathrm{coz}(Tb_1)$. Since T is assumed to be separating it follows that $\mathrm{coz}(b_0) \cap \mathrm{coz}(b_1) \neq \emptyset$ which is a contradiction. □

Proposition 3.7. *(a) Let* $\xi \in \sigma_c \cup \sigma_d$ *and* $b \in \mathcal{B}$ *such that* $\Phi(\xi) \notin \operatorname{supp}(b)$. *Then,* $\langle \xi, T(b) \rangle = 0$.

(b) If $b, b' \in \mathcal{B}$ *and* $\hat{b} = \hat{b}'$ *on an open subset* U *of* $\sigma(\mathcal{B})$*, then for any* $\xi \in \Phi^{-1}(U)$, $\langle \xi, Tb \rangle = \langle \xi, Tb' \rangle$.

Proof. (a) Let $U = \sigma(\mathcal{B}) \backslash \operatorname{supp}(b)$. Then, $\Phi(\xi) \in U$ and so there exists $b_0 \in \mathcal{B}$ such that $\operatorname{coz}(b_0) \subseteq U$ and $\langle \xi, T(b_0) \rangle \neq 0$, that is, $\xi \in \operatorname{coz}^h(Tb_0)$. Since $\operatorname{coz}(b) \subseteq \sigma(\mathcal{B}) \backslash U$ it follows that $\operatorname{coz}(b) \cap \operatorname{coz}(b_0) = \emptyset$ and so $\operatorname{coz}(Tb) \cap \operatorname{coz}(Tb_0) = \emptyset$. Thus, $\operatorname{coz}^h(Tb) \cap \operatorname{coz}^h(Tb_0) = \emptyset$ and consequently $\xi \notin \operatorname{coz}^h(Tb)$, i.e., $\langle \xi, T(b) \rangle = 0$ as desired.

(b) Let $b \in \mathcal{B}$ such that $\hat{b} = 0$ on U and let $\xi \in \Phi^{-1}(U)$. Then, for any $\varphi \in \sigma(\mathcal{B}) \backslash U$ there exists an open neighborhood U_φ of φ, such that $\langle \xi, Tc \rangle = 0$, for each $c \in \mathcal{B}$ with $\operatorname{coz}(c) \subseteq U_\varphi$. Since $\sigma(\mathcal{B}) = \bigcup_{\varphi \in \sigma(\mathcal{B}) \backslash U} U_\varphi \cup U$ we can find $\varphi_i \in \sigma(\mathcal{B}) \backslash U$, $i = 1, \cdots, n$, such that $\sigma(\mathcal{B}) = \bigcup_{i=1}^n U_{\varphi_i} \cup U$. Set $U_i = U_{\varphi_i}$ for $i = 1, \cdots, n$ and $U_{n+1} = U$. Then, for each $i = 1, \cdots, n+1$, there exists $b_i \in \mathcal{B}$, such that $\operatorname{coz}(b_i) \subseteq U_i$, and $\sum_{i=1}^{n+1} b_i = 1_{\mathcal{B}}$. Since $\operatorname{coz}(b_{n+1} b) \subseteq \operatorname{coz}(b_{n+1}) \cap \operatorname{coz}(b) = \emptyset$, we have $b_{n+1} b = 0$ and therefore $\langle \xi, Tb \rangle = \langle \xi, T(\sum_{i=1}^{n+1} b_i b) \rangle =$
$\sum_{i=1}^{n} \langle \xi, T(b_i b) \rangle = 0$ as $\operatorname{coz}(b_i b) \subseteq U_i$. □

Theorem 3.8. *Let* $\mathcal{B}$ *satisfy the Ditkin's condition. Then, there exists a continuous function* $\omega : \sigma_c \longrightarrow \mathbb{C}$ *such that* $\langle \xi, Tb \rangle = \omega(\xi) \Phi(\xi)(b)$ *holds for all* $\xi \in \sigma_c$ *and* $b \in \mathcal{B}$.

Proof. Let $\xi \in \sigma_c$ and define

$$J_\xi = \{ b \in \mathcal{B} : \Phi(\xi) \notin \operatorname{supp}(b) \}$$

and

$$K_\xi = \{ b \in \mathcal{B} : \Phi(\xi)(b) = 0 \}.$$

Since $\mathcal{B}$ satisfies the Ditkin's condition it is easy to see that J_ξ is dense in K_ξ. Let $b \in J_\xi$, then by Proposition 3.7(a), $\langle \xi, T(b) \rangle = 0$, that is, $b \in \ker(\xi \circ T)$. Therefore, $K_\xi = \overline{J_\xi} \subseteq \ker(\xi \circ T)$ and so there exists a non-zero scalar $\omega(\xi) \in \mathbb{C}$ such that $\xi \circ T = \omega(\xi) \cdot \Phi(\xi)$. This gives a non vanishing function $\omega : \sigma_c \longrightarrow \mathbb{C}$ such that $\xi \circ T = \omega(\xi) \cdot \Phi(\xi)$ for all $\xi \in \sigma_c$. Since $\langle \xi, T(1_{\mathcal{B}}) \rangle = \omega(\xi)$, it is clear that ω is continuous on σ_c. □

In the sequel we assume further that $\mathcal{B}$ satisfies the Ditkin's condition.

Lemma 3.9. *If (ξ_α) is a net in $\sigma_c \cup \sigma_d$ such that $\Phi(\xi_\alpha) \neq \Phi(\xi_\beta)$ for each $\alpha \neq \beta$, then* $\limsup \|\xi_\alpha \circ T\| < \infty$. *In particular, $\Phi(\sigma_d)$ is finite and consists of non-isolated points.*

Proof. Assume on the contrary that $\limsup \|\xi_\alpha \circ T\| = \infty$. Then, we can choose a sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $\alpha_n < \alpha_{n+1}$ and $\|\xi_{\alpha_n} \circ T\| \geq n^3 + \|T1_{\mathcal{B}}\|$ for each $n \in \mathbb{N}$. Set $\varphi_n = \Phi(\xi_{\alpha_n})$. We claim that there exist a sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ and a sequence (U_n) of disjoint open subsets of $\sigma(\mathcal{B})$ such that for each $n \in \mathbb{N}$, $\varphi_n \in U_n$, $\|b_n\| \leq n^{-3}$, $\mathrm{coz}(b_n) \subseteq U_n$ and $|\langle \xi_{\alpha_n}, Tb_n\rangle| \geq n^3$. We first note that since for each $n \in \mathbb{N}$, $\|\xi_{\alpha_n} \circ T\| > n^3 + \|T1_{\mathcal{B}}\|$ we can find a sequence (t_n) in $\mathcal{B}$ such that $\|t_n\| \leq 1$ and $|\langle \xi_{\alpha_n}, Tt_n\rangle| \geq n^3 + \|T1_B\|$. Set $z_n = t_n - \varphi_n(t_n) \cdot 1_B$. Then, $\varphi_n(z_n) = 0$ and furthermore $|\langle \xi_{\alpha_n}, Tz_n\rangle| \geq n^3$, since $|\langle \xi_{\alpha_n}, \varphi_n(t_n) \cdot T1_B\rangle| \leq \|\xi_{\alpha_n}\| \cdot \|t_n\| \cdot \|T1_B\| \leq \|T1_B\|$. Now, since $\mathcal{B}$ satisfies the Ditkin's condition, for each n we can find an open neighborhood U_n of φ_n and an element $e_n \in \mathcal{B}$ such that $\widehat{e_n} = 0$ on U_n and $\|z_n - z_n e_n\| \leq n^{-3}$. We may assume that $U_n \cap U_m = \emptyset$ for $n \neq m$. Set $d_n = z_n - z_n e_n$. Then, since $\widehat{d_n} = \widehat{z_n}$ on U_n and $\xi_{\alpha_n} \in \Phi^{-1}(U_n)$, Proposition 3.7(b) implies that $|\langle \xi_{\alpha_n}, Td_n\rangle| = |\langle \xi_{\alpha_n}, Tz_n\rangle| \geq n^3$. For each $n \in \mathbb{N}$, let V_n be an open neighborhood of φ_n such that $\overline{V_n} \subseteq U_n$. Then, by the regularity of $\mathcal{B}$, we can find a sequence (c_n) in $\mathcal{B}$ such that for each $n \in \mathbb{N}$, $\widehat{c_n} = 1$ on V_n and $\widehat{c_n} = 0$ on $\sigma(\mathcal{B}) \backslash U_n$. Now, since $\varphi_n(c_n d_n) = 0$ and $\mathcal{B}$ satisfies the Ditkin's condition we can find a sequence (h_n) in $\mathcal{B}$ such that each $\widehat{h_n}$ vanishes on a neighborhood of φ_n and $\|c_n d_n - h_n c_n d_n\| \leq n^{-3}$. Now, set $b_n = c_n d_n - h_n c_n d_n$. Then, $\|b_n\| \leq n^{-3}$, $\mathrm{coz}(b_n) \subseteq \mathrm{coz}(c_n) \subseteq U_n$ and using Proposition 3.7(b) once again we can easily see that $|\langle \xi_{\alpha_n}, Tb_n\rangle| = |\langle \xi_{\alpha_n}, T(c_n d_n)\rangle| = |\langle \xi_{\alpha_n}, Td_n\rangle| \geq n^3$ which establishes the claim.

Put now $b = \Sigma_{n=1}^{\infty} b_n$, then b is an element of B such that for each $n \in \mathbb{N}$, $\hat{b} = \widehat{b_n}$ on U_n and consequently by Proposition 3.7(b)

$$|\langle \xi_{\alpha_n}, T(b)\rangle| = |\langle \xi_{\alpha_n}, T(b_n)\rangle| \geq n^3.$$

Hence, $\|Tb\| \geq |\langle \xi_{\alpha_n}, T(b)\rangle| \geq n^3$ for each $n \in \mathbb{N}$, which is impossible. Therefore, $\limsup \|\xi_\alpha \circ T\| < \infty$

Assume now that $\Phi(\sigma_d)$ is not finite and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in σ_d such that $\Phi(\xi_n) \neq \Phi(\xi_m)$ for all $n \neq m$. Since for each n, $\xi_n \circ T$ is non-continuous, it follows that $\|\xi_n \circ T\| = \infty$ and so $\limsup \|\xi_n \circ T\| = \infty$ which is a contradiction by the first part.

We shall show that each point in $\Phi(\sigma_d)$ is a non-singular point. Let $\xi \in \sigma_d$ be such that $\Phi(\xi)$ is a singular point. Then, obviously for each $b \in \mathcal{B}$ with $\Phi(\xi)(b) = 0$ we have $\Phi(\xi) \notin \mathrm{supp}\,(b)$ and so by Proposition 3.7(a), $\xi \circ T(b) = 0$. Hence, $\ker(\Phi(\xi)) \subseteq \ker(\xi \circ T)$ and therefore $\xi \circ T = \lambda\Phi(\xi)$ for some non-zero scalar λ, which is impossible since $\xi \circ T$ is not continuous. □

Proposition 3.10. *The subset σ_0 is closed and σ_d is open in $\sigma^h_{\mathcal{A}}(\mathcal{X})$.*

Proof. Since $\sigma_0 = \cap_{b\in\mathcal{B}}\{\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X}) : \langle \xi, Tb\rangle = 0\}$, it is clear that σ_0 is closed. On the other hand $\sup\{|\langle \xi, Tb\rangle| : \xi \in \overline{\sigma_0 \cup \sigma_c}\} = \sup\{|\langle \xi, Tb\rangle| : \xi \in \sigma_0 \cup \sigma_c\} \le \|\omega\| \cdot \|b\|$, where $\|\omega\| = \sup_{\xi\in\sigma_c} |\omega(\xi)| = \sup_{\xi\in\sigma_c} |\langle \xi, T1_{\mathcal{B}}\rangle|$ which is clearly bounded by $\|T1_{\mathcal{B}}\|$. This implies easily that for each $\xi \in \overline{\sigma_0 \cup \sigma_c}$, either $\xi \circ T = 0$ or $\xi \circ T$ is continuous, that is, $\overline{\sigma_0 \cup \sigma_c} = \sigma_0 \cup \sigma_c$. Hence, $\sigma_0 \cup \sigma_c$ is closed and therefore σ_d is open in $\sigma^h_{\mathcal{A}}(\mathcal{X})$. □

If $\mathcal{X}$ is hyper semisimple, then in an analogous way to the Gelfand representation for the Banach algebras, we can identify $\mathcal{X}$ with a subspace of $C(\sigma^h_{\mathcal{A}}(\mathcal{X}) \cup \{0\})$. This identification carries an $\mathcal{A}$-module structure on this subspace. In this case we see that, through this identification, Theorem 3.8 gives a representation of the separating map T on the subset σ_c of $\sigma^h_{\mathcal{A}}(\mathcal{X})$.

In the following we show that under the hyper semisimplicity assumption on $\mathcal{X}$ every bijective separating map $T : \mathcal{B} \longrightarrow \mathcal{X}$ can be considered as a weighted composition operator on the whole $\sigma^h_{\mathcal{A}}(\mathcal{X})$ which is automatically continuous and moreover, its inverse is separating as well.

Theorem 3.11. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{X}$ be as in Theorem 3.8. If $\mathcal{X}$ is hyper semisimple and $T : \mathcal{B} \longrightarrow \mathcal{X}$ is a bijective separating map, then $\sigma_c = \sigma^h_{\mathcal{A}}(\mathcal{X})$. In particular, T is continuous and T^{-1} is separating.*

Proof. Let $\{\sigma_0, \sigma_c, \sigma_d\}$ be the partition given earlier for $\sigma^h_{\mathcal{A}}(\mathcal{X})$. Since T is surjective, it is easily seen that $\sigma_0 = \emptyset$, i.e., $\sigma^h_{\mathcal{A}}(\mathcal{X}) = \sigma_c \cup \sigma_d$. Let ω and Φ be defined as in Theorem 3.8. We prove the theorem via the following steps:

Step I. $\overline{\Phi(\sigma^h_{\mathcal{A}}(\mathcal{X}))} = \overline{\Phi(\sigma_c)} = \sigma(\mathcal{B})$.

We first show that $\overline{\Phi(\sigma^h_{\mathcal{A}}(\mathcal{X}))} = \sigma(\mathcal{B})$. Assume on the contrary that there exist a point $\varphi \in \sigma(\mathcal{B})$ and a neighbourhood U of φ such that

$U \cap \Phi(\sigma^h_{\mathcal{A}}(\mathcal{X})) = \emptyset$. Then, using the regularity of $\mathcal{B}$, we can find a non-zero element $b \in \mathcal{B}$ whose support is contained in U. Thus, for each $\xi \in \sigma^h_{\mathcal{A}}(\mathcal{X})$, $\langle \xi, T(b) \rangle = 0$, by Proposition 3.7(a). Therefore, $T(b) \in \mathrm{rad}_{\mathcal{A}}(\mathcal{X}) = \{0\}$ and hence $b = 0$, which is a contradiction. Now, it suffices to show that $\Phi(\sigma_d) \subseteq \overline{\Phi(\sigma_c)}$. As above assume on the contrary that there exist a point $\xi \in \sigma_d$ and an open neighbourhood U of $\Phi(\xi)$ such that $U \cap \Phi(\sigma_c) = \emptyset$. Since by Lemma 3.9, $\Phi(\sigma_d)$ is finite we can assume that $U \cap \Phi(\sigma_d) = \{\Phi(\xi)\}$. On the other hand, since $\Phi(\sigma_d)$ consists of non-isolated points of $\sigma(\mathcal{B})$ there exists a point $\varphi \in U$ distinct from $\Phi(\xi)$. Then, $U \backslash \{\Phi(\xi)\}$ is a neighbourhood of φ in $\sigma(\mathcal{B})$ with $U \backslash \{\Phi(\xi)\} \cap \Phi(\sigma^h_{\mathcal{A}}(\mathcal{X})) = \emptyset$, that is, $\varphi \notin \overline{\Phi(\sigma^h_{\mathcal{A}}(\mathcal{X}))}$ which is a contradiction. Therefore, $\Phi(\sigma_d) \subseteq \overline{\Phi(\sigma_c)}$ as desired.

Step II. T is continuous.

Let (b_n) be a sequence in $\mathcal{B}$ such that $b_n \to 0$ and $T(b_n) \to x$ for some $x \in \mathcal{X}$. Let $b \in \mathcal{B}$ such that $Tb = x$. Then, for any $\xi \in \sigma_c$ we have $\xi \circ T(b_n) \to 0$ and $\xi \circ T(b_n) \to \xi \circ T(b)$. Thus, $0 = \langle \xi, Tb \rangle = \omega(\xi) \cdot \Phi(\xi)(b)$ for all $\xi \in \sigma_c$. Therefore, $\hat{b} = 0$ on $\Phi(\sigma_c)$ which implies that $b = 0$, since $\hat{b}$ is continuous and $\Phi(\sigma_c)$ is dense in $\sigma(\mathcal{B})$. This shows that T is continuous by the closed graph theorem.

We note that the above argument shows $\sigma_d = \emptyset$ and therefore $\sigma^h_{\mathcal{A}}(\mathcal{X}) = \sigma_c$.

Step III. T^{-1} is separating.

Let $b_1, b_2 \in \mathcal{B}$ such that $\mathrm{coz}(Tb_1) \cap \mathrm{coz}(Tb_2) = \emptyset$. Then, $\widehat{b_1} \cdot \widehat{b_2} = 0$ on $\Phi(\sigma_c) = \Phi(\sigma^h_{\mathcal{A}}(\mathcal{X}))$ which implies, as the above argument, that $b_1 b_2 = 0$. Thus, T^{-1} is separating. □

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References

[1] Y.A. Abramovich, Multiplicative representation of disjointness preserving operators, . *Nederl. Akad. Wetensch. Indag. Math.* **45** (1983) 265-279.

[2] J. Araujo, E. Beckenstein, and L. Narici, Biseparating maps and homeomorphic real-compactifications, *J. Math. Anal. Appl.* **192** (1995) 258-265.

[3] J. Araujo and K. Jarosz, Separating maps on spaces of continuous functions, *Contemp. Math.* **232** (1999) 33-37.

[4] J. Araujo, K. Jarosz, Automatic continuity of biseparating maps, *Studia Math.* **155** (2003) 231-239.

[5] W. Arendt, Spectral properties of Lamperti operators, *Indiana Univ. Math. J.* **32** (1983) 199-215.

[6] S. Banach, *Théorie Des Óperations Linéaires*, Chelsea Publishing Co., New York, 1955.

[7] E. Beckenstein, L. Narici and A. R. Todd, Automatic continuity of linear maps on spaces of continuous functions, *Manuscripta Math.* **62** (1988) 257-275.

[8] J. Bračič, Representations and derivations of modules, *Irish Math. Soc. Bull.* **47** (2001) 27-39.

[9] J. Bračič, Simple multipliers on Banach modules, *Glasg. Math. J.* **45** (2003) 309-322.

[10] J. Bračič, Local operators on Banach modules, *Math. Proc. R. Ir. Acad.* **104 A** (2004) 239-248.

[11] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs, New Series, 24, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 2000.

[12] J. J. Font, Automatic continuity of certain isomorphisms between regular Banach function algebras, *Glasg. Math. J.* **39** (1997) 333-343.

[13] J. J. Font and S. Hernández, On separating maps between locally compact spaces, *Arch. Math. (Basel)* **63** (1994) 158-165.

[14] K. Jarosz, Automatic continuity of separating linear isomorphisms, *Canad. Math. Bull.* **33** (1990) 139-144.

[15] J.-S. Jeang and N.-C. Wong, Weighted composition operators of $C_0(X)$'s, *J. Math. Anal. Appl.* **201** (1996) 981-993.

[16] J.-S. Jeang and N.-C. Wong, Disjointness preserving Fredholm linear operators of $C_0(X)$, *J. Operator Theory* **49** (2003) 61-75.

[17] R. Kantrowitz and M. M. Neumann, Disjointness preserving and local operators on algebras of differentiable functions, *Glasg. Math. J.* **43** (2001) 295-309.

[18] J. Lamperti, On the isometries of certain function-spaces, *Pacific J. Math.* **8** (1958) 459-466.

[19] K.B. Laursen and M.M. Neumann, *An Introduction to Local Spectral Theory*,London Mathematical Society Monographs, New Series, 20, The Clarendon Press, Oxford University Press, New York, 2000.

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