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MODULE COHOMOLOGY GROUP OF INVERSE SEMIGROUP ALGEBRAS

E. NASRABADI* AND A. POURABBAS

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Abstract. Let S be an inverse semigroup and let E be its subsemigroup of idempotents. In this paper we define the n-th module cohomology group of Banach algebras and show that the first module cohomology group $\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(n)})$ is zero, for every odd $n \in \mathbb{N}$. Next, for a Clifford semigroup S we show that $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(n)})$ is a Banach space, for every odd $n \in \mathbb{N}$.

1. **Introduction**

Amini in [1] developed the concept of module amenability for a class of Banach algebras which is in fact a generalization of the Johnson's amenability. For example, for every inverse semigroup S with subsemigroup E of idempotents, he showed that the $\ell^1(E)$-module amenability of $\ell^1(S)$, in the particular case where the left action is trivial and the right action is natural, is equivalent to the amenability of S. Duncan and Namioka in [3] have shown that $\ell^1(S)$ is not amenable, for some amenable semigroup S. In fact, they showed that the amenability of inverse semigroup algebra $\ell^1(S)$ implies that E is finite, but there are many

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*Corresponding author

amenable inverse semigroups including the bicyclic semigroup and Clifford semigroups with an infinite set of idempotents. Amini and Bagha in [2] introduced the concept of weak module amenability and showed that if S is commutative, $\ell^1(S)$ is always weak $\ell^1(E)$-module amenable.

Note that in the group case Johnson [6] showed that a group G is amenable if and only if $L^1(G)$ is amenable and in [7] he showed that $L^1(G)$ is always weakly amenable.

In this paper, we shall be concerned with the structure of the first and second module cohomology group of $\ell^1(S)$ with coefficients in the n-th dual space $\ell^1(S)^{(n)}$, for every odd $n \in \mathbb{N}$.

We begin by recalling some terminology.

Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}$-module with compatible actions, that is,

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad a(\alpha \cdot b) = (a \cdot \alpha)b \quad (\alpha \in \mathfrak{A},\ a, b \in A).$$

If A and B are Banach algebras and Banach $\mathfrak{A}$-modules with compatible actions, an $\mathfrak{A}$ -module map is a mapping $T : A \to B$ with

$$T(a \pm b) = T(a) \pm T(b), T(\alpha \cdot a) = \alpha \cdot T(a), T(a \cdot \alpha) = T(a) \cdot \alpha,$$

where $\alpha \in \mathfrak{A}$ and $a, b \in A$. Note that T is not necessarily linear, so it is not necessarily an $\mathfrak{A}$-module homomorphism.

Let X be a Banach A-module and a Banach $\mathfrak{A}$-module with compatible actions, that is,

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad (a \cdot \alpha) \cdot x = a \cdot (\alpha \cdot x), \quad (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a),$$

where $\alpha \in \mathfrak{A}$, $a \in A$ and $x \in X$ and the same, for the other side action. Then, X is called a Banach A-$\mathfrak{A}$-module, and is called a commutative Banach A-$\mathfrak{A}$-module whenever $\alpha \cdot x = x \cdot \alpha$, for every $\alpha \in \mathfrak{A}$ and $x \in X$. If moreover

$$a \cdot x = x \cdot a \quad (a \in A, x \in X),$$

then X is called a bi-commutative Banach A-$\mathfrak{A}$-module.

Let X be a Banach space with the dual space X'. If X is a (commutative) Banach A-$\mathfrak{A}$-module, then so is X', where the actions of A and $\mathfrak{A}$ on X' are defined by

$$(\alpha \cdot f)(x) = f(x \cdot \alpha), \quad (a \cdot f)(x) = f(x \cdot a),$$

where $\alpha \in \mathfrak{A}, a \in A, f \in X', x \in X$. In particular, if A is a commutative Banach $\mathfrak{A}$-module, then it is a commutative Banach A-$\mathfrak{A}$-module. In this case, the dual space A' is also a commutative Banach A-$\mathfrak{A}$-module.

An $\mathfrak{A}$-module map $D : A \to X$ is called an $\mathfrak{A}$-module derivation, if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A),$$

Note that D is not necessarily linear and if there exists a constant $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for each $a \in A$, then D is bounded and its boundedness implies its norm continuity.

When X is a commutative Banach A-$\mathfrak{A}$-module, each $x \in X$ defines an $\mathfrak{A}$-module derivation

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in A),$$

these are called inner $\mathfrak{A}$-module derivations. If X is a bi-commutative Banach A-$\mathfrak{A}$-module, then the inner derivations are zero.

Definition 1.1. *The Banach algebra A is called $\mathfrak{A}$-module amenable, if for any commutative Banach A-$\mathfrak{A}$-module X, every $\mathfrak{A}$-module derivation $D : A \to X'$ is inner.*

We use the notation $\mathcal{Z}^1_{\mathfrak{A}}(A, X)$ for the set of all $\mathfrak{A}$-module derivations $D : A \to X$ and $\mathcal{B}^1_{\mathfrak{A}}(A, X)$, for those which are inner. The first $\mathfrak{A}$-module cohomology group with coefficients in X is denoted by $\mathcal{H}^1_{\mathfrak{A}}(A, X)$ which is the quotient group $\mathcal{Z}^1_{\mathfrak{A}}(A, X)/\mathcal{B}^1_{\mathfrak{A}}(A, X)$. Hence, A is $\mathfrak{A}$-module amenable if and only if $\mathcal{H}^1_{\mathfrak{A}}(A, X') = 0$, for each commutative Banach A-$\mathfrak{A}$-module X.

Definition 1.2. *The Banach algebra A is called weak $\mathfrak{A}$-module amenable, if $\mathcal{H}^1_{\mathfrak{A}}(A, A')$ is zero.*

Definition 1.3. *A is called n-weak $\mathfrak{A}$ -module amenable, if $\mathcal{H}^1_{\mathfrak{A}}(A, A^{(n)})$ is zero.*

Let $\mathfrak{A}$ and A be Banach algebras such that A be a Banach $\mathfrak{A}$-module and let X be a Banach $\mathfrak{A}$-A-module with compatible actions. An n-$\mathfrak{A}$-module map is a mapping $\phi : A^n \to X$ with the following properties;

$$\begin{aligned}
\phi(a_1, \ldots, a_{i-1}, b \pm c, a_{i+1}, \ldots, a_n) = {} & \phi(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \\
& \pm \phi(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n), \\
\phi(\alpha \cdot a_1, a_2, \ldots, a_n) = {} & \alpha \cdot \phi(a_1, a_2, \ldots, a_n), \\
\phi(a_1, a_2, \ldots, a_n \cdot \alpha) = {} & \phi(a_1, a_2, \ldots, a_n) \cdot \alpha
\end{aligned}$$

and

$$\phi(a_1, \ldots, a_{i-1}, a_i \cdot \alpha, a_{i+1}, \ldots, a_n) = \phi(a_1, \ldots, a_{i-1}, a_i, \alpha \cdot a_{i+1}, \ldots, a_n),$$

where $a_1, \ldots, a_n, b, c \in A$ and $\alpha \in \mathfrak{A}$. Note that ϕ is not necessarily n-linear. The n-$\mathfrak{A}$-module map $\phi : A^n \to X$ is bounded, if there exists a constant $M > 0$ such that

$$|\phi(a_1, a_2, \ldots, a_n)| \leq M \|\phi\| \|a_1\| \cdots \|a_n\|,$$

where $a_1, \ldots, a_n \in A$. We use the notation $\mathcal{C}^n_{\mathfrak{A}}(A, X)$ for the set of all bounded n-$\mathfrak{A}$-module maps from A to X.

For $n \geq 1$, the map $\delta^n : \mathcal{C}^n_{\mathfrak{A}}(A, X) \longrightarrow \mathcal{C}^{n+1}_{\mathfrak{A}}(A, X)$ is given by

$$\begin{aligned} \delta^n T(a_1, \ldots, a_{n+1}) &= a_1 \cdot T(a_2, \ldots, a_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i T(a_1, \ldots, a_i a_{i+1}, \ldots a_{n+1}) \\ &\quad + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1}, \end{aligned}$$

where $T \in \mathcal{C}^n_{\mathfrak{A}}(A, X)$ and $a_1, \ldots, a_{n+1} \in A$.

For every $n \geq 2$, the space $\ker \delta^n$ of all bounded n-$\mathfrak{A}$-module cocycles is denoted by $\mathcal{Z}^n_{\mathfrak{A}}(A, X)$ and the space $\operatorname{Im} \delta^{n-1}$ of all bounded n-$\mathfrak{A}$-module coboundaries is denoted by $\mathcal{B}^n_{\mathfrak{A}}(A, X)$. We see that $\mathcal{B}^n_{\mathfrak{A}}(A, X)$ is included in $\mathcal{Z}^n_{\mathfrak{A}}(A, X)$ and that the n-th $\mathfrak{A}$-module cohomology group $\mathcal{H}^n_{\mathfrak{A}}(A, X)$ is defined by the quotient

$$\mathcal{H}^n_{\mathfrak{A}}(A, X) = \frac{\mathcal{Z}^n_{\mathfrak{A}}(A, X)}{\mathcal{B}^n_{\mathfrak{A}}(A, X)} \quad (n \geq 2).$$

The space $\mathcal{Z}^n_{\mathfrak{A}}(A, X)$ is a Banach space, but in general $\mathcal{B}^n_{\mathfrak{A}}(A, X)$ is not closed, we regard $\mathcal{H}^n_{\mathfrak{A}}(A, X)$ as a complete seminormed space with respect to the quotient seminorm. This seminorm is a norm if and only if $\mathcal{B}^n_{\mathfrak{A}}(A, X)$ is a closed subspace of $\mathcal{C}^n_{\mathfrak{A}}(A, X)$, which means that $\mathcal{H}^n_{\mathfrak{A}}(A, X)$ is a Banach space. It is unknown whether or not $\mathcal{H}^n_{\mathfrak{A}}(A, X)$ is a Banach space, for every $n \in \mathbb{N}$.

Proposition 1.4. [8, Proposition 1.1] *Let X and Y be Banach spaces and let $\Phi : X \to Y$ be a bounded linear map. If there exists a constant M such that, for every $y \in \operatorname{Im} \Phi$ there exists an element $x \in X$ such that $\|x\| \leq M \|y\|$ and $y = \Phi(x)$, then $\operatorname{Im} \Phi$ is closed.*

2. First module cohomology group of inverse semigroup algebras

Let S be a commutative inverse semigroup and E be the set of idempotent elements in S, then E is a commutative inverse semigroup and $\ell^1(S)$ is a commutative Banach $\ell^1(E)$-module with the actions

$$\delta_s \cdot \delta_e = \delta_e \cdot \delta_s = \delta_{se} \qquad (e \in E, s \in S),$$

where δ_s is the point mass at s. Since E is commutative, these actions are module actions.

In this section, for a commutative inverse semigroup S, we will show that the first module cohomology group of $\ell^1(S)$ (as an $\ell^1(E)$-module) with coefficients in $(\ell^1(S))^{(2n+1)}$ is trivial or $\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}') = 0$ where $\mathcal{X} = \ell^{(2n)}(S)$.

Remark 2.1. *Set $\mathcal{X} = (\ell^1(S))^{(2n)}$. We note that $(\ell^1(S))' = \ell^\infty(S)$ is a commutative unital C^*-algebra. Because the second dual of a commutative unital C^*-algebra is a commutative von Neumann algebra, then $\mathcal{X}' = (\ell^1(S))^{(2n+1)}$ is the underlying space of a commutative von Neumann algebra, and hence it is an L^∞-space. The space $\mathcal{X}'_{\mathbb{R}}$ of real-valued functions in $\mathcal{X}'$ forms a complete lattice in the sense that every nonempty subset of $\mathcal{X}'_{\mathbb{R}}$ that is bounded above has a supremum.*

Note that for every $t \in S$ there exists a unique t^* in S such that $tt^*t = t$ and $t^*tt^* = t^*$, we say that t^* is the unique inverse of t.

Lemma 2.2. *Let S be a commutative inverse semigroup and X be as in Remark (2.1). Let D be a $\ell^1(E)$-module derivation. Then, for every $s, t \in S$, we have*

$$\delta_{s^*} \cdot [\delta_{(ts^*)^*} \cdot D(\delta_{ts^*})] \cdot \delta_s = \delta_{t^*} \cdot D(\delta_t) \cdot \delta_{s^*s} + D(\delta_{t^*ts^*}) \cdot \delta_s. \tag{2.1}$$

and

$$\delta_{t^*} \cdot D(\delta_t) \cdot \delta_{s^*s} = \delta_{s^*} \cdot [\delta_{(ts^*)^*} \cdot D(\delta_{ts^*})] \cdot \delta_s + \delta_{t^*ts^*} \cdot D(\delta_s), \tag{2.2}$$

where s^* and t^* are the unique inverses of s and t in S respectively.

Proof. Since D is a $\ell^1(E)$-module derivation, for every $e \in E$ we have $D(\delta_e) = 0$. But, $s^*s \in E$ and hence $\delta_{s^*s} \in \ell^1(E)$, for every $s \in S$, so we have

$$0 = D(\delta_{s^*s}) = \delta_{s^*} \cdot D(\delta_s) + D(\delta_{s^*}) \cdot \delta_s,$$

that is,

$$\delta_{s^*} \cdot D(\delta_s) = -D(\delta_{s^*}) \cdot \delta_s \tag{2.3}$$

Now, for every $s, t \in S$ since t^*t and s^*s are in E by using (2.3), we obtain

$$\begin{aligned}\delta_{t^*} \cdot D(\delta_t) \cdot \delta_{s^*s} &= \delta_{t^*} \cdot D(\delta_{ts^*} * \delta_s) \\ &= \delta_{t^*} \cdot D(\delta_{ts^*}) \cdot \delta_s + \delta_{t^*ts^*} \cdot D(\delta_s) \\ &= \delta_{t^*} \cdot D(\delta_{ts^*ss^*}) \cdot \delta_s + \delta_{t^*ts^*} \cdot D(\delta_s)\end{aligned}$$

since $D(\delta_{ts^*ss^*}) = D(\delta_{s^*sts^*}) = \delta_{s^*s} \cdot D(\delta_{ts^*})$

$$\begin{aligned}&= \delta_{s^*} \cdot [\delta_{(ts^*)^*} \cdot D(\delta_{ts^*})] \cdot \delta_s + \delta_{t^*ts^*} \cdot D(\delta_s) \\ &= \delta_{s^*} \cdot [\delta_{(ts^*)^*} \cdot D(\delta_{ts^*})] \cdot \delta_s - D(\delta_{t^*ts^*}) \cdot \delta_s,\end{aligned}$$

so, we have shown that

$$\delta_{s^*} \cdot [\delta_{(ts^*)^*} \cdot D(\delta_{ts^*})] \cdot \delta_s = \delta_{t^*} \cdot D(\delta_t) \cdot \delta_{s^*s} + D(\delta_{t^*ts^*}) \cdot \delta_s.$$

Using (2.3), we obtain

$$\delta_{t^*} \cdot D(\delta_t) \cdot \delta_{s^*s} = \delta_{s^*} \cdot [\delta_{(ts^*)^*} \cdot D(\delta_{ts^*})] \cdot \delta_s + \delta_{t^*ts^*} \cdot D(\delta_s).$$

□

Lemma 2.3. *Let S be a commutative inverse semigroup and X be as in Remark (2.1). Let D be a $\ell^1(E)$-module derivation. Then, there exists a $\psi \in \mathcal{X}'$ such that for every $s \in S$*

$$D(\delta_s) = \delta_s \cdot \psi - \psi \cdot \delta_s.$$

Proof. Let $D \in \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$. Set

$$\Lambda = \{\operatorname{Re} \delta_{t^*} \cdot D(\delta_t) : t \in S\},$$

where t^* is the unique inverse of t in S. Since Λ is bounded above by $\|D\|$ in $\mathcal{X}'_{\mathbb{R}}$, then $\psi_r = \sup(\Lambda)$ exists in $\mathcal{X}'_{\mathbb{R}}$.

Taking supremum over all $t \in S$ of real part of (2.1) and since for every $s, x \in S$

$$\begin{aligned}&\sup_{t \in S}\{\operatorname{Re} \delta_{t^*} \cdot D(\delta_t) \cdot \delta_{s^*s}(f) + \operatorname{Re} D(\delta_{t^*ts^*}) \cdot \delta_s(f)\} \geq \\ &\qquad \sup_{t \in S}\{\operatorname{Re} \delta_{t^*} \cdot D(\delta_t) \cdot \delta_{s^*s}(f)\} + \operatorname{Re} D(\delta_{x^*xs^*}) \cdot \delta_s(f),\end{aligned}$$

we obtain

$$\begin{aligned}\delta_{s^*}\cdot\psi_r\cdot\delta_s(f) &\geq \sup_{t\in S}\{\operatorname{Re}\delta_{t^*}\cdot D(\delta_t)\cdot\delta_{s^*s}(f)+\operatorname{Re}D(\delta_{t^*ts^*})\cdot\delta_s(f)\}\\ &\geq \sup_{t\in S}\{\operatorname{Re}\delta_{t^*}\cdot D(\delta_t)\cdot\delta_{s^*s}(f)\}+\operatorname{Re}D(\delta_{s^*ss^*})\cdot\delta_s(f)\\ &= \psi_r\cdot\delta_{s^*s}(f)+\operatorname{Re}D(\delta_{s^*ss^*})\cdot\delta_s(f)\\ &= \psi_r\cdot\delta_{s^*s}(f)+\operatorname{Re}D(\delta_{s^*})\cdot\delta_s(f),\end{aligned}$$

so we have shown that

$$\operatorname{Re}D(\delta_{s^*})\cdot\delta_s(f)\leq\delta_{s^*}\cdot\psi_r\cdot\delta_s(f)-\psi_r\cdot\delta_{s^*s}(f). \tag{2.4}$$

Similarly, taking supremum over all $t\in S$ of real part of (2.2) we obtain

$$\begin{aligned}\psi_r\cdot\delta_{s^*s}(f) &\geq \sup_{t\in S}\{\operatorname{Re}\delta_{s^*}\cdot[\delta_{(ts^*)^*}\cdot D(\delta_{ts^*})]\cdot\delta_s(f)\\ &\qquad\qquad+\operatorname{Re}\delta_{t^*ts^*}\cdot D(\delta_s)(f)\}\\ &\geq \sup_{t\in S}\{\operatorname{Re}\delta_{s^*}\cdot[\delta_{(ts^*)^*}\cdot D(\delta_{ts^*})]\cdot\delta_s(f)\}\\ &\qquad\qquad+\operatorname{Re}\delta_{s^*ss^*}\cdot D(\delta_s)(f)\\ &= \delta_{s^*}\cdot\psi_r\cdot\delta_s(f)+\operatorname{Re}\delta_{s^*ss^*}\cdot D(\delta_s)(f)\\ &= \delta_{s^*}\cdot\psi_r\cdot\delta_s(f)+\operatorname{Re}\delta_{s^*}\cdot D(\delta_s)(f),\end{aligned}$$

so we have shown that

$$\operatorname{Re}D(\delta_{s^*})\cdot\delta_s(f)\geq\delta_{s^*}\cdot\psi_r\cdot\delta_s(f)-\psi_r\cdot\delta_{s^*s}(f). \tag{2.5}$$

Now, using (2.4) and (2.5), we obtain

$$\operatorname{Re}D(\delta_{s^*})\cdot\delta_s(f)=\delta_{s^*}\cdot\psi_r\cdot\delta_s(f)-\psi_r\cdot\delta_{s^*s}(f),$$

using (2.3), we obtain

$$\operatorname{Re}\delta_{s^*}\cdot D(\delta_s)(f)=\psi_r\cdot\delta_{s^*s}(f)-\delta_{s^*}\cdot\psi_r\cdot\delta_s(f). \tag{2.6}$$

Now, by replacing f with $f\cdot\delta_s$ in (2.6), we obtain

$$\operatorname{Re}\delta_{ss^*}\cdot D(\delta_s)(f)=\delta_s\cdot\psi_r\cdot\delta_{s^*s}(f)-\delta_{ss^*}\cdot\psi_r\cdot\delta_s(f).$$

Since $\delta_{ss^*}\in\ell^1(E)$ commute with elements of Λ and so with ψ_r and $\delta_{ss^*s}=\delta_s$, therefore we have

$$\operatorname{Re}\delta_{ss^*}\cdot D(\delta_s)(f)=\delta_s\cdot\psi_r(f)-\psi_r\cdot\delta_s(f). \tag{2.7}$$

Similarly, by considering imaginary parts, we obtain ψ_i such that

$$\operatorname{Im}\delta_{ss^*}\cdot D(\delta_s)(f)=\delta_s\cdot\psi_i(f)-\psi_i\cdot\delta_s(f). \tag{2.8}$$

By putting $\psi = \psi_r + i\psi_i \in \mathcal{X}'$ and using (2.7) and (2.8), we obtain

$$\delta_{ss^*} \cdot D(\delta_s)(f) = \delta_s \cdot \psi(f) - \psi \cdot \delta_s(f).$$

On the other hand

$$\delta_{ss^*} \cdot D(\delta_s) = D(\delta_{ss^*} \cdot \delta_s) = D(\delta_s),$$

hence we get

$$D(\delta_s) = \delta_s \cdot \psi - \psi \cdot \delta_s.$$

□

Theorem 2.4. *Let S be a commutative inverse semigroup. Then, $\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$ is zero, where $\mathcal{X} = (\ell^1(S))^{(2n)}$, for every $n \in \mathbb{N}$.*

Proof. Let $D \in \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$. We show that there exists a function ψ in $\mathcal{X}'$ such that $D = ad_\psi$ or $D \in \mathcal{B}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$. By Lemma 2.3 there exists a $\psi \in \mathcal{X}'$ such that for every $s \in S$

$$D(\delta_s) = \delta_s \cdot \psi - \psi \cdot \delta_s. \tag{2.9}$$

Now, we will show that D is linear, for each $s \in S$ and $\lambda \in \mathbb{C}$, since $\lambda\delta_{ss^*} \in \ell^1(E)$, we have

$$D(\lambda\delta_s) = D(\lambda\delta_{ss^*} * \delta_s) = \lambda\delta_{ss^*} \cdot D(\delta_s) = \lambda D(\delta_{ss^*} * \delta_s) = \lambda D(\delta_s),$$

but, D is additive, so we get $D(\lambda g) = \lambda D(g)$, for each $g \in \ell^1(S)$ of finite support. Using the continuity of D and the fact that, all functions of finite support are dense in $\ell^1(S)$, we obtain that D is linear.

Also with the same reason as above for each $g \in \ell^1(S)$ from (2.9) we have

$$D(g) = g \cdot \psi - \psi \cdot g = ad_\psi(g),$$

this shows that $D \in \mathcal{B}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$ and this completes the proof. □

3. Second module cohomology group of inverse semigroup algebras

Let S be an inverse semigroup and let E be the set of idempotent elements in S which is central.

In this section, we state the final result of this paper. We will show that the second $\ell^1(E)$-module cohomology group of $\ell^1(S)$ with coefficients in the $(2n+1)$-th dual space $\ell^1(S)^{(2n+1)}$ is a Banach space.

Why one wish to show that a cohomology group of a Banach algebra is a Banach space? The reason is that, if the algebraic cohomology group is trivial, then this often leads to the conclusion that the space of coboundaries is dense in the space of cocycles. If additionally one can prove that the space of coboundaries is closed, then one has a proof that the cohomology is trivial. This is the method that the second author and others used to show that $\mathcal{H}^3(\ell^1(S), \ell^\infty(S)) = 0$, where S is a semilattice [5].

In the rest of this paper, we set $\mathcal{X} = (\ell^1(S))^{(2n)}$ as we mentioned in Remark 2.1.

Lemma 3.1. *Let S be an inverse semigroup and let $\phi \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$. Then, for every $s \in S$, $e \in E$ and $f \in \mathcal{X}$ with $\|f\| \leq 1$, we have*

$$|\phi(\delta_{es^*}) \cdot \delta_s(f) + \delta_{es^*} \cdot \phi(\delta_s)(f)| \leq 2\,\|\delta\phi\|,$$

where s^* is the unique inverse of s in S.

Proof. Let $\phi \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$. Using the 2-coboundary map, for every $s, t \in S$ and $f \in \mathcal{X}$ with $\|f\| \leq 1$, we have

(3.1)
$$|\delta\phi(\delta_s, \delta_t)(f)| = |\phi(\delta_s) \cdot \delta_t(f) - \phi(\delta_s * \delta_t)(f) + \delta_s \cdot \phi(\delta_t)(f)| \leq \|\delta\phi\|.$$

Using (3.1) with $e \in E$ instead of $s, t \in S$, respectively, we obtain

$$|\phi(\delta_e)(f)| \leq \|\delta\phi\|.$$

Thus, for every $e \in E$ and $s \in S$ we have

$$\begin{aligned}|\phi(\delta_{es^*}) \cdot \delta_s(f) + \delta_{es^*} \cdot \phi(\delta_s)(f)| &\leq |\phi(\delta_{es^*}) \cdot \delta_s(f) + \delta_{es^*} \cdot \phi(\delta_s)(f) \\ &\quad - \phi(\delta_{es^*s})(f)| + |\phi(\delta_{es^*s})(f)| \\ &\leq 2\,\|\delta\phi\|.\end{aligned}$$

□

Corollary 3.2. *Let S be an inverse semigroup and let the idempotents of S are central. For every $\phi \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$, we have*

(3.2)
$$\operatorname{Re} \delta_{t^*} \cdot \phi(\delta_t) \cdot \delta_{s^*s} + \operatorname{Re} \phi(\delta_{t^*ts^*}) \cdot \delta_s \leq \operatorname{Re} \delta_{s^*} \cdot [\delta_{(ts^*)^*} \cdot \phi(\delta_{ts^*})] \cdot \delta_s + 3\,\|\delta\phi\|.$$

Proof. For every $s, t \in S$ and $f \in \mathcal{X}$, since ϕ is a $\ell^1(E)$-module map and using (3.1), by using the centrality of the idempotent s^*s, we obtain

$$\begin{aligned}\operatorname{Re}\delta_{t^*}\cdot\phi(\delta_t)\cdot\delta_{s^*s} &= \operatorname{Re}\delta_{t^*}\cdot\phi(\delta_{ts^*}*\delta_s)\\ &\leq \operatorname{Re}\delta_{t^*}\cdot\phi(\delta_{ts^*})\cdot\delta_s + \operatorname{Re}\delta_{t^*ts^*}\cdot\phi(\delta_s) + \|\delta\phi\|\\ &= \operatorname{Re}\delta_{t^*}\cdot\phi(\delta_{ts^*ss^*})\cdot\delta_s + \operatorname{Re}\delta_{t^*ts^*}\cdot\phi(\delta_s) + \|\delta\phi\|\end{aligned}$$

since $D(\delta_{ts^*ss^*}) = D(\delta_{s^*sts^*}) = \delta_{s^*s}\cdot D(\delta_{ts^*})$ and by previous Lemma

$$\begin{aligned}&= \operatorname{Re}\delta_{s^*}\cdot[\delta_{(ts^*)^*}\cdot\phi(\delta_{ts^*})]\cdot\delta_s + \operatorname{Re}\delta_{t^*ts^*}\cdot\phi(\delta_s) + \|\delta\phi\|\\ &\leq \operatorname{Re}\delta_{s^*}\cdot[\delta_{(ts^*)^*}\cdot\phi(\delta_{ts^*})]\cdot\delta_s - \operatorname{Re}\phi(\delta_{t^*ts^*})\cdot\delta_s + 3\|\delta\phi\|.\end{aligned}$$

□

Lemma 3.3. *Let S be an inverse semigroup and let the idempotents of S are central. Let $\phi \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$, then, there exists a $\psi \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$ such that*

$$|\phi(\delta_s)(f) - [\delta_s\cdot\psi(f) - \psi\cdot\delta_s(f)]| \leq 6\|\delta\phi\|$$

for every $s \in S$.

Proof. Set

$$\Lambda = \{\operatorname{Re}\delta_{t^*}\cdot\phi(\delta_t) : t \in S\},$$

where t^* is the unique inverse of t in S. Since Λ is bounded above by $\|\phi\|$ in $\mathcal{X}'_{\mathbb{R}}$, then $\psi_r = \sup(\Lambda)$ exists in $\mathcal{X}'_{\mathbb{R}}$.

Taking supremum over all $t \in S$ of (3.2) and since for every $s, x \in S$

$$\begin{aligned}&\sup_{t\in S}\{\operatorname{Re}\delta_{t^*}\cdot\phi(\delta_t)\cdot\delta_{s^*s}(f) + \operatorname{Re}\phi(\delta_{t^*ts^*})\cdot\delta_s(f)\}\\ &\qquad\geq \sup_{t\in S}\{\operatorname{Re}\delta_{t^*}\cdot\phi(\delta_t)\cdot\delta_{s^*s}(f)\} + \operatorname{Re}\phi(\delta_{x^*xs^*})\cdot\delta_s(f),\end{aligned}$$

we obtain

$$\begin{aligned}3\|\delta\phi\| + \delta_{s^*}\cdot\psi_r\cdot\delta_s(f) &\geq \sup_{t\in S}\{\operatorname{Re}\delta_{t^*}\cdot\phi(\delta_t)\cdot\delta_{s^*s}(f)\\ &\quad + \operatorname{Re}\phi(\delta_{t^*ts^*})\cdot\delta_s(f)\}\\ &\geq \sup_{t\in S}\{\operatorname{Re}\delta_{t^*}\cdot\phi(\delta_t)\cdot\delta_{s^*s}(f)\}\\ &\quad + \operatorname{Re}\phi(\delta_{s^*ss^*})\cdot\delta_s(f)\\ &= \psi_r\cdot\delta_{s^*s}(f) + \operatorname{Re}\phi(\delta_{s^*ss^*})\cdot\delta_s(f)\\ &= \psi_r\cdot\delta_{s^*s}(f) + \operatorname{Re}\phi(\delta_{s^*})\cdot\delta_s(f),\end{aligned}$$

so we have shown that

$$\operatorname{Re}\phi(\delta_{s^*})\cdot\delta_s(f)-[\delta_{s^*}\cdot\psi_r\cdot\delta_s(f)-\psi_r\cdot\delta_{s^*s}(f)]\leq 3\,\|\delta\phi\|\,. \tag{3.3}$$

Now, by replacing f with $\delta_{s^*}\cdot f$ in (3.3), we obtain

$$\operatorname{Re}\phi(\delta_{s^*})\cdot\delta_{ss^*}(f)-[\delta_{s^*}\cdot\psi_r\cdot\delta_{ss^*}(f)-\psi_r\cdot\delta_{s^*ss^*}(f)]\leq 3\,\|\delta\phi\|\,. \tag{3.4}$$

And by replacing f with $(-\delta_s^*)\cdot f$ in (3.3), we obtain

$$-3\,\|\delta\phi\|\leq\operatorname{Re}\phi(\delta_{s^*})\cdot\delta_{ss^*}(f)-[\delta_{s^*}\cdot\psi_r\cdot\delta_{ss^*}(f)-\psi_r\cdot\delta_{s^*ss^*}(f)]. \tag{3.5}$$

From (3.4) and (3.5), we have

$$|\operatorname{Re}\phi(\delta_{s^*})\cdot\delta_{ss^*}(f)-[\delta_{s^*}\cdot\psi_r\cdot\delta_{ss^*}(f)-\psi_r\cdot\delta_{s^*ss^*}(f)]|\leq 3\,\|\delta\phi\|\,,$$

since $\delta_{ss^*}\in\ell^1(E)$, we have

$$\phi(\delta_{s^*})\cdot\delta_{ss^*}=\phi(\delta_{s^*ss^*})=\phi(\delta_{s^*}),$$

thus

$$|\operatorname{Re}\phi(\delta_{s^*})(f)-[\delta_{s^*}\cdot\psi_r\cdot\delta_{ss^*}(f)-\psi_r\cdot\delta_{s^*ss^*}(f)]|\leq 3\,\|\delta\phi\|\,.$$

Since the idempotent ss^* is central, then $\delta_{ss^*}\in\ell^1(E)$ commute with elements of Λ and so with ψ_r and $\delta_{s^*ss^*}=\delta_{s^*}$, therefore we have

$$|\operatorname{Re}\phi(\delta_{s^*})(f)-[\delta_{s^*}\cdot\psi_r(f)-\psi_r\cdot\delta_{s^*}(f)]|\leq 3\,\|\delta\phi\|\,, \tag{3.6}$$

similarly by considering imaginary parts, we obtain ψ_i such that

$$|\operatorname{Im}\phi(\delta_{s^*})(f)-[\delta_{s^*}\cdot\psi_i(f)-\psi_i\cdot\delta_{s^*}(f)]|\leq 3\,\|\delta\phi\|\,. \tag{3.7}$$

By putting $\psi=\psi_r+i\psi_i$ and using (3.6) and (3.7), we obtain

$$|\phi(\delta_{s^*})(f)-[\delta_{s^*}\cdot\psi(f)-\psi\cdot\delta_{s^*}(f)]|\leq 6\,\|\delta\phi\|\,. \tag{3.8}$$

□

Theorem 3.4. *Let S be an inverse semigroup and let the idempotents of S are central. Then, $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S),\mathcal{X}')$ is a Banach space, where $\mathcal{X}=(\ell^1(S))^{(2n)}$, for every $n\in\mathbb{N}$.*

Proof. Let $\phi\in\mathcal{C}^1_{\ell^1(E)}(\ell^1(S),\mathcal{X}')$. We show that there exists a constant M and $\bar\psi\in\mathcal{C}^1_{\ell^1(E)}(\ell^1(S),\mathcal{X}')$ such that $\delta\bar\psi=\delta\phi$ and $\|\bar\psi\|\leq M\,\|\delta\phi\|$.

By Lemma 3.3 there exists a $\psi\in\mathcal{C}^1_{\ell^1(E)}(\ell^1(S),\mathcal{X}')$ such that

$$|\phi(\delta_s)(f)-[\delta_s\cdot\psi(f)-\psi\cdot\delta_s(f)]|\leq 6\,\|\delta\phi\|\,, \tag{3.9}$$

for every $s\in S$.

Now, we will show that ϕ is linear, for each $s \in S$ and $\lambda \in \mathbb{C}$, since $\lambda\delta_{ss^*} \in \ell^1(E)$, we have

$$\phi(\lambda\delta_s) = \phi(\lambda\delta_{ss^*} * \delta_s) = \lambda\delta_{ss^*} \cdot \phi(\delta_s) = \lambda\phi(\delta_{ss^*} * \delta_s) = \lambda\phi(\delta_s),$$

since ϕ is additive, we get $\phi(\lambda g) = \lambda\phi(g)$, for each $g \in \ell^1(S)$ of finite support. But, ϕ is continuous and functions of finite support are dense in $\ell^1(S)$, hence ϕ is linear.

Also by the same reason for each $g \in \ell^1(S)$ from (3.9) we have

$$|\phi(g)(f) - (g \cdot \psi - \psi \cdot g)(f)| \leq 6 \|\delta\phi\| \|g\| .$$

Define $\bar{\psi} \in \mathcal{C}^1_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$ by

$$\bar{\psi}(g) = \phi(g) - (g \cdot \psi - \psi \cdot g),$$

then $\delta\bar{\psi} = \delta\phi$ and $\left|\bar{\psi}(g)(f)\right| \leq 6 \|\delta\phi\|$, for every $g \in \ell^1(S)$ with $\|g\| \leq 1$ and $f \in \mathcal{X}$ with $\|f\| \leq 1$. So, $\left\|\bar{\psi}\right\| \leq 6 \|\delta\phi\|$ and by Proposition 1.4, $\operatorname{Im}\delta = \mathcal{B}^2_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$ is closed, which means that $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$ is a Banach space and this completes the proof. □

Corollary 3.5. *Let S be a Clifford semigroup and let E be the set of idempotent elements in S. Then, $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$ is a Banach space, where $\mathcal{X} = (\ell^1(S))^{(2n)}$, for every $n \in \mathbb{N}$.*

Proof. By [4, Theorem 4.2.1] every Clifford semigroup is an inverse semigroup, where its idempotents are central. So, by the previous theorem $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \mathcal{X}')$ is a Banach space. □

References

[1] M. Amini, Module amenability for semigroup algebras, *Semigroup Forum* **69** (2004) 243-254.

[2] M. Amini, D. E. Bagha, Weak Module amenability for semigroup algebras, *Semigroup Forum* **71** (2005) 18-26.

[3] J. Duncan, I. Namioka, Amenability of inverse semigroups and their semigroup algebras, *Proc. Roy. Soc. Edinburgh Sect. A* **80** (1978) 309-321.

[4] J. M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, New Series, 12, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.

[5] F. Gourdeau, A. Pourabbas and M. C. White, Simplicial cohomology of some semigroup algebras, *Canad. Math. Bull.* **50**(1) (2007) 56-70.

[6] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972), iii+96 pp.

[7] B. E. Johnson, Weak amenability of group algebras, *Bull. London Math. Soc.* **23** (1991) 281-284.
[8] A. Pourabbas, Second cohomology group of group algrbras with coefficients in iterated duals, *Proc. Amer. Math. Soc.* **132**(5) (2004) 1403-1410.

E. Nasrabadi

Department of Mathematics, Birjand University, Birjand 9717851367, Iran

Email: `enasrabadi@birjand.ac.ir`

A. Pourabbas

Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, Tehran 15914, Iran

Email: `arpabbas@aut.ac.ir`