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G-FRAMES AND HILBERT-SCHMIDT OPERATORS

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ABSTRACT. In this paper we introduce and study Besselian g-frames. We show that the kernel of associated synthesis operator for a Besselian g-frame is finite dimensional. We also introduce α-dual of a g-frame and we get some results when we use the Hilbert-Schmidt norm for the members of a g-frame in a finite dimensional Hilbert space.

1. Introduction

Frames for Hilbert spaces introduced by Duffin and schaeffer in 1952 [4]. A sequence $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a *frame* for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \tag{1.1}$$

for all $f \in \mathcal{H}$. The numbers A, B are called *frame bounds*. Various generalizations of frames in Hilbert spaces have been proposed and studied recently. For example, frame of subspaces [2], Pseudo frames for subspaces[7], Bounded quasi-projectors [5], oblique frames [3] etc. Wenchang Sun in his paper [12] introduced the concept of g-frames which include all mentioned generalizations. Members of ordinary frames are

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vectors of a Hilbert space, while members of g-frames are bounded operators. Besselian frames and near-Riesz bases in Hilbert spaces introduced by Holub [6]. Also, Besselian frame of subspaces introduced and discussed in [9]. The authors of this paper in [1], introduced the concept of near g-Riesz bases and they showed that a near g-Riesz basis is a Besselian g-frame.
In this paper by using the concept of Besselian frame and g-frame we define Besselian g-frame and investigate some of their properties. In section 2, we give the basic definitions and known results needed. In section 3, we investigate some properties of Besselian g-frames. In particular, we show that under some conditions, the kernel of associated synthesis operator for a Besselian g-frame is finite dimensional. In section 4, we introduce α-dual of a g-frame and we get some results when we use the Hilbert-Schmidt norm for the members of a g-frame in a finite dimensional Hilbert space.

2. **Preliminaries**

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space and $\{\mathcal{H}_i\}_{i\in I}$ is a sequence of separable Hilbert spaces, where I is a subset of $\mathbb{N}$.

Definition 2.1. *The sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g-Bessel sequence if there exists $B > 0$ such that*

$$\sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2 \tag{2.1}$$

for all $f \in \mathcal{H}$.

Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be given. Let us define

$$\left(\sum_{i\in I} \oplus\mathcal{H}_i\right)_{l_2} = \left\{\{g_i\} : g_i \in \mathcal{H}_i, \sum_{i\in I} \|g_i\|^2 < \infty\right\}$$

with the inner product given by $\langle\{f_i\}, \{g_i\}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$. It is clear that $\left(\sum_{i\in I} \oplus\mathcal{H}_i\right)_{l_2}$ is a Hilbert space with respect to the pointwise operations. It is proved in [10] , if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Bessel sequence for $\mathcal{H}$, then the operator

$$T : \left(\sum_{i\in I} \oplus\mathcal{H}_i\right)_{l_2} \to H$$

defined by

$$T(\{g_i\}) = \sum_{i\in I} \Lambda_i^*(g_i) \tag{2.2}$$

is well defined, bounded and $T^*f = \{\Lambda_i f\}_{i\in I}$.

Definition 2.2. *We call a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist two positive constants A and B such that*

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2 \tag{2.3}$$

for all $f \in \mathcal{H}$. We call A and B the lower and upper $g-$frame bounds, respectively.
We call $\{\Lambda_i\}_{i\in I}$ a tight g-frame if $A = B$ and Parseval g-frame if $A = B = 1$.

The sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ if and only if the operator T defined by (2.2) is bounded and onto (see [10]). The operators T and T^* are called the synthesis and analysis operators, respectively.

Proposition 2.3. [12] *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$. The operator*

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i\in I} \Lambda_i^* \Lambda_i f$$

is a positive, bounded and invertible operator.

Proposition 2.3 implies that every $f \in \mathcal{H}$ can be represented as

$$f = SS^{-1}f = \sum_{i\in I} \Lambda_i^*\Lambda_i S^{-1}f, \qquad f = S^{-1}Sf = \sum_{i\in I} S^{-1}\Lambda_i^*\Lambda_i f. \tag{2.4}$$

The operator S is called the g-frame operator of $\{\Lambda_i\}_{i\in I}$.
It is easy to check that if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Bessel sequence, then S is well defined and $S = TT^*$. We end this section by definition of g-Riesz basis.

Definition 2.4. *A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist two positive constants*

A and B such that for any finite subset $F \subseteq I$ and $g_i \in \mathcal{H}_i,\ i \in F$,

$$A \sum_{i\in F} \|g_i\|^2 \leq \| \sum_{i\in F} \Lambda_i^* g_i \|^2 \leq B \sum_{i\in F} \|g_i\|^2, \tag{2.5}$$

and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is g-complete, i.e., $\{f : \Lambda_i f = 0, i \in I\} = 0$.

It is proved in [10], that $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is g-complete if and only if $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in I} = \mathcal{H}$.

3. Besselian g-frames

As usual, we denote by $l^2(I)$ the Hilbert space of all square-summable sequences of scalars $\{c_i\}_{i\in I}$. If $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$, then $\sum_{i\in I} c_i f_i$ converges if $\{c_i\}_{i\in I} \in l^2(I)$. But the converse is not true in general (see [6]). We say that a frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$ is

- Besselian if, whenever $\sum_{i\in I} c_i f_i$ converges, then $\{c_i\}_{i\in I} \in l^2(I)$;
- a near-Riesz basis, if there is a finite set σ for which $\{f_i\}_{i\in I\setminus\sigma}$ is a Riesz basis for H.

We recall the following characterization of frames which are near-Riesz bases.

Theorem 3.1. [6] *If $\{f_i\}_{i\in I}$ is a frame in $\mathcal{H}$, the following are equivalent:*

(i) *$\{f_i\}_{i\in I}$ is a near-Riesz basis for $\mathcal{H}$;*
(ii) *$\{f_i\}_{i\in I}$ is Besselian;*
(iii) *$\sum_{i\in I} c_i f_i$ converges if and only if $\{c_i\}_{i\in I} \in l^2(I)$.*

Besselian frame of subspaces introduced and discussed in [9]. Here we introduce the concept of Besselian g-frames.

Definition 3.2. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. We call Λ a Besselian g-frame if, whenever $\sum_{i\in I} \Lambda_i^* g_i$ converges, then*

$$\{g_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus \mathcal{H}_i \right)_{l_2}.$$

Let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$ and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be given. Then $\{\Lambda_i^* e_{ij}\}_{i\in I, j\in J_i}$ is a frame (res. Riesz basis) for $\mathcal{H}$ if and only if $\{\Lambda_i\}_{i\in I}$ is a g-frame (res. g-Riesz basis) for $\mathcal{H}$ (see [12]).

Theorem 3.3. *Suppose that* $\dim \mathcal{H}_i < \infty$ *for each* $i \in I$. *Let* $\Lambda = \{\Lambda_i\}_{i\in I}$ *be a Besselian g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in I}$ *and* T *be the associated synthesis operator for* Λ. *Then* KerT *is finite dimensional.*

Proof. Let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$. Then $\{\Lambda_i^* e_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$. Suppose that $\sum_{i\in I}\sum_{j\in J_i} c_{ij}\Lambda_i^* e_{ij}$ converges. Since Λ is a Besselian g-frame, we get $\left\{\sum_{j\in J_i} c_{ij}e_{ij}\right\}_{i\in I} \in \left(\sum_{i\in I}\oplus\mathcal{H}_i\right)_{l_2}$. So

$$\sum_{i\in I}\sum_{j\in J_i}|c_{ij}|^2 = \sum_{i\in I}\Big\|\sum_{j\in J_i} c_{ij}e_{ij}\Big\|^2 < \infty.$$

Hence $\{\Lambda_i^* e_{ij}\}_{i\in I, j\in J_i}$ is Besselian. Let Q be the associated synthesis operator for $\{\Lambda_i^* e_{ij}\}_{i\in I, j\in J_i}$, then $\dim \mathrm{KerQ} < \infty$ [6, *Theorem 2.3*]. Let us define $E_{ij} \in \left(\sum_{i\in I}\oplus\mathcal{H}_i\right)_{l_2}$ by

$$(E_{ij})_k = \begin{cases} e_{ij}, & i = k \\ 0, & i \neq k \end{cases} \tag{3.1}$$

for all $i, j, k \in I$. It is easy to check that $\{E_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\left(\sum_{i\in I}\oplus\mathcal{H}_i\right)_{l_2}$ (see[10]). By the definition of Q and T, it is clear that

$$Q(\{c_{ij}\}_{i\in I, j\in J_i}) = \sum_{i\in I}\sum_{j\in J_i} c_{ij}\Lambda_i^* e_{ij} = T\Big(\sum_{i\in I}\sum_{j\in J_i} c_{ij}E_{ij}\Big).$$

Now we consider the mapping

$$\varphi : \mathrm{KerQ} \to \mathrm{KerT}, \quad \varphi(\{c_{ij}\}_{i\in I, j\in J_i}) = \sum_{i\in I}\sum_{j\in J_i} c_{ij}E_{ij}.$$

It is obvious that φ is linear and injective. We claim that φ is surjective. Let $\{g_i\}_{i\in I} \in \mathrm{KerT}$. Then $g_i \in \mathcal{H}_i$ and $g_i = \sum_{j\in J_i}\lambda_{ij}e_{ij}$ for each $i \in I$. Since $\|g_i\|^2 = \sum_{j\in J_i}|\lambda_{ij}|^2$, we have $\sum_{i\in I}\sum_{j\in J_i}|\lambda_{ij}|^2 = \sum_{i\in I}\|g_i\|^2 <$

∞. Therefore $\{\lambda_{ij}\}_{i\in I, j\in J_i} \in l^2$ and

$$Q\big(\{\lambda_{ij}\}_{i\in I, j\in J_i}\big) = T\Big(\sum_{i\in I}\sum_{j\in J_i} \lambda_{ij} E_{ij}\Big) = T\big(\{g_i\}_{i\in I}\big) = 0,$$

$$\varphi\big(\{\lambda_{ij}\}_{i\in I, j\in J_i}\big) = \sum_{i\in I}\sum_{j\in J_i} \lambda_{ij} E_{ij} = \{g_i\}_{i\in I}.$$

Hence dim KerT = dim KerQ $< \infty$ and the proof is completed. □

In the next theorem we get characterizations of generalized frames, Riesz bases, and frames.

Theorem 3.4. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be given and let $\{g_{ij}\}_{j\in K_i}$ be a frame (res. Riesz basis) for $\mathcal{H}_i$ with bounds A_i, B_i for each $i \in I$ such that $0 < A = \inf_{i\in I} A_i$ and $B = \sup_{i\in I} B_i < \infty$. Then $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in K_i}$ is a frame (res. Riesz basis) for $\mathcal{H}$ if and only if $\{\Lambda_i\}_{i\in I}$ is a g-frame (res. g-Riesz basis) for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$.*

Proof. (1) Let $f \in \mathcal{H}$ and $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in K_i}$ be a frame for $\mathcal{H}$ with bounds $0 < C \leq D$. Then

$$\begin{aligned} A\sum_{i\in I} \|\Lambda_i f\|^2 \leq \sum_{i\in I} A_i \|\Lambda_i f\|^2 \leq & \sum_{i\in I}\sum_{j\in K_i} |\langle g_{ij}, \Lambda_i f\rangle|^2 \\ = & \sum_{i\in I}\sum_{j\in K_i} |\langle \Lambda_i^* g_{ij}, f\rangle|^2 \leq D\|f\|^2 \end{aligned}$$

and

$$\begin{aligned} C\|f\|^2 \leq \sum_{i\in I}\sum_{j\in K_i} |\langle \Lambda_i^* g_{ij}, f\rangle|^2 = & \sum_{i\in I}\sum_{j\in K_i} |\langle g_{ij}, \Lambda_i f\rangle|^2 \\ \leq & \sum_{i\in I} B_i \|\Lambda_i f\|^2 \leq B\sum_{i\in I} \|\Lambda_i f\|^2. \end{aligned}$$

Hence

$$\frac{C}{B}\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq \frac{D}{A}\|f\|^2.$$

Next we assume that $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with bounds $0 < C_0 \leq D_0$. By the same argument we have

$$AC_0\|f\|^2 \leq \sum_{i\in I}\sum_{j\in K_i} |\langle \Lambda_i^* g_{ij}, f\rangle|^2 \leq BD_0\|f\|^2$$

for all $f \in \mathcal{H}$.

(2) Suppose that $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in K_i}$ is a Riesz basis for $\mathcal{H}$ with Riesz basis bounds $0 < C \leq D$. Let $F \subseteq I$ be a finite subset of I and $g_i \in \mathcal{H}_i$ for each $i \in F$. Then we have $g_i = \sum_{j\in J_i} \lambda_{ij} g_{ij}$ where $\{\lambda_{ij}\} \in l^2(K_i)$. Since $\{g_{ij}\}_{j\in K_i}$ is a Riesz basis for $\mathcal{H}_i$, we have

$$\begin{aligned} A\sum_{j\in K_i} |\lambda_{ij}|^2 \leq A_i \sum_{j\in K_i} |\lambda_{ij}|^2 \leq \Big\| \sum_{j\in K_i} \lambda_{ij} g_{ij} \Big\|^2 = \|g_i\|^2, \\ \Big\| \sum_{j\in K_i} \lambda_{ij} g_{ij} \Big\|^2 \leq B_i \sum_{j\in K_i} |\lambda_{ij}|^2 \leq B \sum_{j\in K_i} |\lambda_{ij}|^2. \end{aligned} \tag{3.2}$$

Therefore

$$\frac{C}{B}\sum_{i\in F} \|g_i\|^2 \leq C \sum_{i\in F}\sum_{j\in K_i} |\lambda_{ij}|^2 \leq \Big\| \sum_{i\in F}\sum_{j\in K_i} \lambda_{ij}\Lambda_i^* g_{ij} \Big\|^2 = \|\sum_{i\in F} \Lambda_i^* g_i\|^2,$$

$$\Big\| \sum_{i\in F}\sum_{j\in K_i} \lambda_{ij}\Lambda_i^* g_{ij} \Big\|^2 \leq D \sum_{i\in F}\sum_{j\in K_i} |\lambda_{ij}|^2 \leq \frac{D}{A}\sum_{i\in F} \|g_i\|^2.$$

Since $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in K_i}$ is a Riesz basis for $\mathcal{H}$, we have $\overline{\text{span}}\{\Lambda_i^* g_{ij}\}_{i\in I, j\in K_i}$ $= \mathcal{H}$ and so $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in I} = \mathcal{H}$. Hence $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H}$ with respect to$\{\mathcal{H}_i\}_{i\in I}$.

Conversely, let $\{\Lambda_i\}_{i\in I}$ be a g-Riesz basis for $\mathcal{H}$ with bounds $0 < C_0 \leq D_0$ and $\{c_{ij}\}$ be a finite scalar sequence. Then

$$C_0 \sum_i \Big\| \sum_j c_{ij} g_{ij} \Big\|^2 \leq \Big\| \sum_i \Lambda_i^* (\sum_j c_{ij} g_{ij}) \Big\|^2 \leq D_0 \sum_i \Big\| \sum_j c_{ij} g_{ij} \Big\|^2$$

and

$$A\sum_j |c_{ij}|^2 \leq A_i \sum_j |c_{ij}|^2 \leq \Big\| \sum_j c_{ij} g_{ij} \Big\|^2 \leq B_i \sum_j |c_{ij}|^2 \leq B \sum_j |c_{ij}|^2.$$

Hence

$$AC_0 \sum_{i,j} |c_{ij}|^2 \leq \Big\| \sum_{i,j} c_{ij}\Lambda_i^* g_{ij} \Big\|^2 \leq BD_0 \sum_{i,j} |c_{ij}|^2.$$

Moreover, we have $\mathcal{H} = \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in I} = \overline{\text{span}}\{\Lambda_i^* g_{ij}\}_{i\in I, j\in K_i}$. So $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in K_i}$ is a Riesz basis for $\mathcal{H}$.

For any sequence $\{\mathcal{H}_i\}_{i\in I}$ of Hilbert spaces, we can find a Hilbert space $\mathcal{K}$ to contain all the $\mathcal{H}_i$ by setting $\mathcal{K} = \left(\sum_{i\in I} \oplus \mathcal{H}_i\right)_{l_2}$. $\square$

Proposition 3.5. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ and $E \subseteq I$ such that*

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle, \quad g_i \in \mathcal{H}_i,\ g_j \in \mathcal{H}_j, \quad i,j \in E.$$

Then $f = \sum_{i\in E} \Lambda_i^*\Lambda_i f$ for all $f \in \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$.

Proof. First of all, the series $\sum_{i\in E} \Lambda_i^*\Lambda_i f$ are convergent for all $f \in \mathcal{H}$. To see this, let J be a finite subset of E. Then

$$\Big\| \sum_{i\in J} \Lambda_i^*\Lambda_i f\Big\|^2 = \sum_{i\in J} \|\Lambda_i f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2$$

for all $f \in \mathcal{H}$. Since $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$, we get $\sum_{i\in E} \Lambda_i^*\Lambda_i f$ converges. Let $f \in \text{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$, then $f = \sum_{i\in E} \Lambda_i^* g_i$ where $g_i \in \mathcal{H}_i$ and the set $\{\, i \in E : \Lambda_i^* g_i \neq 0\,\}$ is finite. We show that $g_i = \Lambda_i f$ for $i \in E$. Let $h \in \mathcal{H}_i$, then

$$\begin{aligned}\langle \Lambda_i f, h\rangle = \langle \sum_{k\in E} \Lambda_i\Lambda_k^* g_k, h\rangle &= \sum_{k\in E} \langle \Lambda_k^* g_k, \Lambda_i^* h\rangle \\ &= \langle \Lambda_i^* g_i, \Lambda_i^* h\rangle = \langle g_i, h\rangle.\end{aligned}$$

So $g_i = \Lambda_i f$ for $i \in E$ and $f = \sum_{i\in E} \Lambda_i^*\Lambda_i f$.

For the case $f \in \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$, there exists a sequence $\{f_n\}$ in $\text{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$ such that $f_n \to f$ as $n \to \infty$. Let B be the upper g-frame bound for Λ. We have

$$\begin{aligned}\Big\| \sum_{i\in E} \Lambda_i^*\Lambda_i f_n - \sum_{i\in E} \Lambda_i^*\Lambda_i f\Big\|^2 &= \Big\| \sum_{i\in E} \Lambda_i^*\Lambda_i (f_n - f)\Big\|^2 \\ &= \sum_{i\in E} \|\Lambda_i(f_n - f)\|^2 \\ &\leq B\|f_n - f\|^2 \to 0.\end{aligned}$$

Hence $f = \sum_{i\in E} \Lambda_i^*\Lambda_i f$. □

Definition 3.6. *A g-frame $\{\Lambda_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ is called a g-Riesz frame if for every $J \subseteq I$, $\{\Lambda_i\}_{i\in J}$ is a g-frame for $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in J}$ with uniform g-frame bounds A, B.*

Proposition 3.7. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with bounds $0 < C \leq D$ such that*

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle, \quad g_i \in \mathcal{H}_i,\ g_j \in \mathcal{H}_j, \quad i,j \in I. \tag{3.3}$$

Then Λ is a g-Riesz frame with bounds 1 *and* D. *Moreover, if* $\{g_{ij}\}_{j\in K_i}$ *is a Riesz frame for* $\mathcal{H}_i$ *with bounds* A_i, B_i *for each* $i \in I$ *and* $0 < A = \inf_{i\in I} A_i,\ \ B = \sup_{i\in I} B_i < \infty$, *then* $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in K_i}$ *is a Riesz frame for* $\mathcal{H}$.

Proof. Let $E \subseteq I$ and $W = \overline{\operatorname{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$. By Proposition 3.5 we have

$$\|f\|^2 = \Big\| \sum_{i\in E} \Lambda_i^* \Lambda_i f \Big\|^2 = \sum_{i\in E} \|\Lambda_i f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq D\|f\|^2$$

for all $f \in W$. Now we assume that $\{g_{ij}\}_{j\in K_i}$ is a Riesz frame for $\mathcal{H}_i$ and $I_0 \subseteq I$. We show that $\{\Lambda_i^* g_{ij}\}_{i\in I_0, j\in K_i^1}$ is a frame for $\overline{\operatorname{span}}\{\Lambda_i^* g_{ij}\}_{i\in I_0, j\in K_i^1}$ with uniform frame bounds A and BD, where $K_i^1 \subseteq K_i$ for each $i \in I_0$. Let $f \in \operatorname{span}\{\Lambda_i^* g_{ij}\}_{i\in I_0, j\in K_i^1}$ and $k \in I_0$. Then there is a finite scalar sequence $\{c_{ij}\}$ such that $\Lambda_k f = \sum_{i,j} c_{ij} \Lambda_k \Lambda_i^* g_{ij}$. It follows from (3.3) that

$$\Lambda_l f = \sum_j c_{lj} \Lambda_l \Lambda_l^* g_{lj} = \sum_j c_{lj} g_{lj}.$$

Therefore $\Lambda_l f \in \overline{\operatorname{span}}\{g_{lj}\}_{j\in K_l^1}$ for all $f \in \overline{\operatorname{span}}\{\Lambda_i^* g_{ij}\}_{i\in I_0, j\in K_i^1}$ and all $l \in I_0$. Since $\{\Lambda_i\}_{i\in I}$ is a g-Riesz frame we have

$$\|f\|^2 \leq \sum_{i\in I_0} \|\Lambda_i f\|^2 \leq D\|f\|^2 \tag{3.4}$$

for all $f \in \overline{\operatorname{span}}\{\Lambda_i^* g_{ij}\}_{i\in I_0, j\in K_i^1}$. Also $\{g_{ij}\}_{j\in K_i}$ is a Riesz frame for $\mathcal{H}_i$ so

$$A_i \|\Lambda_i f\|^2 \leq \sum_{j\in K_i^1} |\langle \Lambda_i f, g_{ij}\rangle|^2 \leq B_i \|\Lambda_i f\|^2 \tag{3.5}$$

for all $f \in \mathcal{H}$. Therefore (3.4) and (3.5) imply that

$$\begin{aligned} A\|f\|^2 &\leq \sum_{i\in I_0} A_i \|\Lambda_i f\|^2 \\ &\leq \sum_{i\in I_0} \sum_{j\in K_i^1} |\langle f, \Lambda_i^* g_{ij}\rangle|^2 \leq \sum_{i\in I_0} B_i \|\Lambda_i f\|^2 \leq BD\|f\|^2 \end{aligned}$$

for all $f \in \overline{\operatorname{span}}\{\Lambda_i^* g_{ij}\}_{i\in I_0, j\in K_i^1}$ □

4. α-dual of g-frames and Hilbert-Schmidt operators

In this section $\mathcal{H}$ denotes a finite dimensional Hilbert space. We also denote the norm of a Hilbert-Schmidt operator T by $\|T\|_2$.

Definition 4.1. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be g-frames for $\mathcal{H}$. We say that $\{\Theta_i\}_{i\in I}$ is a dual g-frame (or simply dual) of $\{\Lambda_i\}_{i\in I}$ if*

$$f = \sum_{i\in I} \Lambda_i^* \Theta_i f$$

holds for all $f \in \mathcal{H}$.

It is easy to show that if $\{\Theta_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$, then $\{\Lambda_i\}_{i\in I}$ will be a dual g-frame of $\{\Theta_i\}_{i\in I}$.

Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with g-frame operator S. Then (2.4) shows that $\{ \Lambda_i S^{-1}\}_{i\in I}$ is a dual g-frame of $\{ \Lambda_i \}_{i\in I}$. $\{ \Lambda_i S^{-1} \}_{i\in I}$ is called *canonical dual* g-frame of $\{ \Lambda_i \}_{i\in I}$.

Proposition 4.2. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a dual of g-frame $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$. Then*

$$\sum_{i\in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i\in I} \|\Lambda_i\|_2^2 + \sum_{i\in I} \|\Theta_i\|_2^2 - 2\dim \mathcal{H}.$$

Proof. Suppose that $\{e_n\}_{n=1}^M$ is an orthonormal basis for $\mathcal{H}$. We have

$$\begin{aligned}
\sum_{i\in I} \|\Lambda_i - \Theta_i\|_2^2 &= \sum_{i\in I}\sum_{n} \|(\Lambda_i - \Theta_i)e_n\|^2 \\
&= \sum_{n}\sum_{i\in I} \langle(\Lambda_i - \Theta_i)e_n, (\Lambda_i - \Theta_i)e_n\rangle \\
&= \sum_{i\in I}\sum_{n} \|\Lambda_i e_n\|^2 + \sum_{i\in I}\sum_{n} \|\Theta_i e_n\|^2 \\
&\quad - \sum_{n}\sum_{i\in I} \langle \Lambda_i^*\Theta_i e_n, e_n\rangle \quad - \sum_{n}\sum_{i\in I} \langle e_n, \Lambda_i^*\Theta_i e_n\rangle \\
&= \sum_{i\in I} \|\Lambda_i\|_2^2 + \sum_{i\in I} \|\Theta_i\|_2^2 - 2\dim \mathcal{H}.
\end{aligned}$$

□

Corollary 4.3. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two Parseval g-frames for $\mathcal{H}$. If $\{\Theta_i\}_{i\in I}$ is a dual of $\{\Lambda_i\}_{i\in I}$, then $\Lambda_i = \Theta_i$ for all $i \in I$.*

Proof. Since $\mathcal{H}$ is a finite dimensional and $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame for $\mathcal{H}$, we have $\sum_{i\in I} \|\Lambda_i\|_2^2 = \dim \mathcal{H}$ (see [11]). Hence the result follows by Proposition 4.2. □

Corollary 4.4. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a dual of $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$, where $\{\Theta_i\}_{i\in I}$ and $\{\Lambda_i\}_{i\in I}$ are two tight g-frames for $\mathcal{H}$ with bounds B_Θ and B_Λ, respectively. Then $B_\Theta + B_\Lambda = 2$ if and only if $\Lambda_i = \Theta_i$ for all $i \in I$.*

Proof. Since $\sum_{i\in I} \|\Lambda_i\|_2^2 = B_\Lambda \dim \mathcal{H}$ and $\sum_{i\in I} \|\Theta_i\|_2^2 = B_\Theta \dim \mathcal{H}$, the result follows from Proposition 4.2. □

Remark 4.5. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two g-frames for $\mathcal{H}$ with the associated synthesis operators T_Λ and T_Θ, respectively. Using the proof of Proposition 4.2, we get*

$$\sum_{i\in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i\in I} \|\Lambda_i\|_2^2 + \sum_{i\in I} \|\Theta_i\|_2^2 - \mathrm{tr}(\mathrm{T}_\Theta \mathrm{T}_\Lambda^*) - \mathrm{tr}(\mathrm{T}_\Lambda \mathrm{T}_\Theta^*).$$

Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with the frame operator S. It is clear that $\{\Lambda_i S^{\alpha-1}\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with the property $\sum_{i\in I} \Lambda_i^* \Lambda_i S^{\alpha-1} = S^\alpha f$ for all $f \in \mathcal{H}$. For $\alpha = 0$ we get the canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Definition 4.6. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with g-frame operator S_Λ. A g-frame $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a α-dual of $\{\Lambda_i\}_{i\in I}$ if $\sum_{i\in I} \Lambda_i^* \Theta_i = S_\Lambda^\alpha f$ for all $f \in \mathcal{H}$.*

It is clear that $\{\Lambda_i S_\Lambda^{\alpha-1}\}_{i\in I}$ is a α-dual of $\{\Lambda_i\}_{i\in I}$. The canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$ has some interesting properties between other dual g-frames of $\{\Lambda_i\}_{i\in I}$ (see [12]). We will show that the α-dual frame $\{\Lambda_i S_\Lambda^{\alpha-1}\}_{i\in I}$ has some minimal properties between other α-dual frames of $\{\Lambda_i\}_{i\in I}$.

Proposition 4.7. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a α-dual of $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ with g-frame operator S_Θ. Then*

$$\sum_{i\in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 = \left\| S_\Lambda^{\frac{2\alpha-1}{2}} \right\|_2^2 \leq \|S_\Theta^{\frac{1}{2}}\|_2^2 = \sum_{i\in I} \|\Theta_i\|_2^2. \tag{4.1}$$

The equality in (4.1) holds if and only if $\Theta_i = \Lambda_i S_\Lambda^{\alpha-1}$ *for all* $i \in I$.

Proof. Let $\{e_n\}_{n=1}^M$ be an orthonormal basis for $\mathcal{H}$. We have

$$\begin{aligned}
\sum_{i\in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 &= \sum_{i\in I}\sum_n \langle \Lambda_i S_\Lambda^{\alpha-1} e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n\rangle \\
&= \sum_{i\in I}\sum_n \langle \Lambda_i^*\Lambda_i S_\Lambda^{\alpha-1} e_n, S_\Lambda^{\alpha-1} e_n\rangle \\
&= \sum_n \langle e_n, S_\Lambda^{2\alpha-1} e_n\rangle \\
&= \sum_n \langle S_\Lambda^{\frac{2\alpha-1}{2}} e_n, S_\Lambda^{\frac{2\alpha-1}{2}} e_n\rangle = \left\| S_\Lambda^{\frac{2\alpha-1}{2}} \right\|_2^2, \\
\sum_{i\in I} \|\Theta_i\|_2^2 &= \sum_{i\in I}\sum_n \langle \Theta_i^*\Theta_i e_n, e_n\rangle = \sum_n \langle S_\Theta e_n, e_n\rangle = \|S_\Theta^{\frac{1}{2}}\|_2^2.
\end{aligned}$$

On the other hand

(4.2)
$$\begin{aligned}
\sum_{i\in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 &= \sum_{i\in I}\sum_n \langle \Lambda_i S_\Lambda^{\alpha-1} e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n\rangle \\
&= \sum_n \langle S_\Lambda^{\alpha} e_n, S_\Lambda^{\alpha-1} e_n\rangle \\
&= \sum_n \sum_{i\in I} \langle \Lambda_i^*\Theta_i e_n, S_\Lambda^{\alpha-1} e_n\rangle \\
&= \sum_n \sum_{i\in I} \langle \Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n\rangle \\
&\le \left(\sum_n \sum_{i\in I} \|\Lambda_i S_\Lambda^{\alpha-1} e_n\|^2\right)^{\frac{1}{2}} \left(\sum_n \sum_{i\in I} \|\Theta_i e_n\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

So $\sum_{i\in I}\|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 \leq \sum_{i\in I}\|\Theta_i\|_2^2$ and we obtain (4.1).
If $\sum_{i\in I}\|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 = \sum_{i\in I}\|\Theta_i\|_2^2$, then it follows from (4.2) that

$$\begin{aligned}\sum_n\sum_{i\in I}\langle\Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1}e_n\rangle &= \sum_n\sum_{i\in I}\left|\langle\Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1}e_n\rangle\right| \\ &= \left(\sum_n\sum_{i\in I}\|\Lambda_i S_\Lambda^{\alpha-1}e_n\|^2\right)^{\frac{1}{2}} \\ &\quad \left(\sum_n\sum_{i\in I}\|\Theta_i e_n\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

So $\langle\Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1}e_n\rangle \geq 0$ and $\langle\Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1}e_n\rangle = \|\Theta_i e_n\|\|\Lambda_i S_\Lambda^{\alpha-1}e_n\|$ for all i, n. Therefore there exist $\lambda, \lambda_{i,n} \geq 0$ such that

$$\Lambda_i S_\Lambda^{\alpha-1}e_n = \lambda_{i,n}\Theta_i e_n, \quad \|\Lambda_i S_\Lambda^{\alpha-1}e_n\| = \lambda\|\Theta_i e_n\|$$

for all i, n. Hence $\lambda_{i,n} = \lambda$ and we conclude that $\Lambda_i S_\Lambda^{\alpha-1}e_n = \lambda\Theta_i e_n$ for all i, n. Since $\sum_{i\in I}\|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 = \sum_{i\in I}\|\Theta_i\|_2^2$, we get $\lambda = 1$ and so $\Lambda_i S_\Lambda^{\alpha-1}e_n = \Theta_i e_n$ for all i, n. □

Corollary 4.8. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with g-frame operator S_Λ. Then*

$$\sum_{i\in I}\|\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}\|_2^2 =$$
$$\min\left\{\sum_{i\in I}\|\Lambda_i - \Theta_i\|_2^2 : \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\} \text{ is a } \alpha\text{-dual of } \{\Lambda_i\}_{i\in I}\right\}.$$

Moreover, if $\{\Theta_i\}_{i\in I}$ is a α-dual of $\{\Lambda_i\}_{i\in I}$, then $\sum_{i\in I}\|\Lambda_i - \Theta_i\|_2^2 = \sum_{i\in I}\|\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}\|_2^2$ if and only if $\Theta_i = \Lambda_i S_\Lambda^{\alpha-1}$ for all $i \in I$.

Proof. Since

$$\sum_{i\in I}\sum_n\langle\Theta_i e_n, \Lambda_i e_n\rangle = \sum_n\langle S_\Lambda^\alpha e_n, e_n\rangle = \sum_{i\in I}\sum_n\langle\Lambda_i S_\Lambda^{\alpha-1}e_n, \Lambda_i e_n\rangle,$$

by Proposition 4.7 we have

$$\begin{aligned}\sum_{i\in I}\|\Lambda_i-\Theta_i\|_2^2 &=\sum_{i\in I}\sum_n\big(\|\Lambda_i e_n\|^2+\|\Theta_i e_n\|^2-2\Re\langle\Theta_i e_n,\Lambda_i e_n\rangle\big)\\ &\geq\sum_{i\in I}\sum_n\big(\|\Lambda_i e_n\|^2+\|\Lambda_i S_\Lambda^{\alpha-1}e_n\|^2-2\Re\langle\Lambda_i S_\Lambda^{\alpha-1}e_n,\Lambda_i e_n\rangle\big)\\ &=\sum_{i\in I}\sum_n\|(\Lambda_i-\Lambda_i S_\Lambda^{\alpha-1})e_n\|^2\\ &=\sum_{i\in I}\|\Lambda_i-\Lambda_i S_\Lambda^{\alpha-1}\|_2^2.\end{aligned}$$

Therefore the above inequality implies that $\sum_{i\in I}\|\Lambda_i-\Theta_i\|_2^2=\sum_{i\in I}\|\Lambda_i-\Lambda_i S_\Lambda^{\alpha-1}\|_2^2$ if and only if $\sum_{i\in I}\sum_n\|\Lambda_i S_\Lambda^{\alpha-1}e_n\|^2=\sum_{i\in I}\sum_n\|\Theta_i e_n\|^2$. Hence by Proposition 4.7, $\sum_{i\in I}\|\Lambda_i-\Theta_i\|_2^2=\sum_{i\in I}\|\Lambda_i-\Lambda_i S_\Lambda^{\alpha-1}\|_2^2$ if and only if $\Theta_i=\Lambda_i S_\Lambda^{\alpha-1}$ for all $i\in I$. □

Corollary 4.9. *Let $\{\Theta_i\in B(\mathcal{H},\mathcal{H}_i):i\in I\}$ is a dual of g-frame $\{\Lambda_i\in B(\mathcal{H},\mathcal{H}_i):i\in I\}$ for $\mathcal{H}$. Then*

$$\sum_{i\in I}\|\Theta_i\|_2^2=\|S_\Theta^{\frac{1}{2}}\|_2^2\geq\|S_\Lambda^{-\frac{1}{2}}\|_2^2=\sum_{i\in I}\|\Lambda_i S_\Lambda^{-1}\|_2^2 \tag{4.3}$$

where S_Λ and S_Θ are the g-frame operators of $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$, respectively. Moreover, the following are equivalent

(*i*) $\sum_{i\in I}\|\Theta_i\|_2^2=\sum_{i\in I}\|\Lambda_i S_\Lambda^{-1}\|_2^2$;
(*ii*) $\Theta_i=\Lambda_i S_\Lambda^{-1}$ *for all* $i\in I$;
(*iii*) $S_\Theta=S_\Lambda^{-1}$.

Proposition 4.10. *Let $\{\Lambda_i\in B(\mathcal{H},\mathcal{H}_i):i\in I\}$ be a g-frame and $\{\Theta_i\in B(\mathcal{H},\mathcal{H}_i):i\in I\}$ be a Parseval g-frame for $\mathcal{H}$. Then*

(i) $\mathrm{tr}(\mathrm{T}_\Theta\mathrm{T}_\Lambda^*)+\mathrm{tr}(\mathrm{T}_\Lambda\mathrm{T}_\Theta^*)\leq 2\|\mathrm{S}_\Lambda^{\frac{1}{4}}\|_2^2$;
(ii) $\mathrm{tr}(\mathrm{T}_\Theta\mathrm{T}_\Lambda^*)+\mathrm{tr}(\mathrm{T}_\Lambda\mathrm{T}_\Theta^*)=2\|\mathrm{S}_\Lambda^{\frac{1}{4}}\|_2^2$ *if and only if* $\Theta_i=\Lambda_i S_\Lambda^{-\frac{1}{2}}$ *for all* $i\in I$.

Proof. By Remark 4.5, $\mathrm{tr}(\mathrm{T}_\Theta\mathrm{T}_\Lambda^*)+\mathrm{tr}(\mathrm{T}_\Lambda\mathrm{T}_\Theta^*)$ is real. Since $\{\Theta_i\}_{i\in I}$ is a Parseval g-frame for $\mathcal{H}$, we have $\|T_\Theta\|=\|T_\Theta^*\|=1$. Let us denote the trace-class norm by $\|.\|_1$. By a simple computation we get $\|T_\Lambda^*\|_1=\|S_\Lambda^{\frac{1}{4}}\|_2^2$. By [8; Theorems 2.4.14 and 2.4.16], we get

$$\big|\mathrm{tr}(\mathrm{T}_\Theta\mathrm{T}_\Lambda^*)\big|,\big|\mathrm{tr}(\mathrm{T}_\Lambda\mathrm{T}_\Theta^*)\big|\leq\|\mathrm{S}_\Lambda^{\frac{1}{4}}\|_2^2.$$

Therefore (i) is proved. To prove (ii), let $\mathrm{tr}(\mathrm{T}_\Theta \mathrm{T}_\Lambda^*) + \mathrm{tr}(\mathrm{T}_\Lambda \mathrm{T}_\Theta^*) = 2\|\mathrm{S}_\Lambda^{\frac{1}{4}}\|_2^2$. It follows from (i) that $\mathrm{tr}(\mathrm{T}_\Theta \mathrm{T}_\Lambda^*) = \|\mathrm{S}_\Lambda^{\frac{1}{4}}\|_2^2$. Hence we get the result by Corollary 2.6 of [11]. □

Corollary 4.11. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ with g-frame operator S_Λ. Then*

$$\max\{\Re\mathrm{tr}(\mathrm{T}_\Theta \mathrm{T}_\Lambda^*) : \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\} \text{ is a Parseval g-frame for } \mathcal{H}\} = \|S_\Lambda^{\frac{1}{4}}\|_2^2.$$

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