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ON THE ISHIKAWA ITERATION PROCESS IN CAT(0) SPACES

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Abstract. In this paper, several Δ and strong convergence theorems are established for the Ishikawa iterations for nonexpansive mappings in the framework of CAT(0) spaces. Our results extend and improve the corresponding results given by many authors.

1. **Introduction**

Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C.$$

It has been shown that if X is uniformly convex then every nonexpansive mapping $T : C \to C$ has a fixed point (see Browder [2], cf. also Kirk [17]). In 1974, Ishikawa [16] introduced a new iteration procedure for approximating fixed points of pseudo-contractive compact mappings in Hilbert spaces as follows.

$$\text{(IS)} \quad x_1 \in C, \; x_{n+1} = t_n T[s_n T x_n + (1 - s_n)x_n] + (1 - t_n)x_n, \; n \geq 1,$$

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where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0,1]$ satisfying certain conditions. Note that the normal Mann iteration procedure [26],

$$\text{(M)} \qquad x_1 \in C, \;\; x_{n+1} = t_n T x_n + (1-t_n)x_n, \;\; n \geq 1,$$

where $\{t_n\}$ is a sequence in $[0,1]$, is a special case of the Ishikawa one. In 1993, Tan and Xu [33] showed weak and strong convergence of the Ishikawa iterations for nonexpansive mappings in uniformly convex Banach spaces. Precisely, they proved the following result.

Theorem 1.1. *Let X be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable, C be a bounded closed convex subset of X and $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is the iterative scheme defined by (IS) with conditions $(T1)$ $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$ $(T2)$ $\sum_{n=1}^{\infty}(1-t_n)s_n < \infty$, and $(T3)$ $\limsup_n s_n < 1$, then $\{x_n\}$ converges weakly to a fixed point of T.*

In 2002, Zhou et. al. [35] generalized Theorem 1.1 by removing condition $(T3)$ and weaken condition $(T2)$ to $(Z2)$ as the following result.

Theorem 1.2. *Let X be as in Theorem 1.1, C be a closed convex (not necessary bounded) subset of X and $T : C \to C$ be a nonexpansive mapping with nonempty fixed point set $F(T)$. If $\{x_n\}$ is the iterative scheme defined by (IS) with conditions $(Z1)$ $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$ $(Z2)$ $\sum_{n=1}^{\infty} \tau_n < \infty$ where $\tau_n = min\{t_n, 1-t_n\}s_n$, then $\{x_n\}$ converges weakly to a fixed point of T.*

In 2004, Zhou [34] improved Theorem 1.2 by weaken condition $(Z2)$ to $\sum_{n=1}^{\infty} t_n(1-t_n)s_n < \infty$ as the following result.

Theorem 1.3. *Let X, C, T be as in Theorem 1.2. If $\{x_n\}$ is the iterative scheme defined by (IS) with conditions (1) $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$ (2) $\sum_{n=1}^{\infty} t_n(1-t_n)s_n < \infty$, then $\{x_n\}$ converges weakly to a fixed point of T.*

The purpose of this paper is to study the Ishikawa iteration process in the framework of CAT(0) spaces and give analogs of some results in [34], specially to Theorem 1.3, in this setting. It is worth mentioning that our results also extend some results in [11].

2. **CAT(0) spaces**

A metric space X is a CAT(0) space, if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $\mathbb{R}-$trees (see [1]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [14]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [1]. Burago et. al. [5] contains a somewhat more elementary treatment, and Gromov [15] a deeper study.

Fixed point theory in a CAT(0) space was first studied by Kirk (see [18] and [19]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and many of papers have appeared (see e. g., [20, 8, 6, 10, 13, 7, 9, 11, 32, 12, 22, 23, 31, 27, 28, 29]). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(κ) space with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \geq \kappa$ (see [1], p. 165).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$, for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space*, if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic*, if there is exactly one geodesic joining x and y, for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex*, if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$, for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space, if all geodesic triangles satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the CAT(0) *inequality*, if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

Let $x, y \in X$, by Lemma 2.1(iv) of [11], for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1 - t)d(x, y). \tag{2.1}$$

From now on, we will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.1). By using this notation Dhompongsa and Panyanak [11] obtained the following lemma which will be used frequently in the proof of our results.

Lemma 2.1. *Let X be a CAT(0) space. Then,*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z), \tag{2.2}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

The following lemma is also needed for proving our main theorem which can be found in [19].

Lemma 2.2. *Let p, x, y be points of a CAT(0) space X, and let $\alpha \in [0, 1]$. Then,*

$$d((1 - \alpha)p \oplus \alpha x, (1 - \alpha)p \oplus \alpha y) \leq \alpha d(x, y).$$

If x, y_1, y_2 are points in a CAT(0) space and if $y_0 = \frac{1}{2}y_1 \oplus \frac{1}{2}y_2$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \tfrac{1}{2}d(x, y_1)^2 + \tfrac{1}{2}d(x, y_2)^2 - \tfrac{1}{4}d(y_1, y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [4]. In fact, (cf. [1], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies (CN).

The following lemma is a generalization of the (CN) inequality which can be found in [11].

Lemma 2.3. *Let (X, d) be a CAT(0) space. Then,*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2, \tag{2.3}$$

for all $t \in [0, 1]$ and $x, y, z \in X$.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [10] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

We now give the definition of $\Delta-$convergence.

Definition 2.4. ([21, 25]) A sequence $\{x_n\}$ in a CAT(0) space X is said to $\Delta-$*converges* to $x \in X$, if x is the unique asymptotic center of $\{u_n\}$, for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta-\lim_n x_n = x$ and call x the $\Delta-$*limit* of $\{x_n\}$.

The notion of $\Delta-$convergence was first studied in a general metric space by Lim [25]. Kirk and Panyanak [21] specialized Lim's concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogues in this setting. Since then the notion of $\Delta-$convergence has been widely studied and a number of papers have appeared (see e. g., [11, 12, 22, 23, 27, 28]).

Definition 2.5. Let C be a nonempty subset of a CAT(0) space X and $T : C \to X$ be a mapping. T is called *nonexpansive*, if for each $x, y \in C$,

$$d(Tx, Ty) \le d(x, y).$$

A point $x \in C$ is called a fixed point of T, if $x = Tx$. We denote with $F(T)$ the set of fixed points of T.

For arbitrary $x_1 \in C$, the Ishikawa iteration scheme $\{x_n\}$ is defined by

$$\text{(ISCAT)} \qquad \begin{cases} y_n = s_n T x_n \oplus (1 - s_n) x_n \\ x_{n+1} = t_n T y_n \oplus (1 - t_n) x_n, \ \ n \ge 1, \end{cases}$$

where $\{s_n\}$ and $\{t_n\}$ are sequences in $[0, 1]$.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.6. ([21]) *Every bounded sequence in a complete CAT(0) space always has a* $\Delta-$*convergent subsequence.*

Lemma 2.7. ([9]) *If* C *is a closed convex subset of a complete CAT(0) space and if* $\{x_n\}$ *is a bounded sequence in* C*, then the asymptotic center of* $\{x_n\}$ *is in* C*.*

Lemma 2.8. ([21]) *Let* C *be a closed convex subset of a complete CAT(0) space* X*, and let* $T : C \to X$ *be a nonexpansive mapping. Then, the conditions* $\{x_n\}$ $\Delta-$*converges to* x *and* $d(x_n, Tx_n) \to 0$*, imply* $x \in C$ *and* $Tx = x$*.*

The following lemma is also needed for proving our main results which can be found in [34, 35].

Lemma 2.9. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 + b_n)a_n \text{ for all } n \geq 1. \tag{2.4}$$

If $\sum_{n=1}^{\infty} b_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exists. In particular, if there is a subsequence of* $\{a_n\}$ *which converges to* 0*, then* $\lim_{n\to\infty} a_n = 0$*.*

3. Main results

Lemma 3.1. *Let* C *be a nonempty convex subset of a complete CAT(0) space* X *and* $T : C \to C$ *be a nonexpansive mapping. If* $\{x_n\}$ *is defined by (ISCAT), then*

$$d(Tx_{n+1}, x_{n+1}) \leq [1 + 4t_n(1 - t_n)s_n]d(Tx_n, x_n) \text{ for all } n \geq 1.$$

Proof. By Lemma 2.2 and the nonexpansiveness of T, we have

$$\begin{aligned} d\left(Tx_{n+1}, x_{n+1}\right) &\leq d\left(Tx_{n+1}, T(t_nTx_n \oplus (1 - t_n)x_n)\right) \\ &\quad + d\left(T(t_nTx_n \oplus (1 - t_n)x_n), Tx_n\right) \\ &\quad + d\left(Tx_n, t_nTx_n \oplus (1 - t_n)x_n\right) \\ &\quad + d\left(t_nTx_n \oplus (1 - t_n)x_n, x_{n+1}\right) \\ &\leq 2d\left(t_nTx_n \oplus (1 - t_n)x_n, x_{n+1}\right) \\ &\quad + d\left(t_nTx_n \oplus (1 - t_n)x_n, x_n\right) \end{aligned}$$

$$
\begin{aligned}
&+ (1-t_n)d(Tx_n, x_n) \\
&\le 2t_n d\left(Tx_n, Ty_n\right) + d(Tx_n, x_n) \\
&\le 2t_n d(x_n, y_n) + d(Tx_n, x_n) \\
&\le (1+2t_n s_n) d(Tx_n, x_n)
\end{aligned}
$$

and hence,

$$
(1-t_n)d\left(Tx_{n+1}, x_{n+1}\right) \le [1-t_n+2t_n(1-t_n)s_n]d(Tx_n, x_n). \tag{3.1}
$$

On the other hand,

$$
\begin{aligned}
d\left(Tx_{n+1}, x_{n+1}\right) &\le d\left(Tx_{n+1}, T(t_nTy_n \oplus (1-t_n)y_n)\right) \\
&+ d\left(T(t_nTy_n \oplus (1-t_n)y_n), Ty_n\right) \\
&+ d\left(Ty_n, t_nTy_n \oplus (1-t_n)y_n\right) \\
&+ d\left(t_nTy_n \oplus (1-t_n)y_n, x_{n+1}\right) \\
&\le 2d\left(t_nTy_n \oplus (1-t_n)y_n, x_{n+1}\right) \\
&+ d\left(t_nTy_n \oplus (1-t_n)y_n, y_n\right) \\
&+ (1-t_n)d(Ty_n, y_n) \\
&\le 2(1-t_n)d(x_n, y_n) + d(Ty_n, y_n) \\
&\le 2(1-t_n)d(x_n, y_n) + d(Ty_n, Tx_n) + d(Tx_n, y_n) \\
&\le 2(1-t_n)d(x_n, y_n) + d(y_n, x_n) + d(Tx_n, y_n) \\
&\le 2(1-t_n)s_n d(x_n, Tx_n) \\
&+ s_n d(Tx_n, x_n) + (1-s_n)d(Tx_n, x_n) \\
&\le (1+2(1-t_n)s_n)d(Tx_n, x_n).
\end{aligned}
$$

Thus,

$$
t_n d\left(Tx_{n+1}, x_{n+1}\right) \le [t_n + 2t_n(1-t_n)s_n]d(Tx_n, x_n). \tag{3.2}
$$

Combining (3.1) with (3.2), we can obtain the desired result. □

Lemma 3.2. *Let C be a nonempty closed convex (not necessarily bounded) subset of a complete CAT(0) space X and $T : C \to C$ be a nonexpansive mapping with nonempty fixed point set $F(T)$ and let $\{x_n\}$ be defined by (ISCAT) with the restrictions that $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$ and $\sum_{n=1}^{\infty} t_n(1-t_n)s_n < \infty$. Then,*

$$
\lim_{n\to\infty} d(Tx_n, x_n) = 0.
$$

Proof. It follows from Lemmas 2.9 and 3.1 that $\lim_n d(Tx_n, x_n)$ exists. Fix $p \in F(T)$, by Lemma 2.3, we have

$$\begin{aligned} d(x_{n+1}, p)^2 &= d(t_n Ty_n \oplus (1-t_n)x_n, p)^2 \\ &\leq t_n d(Ty_n, p)^2 + (1-t_n)d(x_n, p)^2 - t_n(1-t_n)d(Ty_n, x_n)^2 \\ &\leq t_n d(y_n, p)^2 + (1-t_n)d(x_n, p)^2 - t_n(1-t_n)d(Ty_n, x_n)^2. \end{aligned}$$

That is,

$$d(x_{n+1}, p)^2 \leq t_n d(y_n, p)^2 + (1-t_n)d(x_n, p)^2 - t_n(1-t_n)d(Ty_n, x_n)^2. \tag{3.3}$$

On the other hand,

$$\begin{aligned} d(y_n, p)^2 &= d(s_n Tx_n \oplus (1-s_n)x_n, p)^2 \\ &\leq s_n d(Tx_n, p)^2 + (1-s_n)d(x_n, p)^2 - s_n(1-s_n)d(Tx_n, x_n)^2 \\ &\leq s_n d(x_n, p)^2 + (1-s_n)d(x_n, p)^2 - s_n(1-s_n)d(Tx_n, x_n)^2 \\ &= d(x_n, p)^2 - s_n(1-s_n)d(Tx_n, x_n)^2 \\ &\leq d(x_n, p)^2. \end{aligned}$$

Thus,

$$d(y_n, p)^2 \leq d(x_n, p)^2. \tag{3.4}$$

From (3.3) and (3.4), we get

$$d(x_{n+1}, p)^2 \leq d(x_n, p)^2 - t_n(1-t_n)d(Ty_n, x_n)^2. \tag{3.5}$$

This implies

$$\sum_{n=1}^{\infty} t_n(1-t_n)d(Ty_n, x_n)^2 \leq d(x_1, p)^2 < \infty. \tag{3.6}$$

Since $\sum_{n=1}^{\infty} t_n(1-t_n)s_n < \infty$, combining this with (3.6) we have

$$\sum_{n=1}^{\infty} t_n(1-t_n)\left[d(Ty_n, x_n)^2 + s_n\right] < \infty.$$

Since $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$, we have

$$\liminf_{n\to\infty} \left[d(Ty_n, x_n)^2 + s_n\right] = 0.$$

There exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{k\to\infty} d(Ty_{n_k}, x_{n_k}) = 0 \quad \text{and} \quad \lim_{k\to\infty} s_{n_k} = 0. \tag{3.7}$$

On the other hand,

$$\begin{aligned} d(Tx_{n_k}, x_{n_k}) &\leq d(Tx_{n_k}, Ty_{n_k}) + d(Ty_{n_k}, x_{n_k}) \\ &\leq d(x_{n_k}, y_{n_k}) + d(Ty_{n_k}, x_{n_k}) \\ &= s_{n_k} d(Tx_{n_k}, x_{n_k}) + d(Ty_{n_k}, x_{n_k}), \end{aligned}$$

that is,

$$(1 - s_{n_k}) d(Tx_{n_k}, x_{n_k}) \leq d(Ty_{n_k}, x_{n_k}). \tag{3.8}$$

By (3.7) and (3.8), we have

$$\lim_{k \to \infty} d(Tx_{n_k}, x_{n_k}) = 0.$$

But, since $\lim_n d(Tx_n, x_n)$ exists, we have $\lim_n d(Tx_n, x_n) = 0$ as desired. □

Now, we are ready to prove our main result.

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T : C \to C$ be a nonexpansive mapping with nonempty fixed point set $F(T)$ and let $\{x_n\}$ be defined by (ISCAT) with the restrictions that $\sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$ and $\sum_{n=1}^{\infty} t_n(1 - t_n)s_n < \infty$. Then, $\{x_n\}$ $\Delta-$converges to a fixed point of T.*

Proof. By Lemma 3.2, $\lim_{n \to \infty} d(Tx_n, x_n) = 0$. We now let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subset F(T)$. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.6 and 2.7 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. Since $\lim_n d(Tv_n, v_n) = 0$, then $v \in F(T)$ by Lemma 2.8 and $\lim_n d(x_n, v)$ exists by (3.5). We claim that $u = v$. Suppose not, by the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_n d(v_n, v) &< \limsup_n d(v_n, u) \\ &\leq \limsup_n d(u_n, u) \\ &< \limsup_n d(u_n, v) \\ &= \limsup_n d(x_n, v) \\ &= \limsup_n d(v_n, v) \end{aligned}$$

a contradiction, and hence $u = v \in F(T)$. To show that $\{x_n\}$ $\Delta-$converges to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 2.6 and 2.7 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in F(T)$. We can complete the proof by showing that $x = v$. Suppose not, since $\{d(x_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_n d(v_n, v) &< \limsup_n d(v_n, x) \\ &\leq \limsup_n d(x_n, x) \\ &< \limsup_n d(x_n, v) \\ &= \limsup_n d(v_n, v)\end{aligned}$$

a contradiction, and hence the conclusion follows. □

Finally, we give a strong convergence theorem of the Ishikawa iteration which is an analog of Theorem 2.4 of [34]. Recall that a mapping $T : C \to C$ is said to satisfy *Condition I* [30], if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in C$, where $d(x, F(T)) = \inf_{z \in F(T)} d(x, z)$.

Theorem 3.4. *Let X, C, T and $\{x_n\}$ be as in Theorem 3.3. Suppose that T satisfies Condition I. Then, $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Condition I, we have

$$d(x_n, Tx_n) \geq f\left(d(x_n, F(T))\right) \quad \text{for all} \quad n \geq 1.$$

It follows from (3.5) that the sequence $\{d(x_n, F(T))\}$ is decreasing, and hence $\lim_n d(x_n, F(T)) = 0$ by Lemma 3.2. We can prove, by using a standard argument, that $\{x_n\}$ converges strongly to a fixed point of T. □

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