BEST PROXIMITY PAIR AND COINCIDENCE POINT THEOREMS FOR NONEXPANSIVE SET-VALUED MAPS IN HILBERT SPACES

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Abstract. This paper is concerned with the best proximity pair problem in Hilbert spaces. Given two subsets $A$ and $B$ of a Hilbert space $H$ and the set-valued maps $F : A \to 2^B$ and $G : A_0 \to 2^{A_0}$, where $A_0 = \{x \in A : \|x - y\| = d(A, B) \text{ for some } y \in B\}$, best proximity pair theorems provide sufficient conditions that ensure the existence of an $x_0 \in A$ such that 

$$d(G(x_0), F(x_0)) = d(A, B).$$

1. Introduction

Let $(M, d)$ be a metric space and let $A$ and $B$ be nonempty subsets of $M$. Let $d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$. Let

$$B_0 := \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\},$$

and

$$A_0 := \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}.$$

Let $G : A_0 \to 2^{A_0}$ and $F : A \to 2^B$ be set valued maps. $(G(x_0), F(x_0))$ is called a best proximity pair, if $d(G(x_0), F(x_0)) = d(A, B)$. Best proximity pair theorems analyse the conditions on $F$, $G$, $A$ and $B$ under which the problem of minimizing the real valued function $x \to d(G(x), F(x))$


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has a solution. In the setting of normed spaces and hyperconvex metric spaces, the best proximity pair problem has been studied by many authors, see [1, 2, 3, 5, 7, 8].

Let $H$ be a Hilbert space and $A, B \subseteq H$. It is well-known that if $A$ and $B$ are compact subsets of $M$, then there exist $a_0 \in A$ and $b_0 \in B$ such that $d(A, B) = d(a_0, b_0)$. Therefore, in this case

$$d(A, B) = 0 \iff A \cap B \neq \emptyset.$$

Let $M$ be a metric space and let $M$ denote the family of nonempty, closed bounded subsets of $M$. Let $A, B \in M$. The Hausdorff metric $d_H$ on $M$ defined by

$$d_H(A, B) = \inf \{ \epsilon > 0 : A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A) \},$$

where $N_\epsilon(A)$ denotes the closed $\epsilon$-neighborhood of $A$, that is, $N_\epsilon(A) = \{ x \in M : d(x, A) \leq \epsilon \}$. Let $X$ and $Y$ be topological spaces with $C \subseteq Y$. Let $G : X \to 2^Y$ be a set-valued map with nonempty values. The inverse image of $C$ under $G$ is

$$G^{-}(C) = \{ x \in X : G(x) \cap B \neq \emptyset \}.$$

A set-valued map $F : A \to 2^B$ is said to be nonexpansive, if for each $x, y \in A$

$$d_H(F(x), F(y)) \leq \| x - y \|.$$

Given a nonempty closed convex subset $A$ of a Hilbert space $H$, $P_A$ will always denote the nearest point projection of $H$ onto $A$. We will use the well-known fact that $P_A$ is nonexpansive and so is continuous.

**Lemma 1.1.** ([5, Lemma 3.1]) Let $A$ be a nonempty closed convex subset of a Hilbert space $H$. If $C$ and $D$ are nonempty closed and bounded subsets of $H$, then

$$d_H(P_A(C), P_A(D)) \leq d_H(C, D).$$

Let $(X, \| \cdot \|)$ be a reflexive Banach space and $A \subseteq X$ be nonempty, closed, convex and bounded. It is well-known that for each $x \in X$, $P_A(x) \neq \emptyset$. Here we give the proof for the completeness. For each $n \in \mathbb{N}$, let $A_n(x) = \{ y \in A : d(x, y) \leq d(x, A) + \frac{1}{n} \}$. Notice that $(A_n(x))$ is a decreasing sequence of nonempty closed, convex bounded subsets of the reflexive Banach space $X$, so by Šmulina theorem we have [4, page 433]

$$P_A(x) = \bigcap_{n=1}^{\infty} A_n(x) \neq \emptyset.$$
Lemma 1.2. ([5, Lemma 3.2]) Let $X$ be a reflexive Banach space. Let $A$ be a nonempty bounded closed convex subset of $X$, and let $B$ be a nonempty closed convex subset of $X$. Then, $A_0$ and $B_0$ are nonempty and satisfy

$$P_B(A_0) \subseteq B_0 \text{ and } P_A(B_0) \subseteq A_0.$$  

Recall that a Banach space $X$ is uniformly convex, if given $\epsilon > 0$ there is a $\delta > 0$ such that whenever $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Theorem 1.3. ([6]) Let $X$ be a uniformly convex Banach space, Let $K$ be a bounded, closed and convex subset of $X$, and suppose $F : K \to 2^K$ is a compact-valued, nonexpansive set-valued map. Then, $F$ has a fixed point.

2. Main results

We first present a coincidence point theorem for nonexpansive set-valued self maps.

Theorem 2.1. Let $H$ be a Hilbert space and $K$ be a closed, bounded convex subset of $H$. Let $F : K \to 2^K$ be a nonexpansive set-valued map with nonempty compact values. Let $G : K \to 2^K$ be an onto, set-valued map for which $G^{-1}(C)$ is compact for each compact set $C \subseteq K$. Assume that for each compact subsets $C$ and $D$ of $K$

$$d_H(G^{-1}(C), G^{-1}(D)) \leq d_H(C, D).$$

Then, there exists a $x_0 \in K$ with

$$F(x_0) \cap G(x_0) \neq \emptyset.$$  

Proof. Since

$$F(x_0) \cap G(x_0) \neq \emptyset \Leftrightarrow x_0 \in G^{-1}(F(x_0)) = \{x \in H : G(x) \cap F(x_0) \neq \emptyset\},$$

then, the conclusion follows, if we show that the set-valued map $J(x) = G^{-1}(F(x)) : K \to 2^K$ has a fixed point. Since $G$ is onto, then $J(x) \neq \emptyset$. For each $x \in K$, since $F(x)$ is compact, then $J(x) = G^{-1}(F(x))$ is compact. Now, we show that $J$ is nonexpansive. For each $x, y \in K$ we have

$$d_H(J(x), J(y)) = d_H(G^{-1}(F(x)), G^{-1}(F(y))) \leq d_H(F(x), F(y)) \leq \|x - y\|. $$
Therefore, \( J \) satisfies all conditions of Theorem 1.3 and so has a fixed point. \( \square \)

Now, we obtain a best proximity pair theorem for nonexpansive set-valued maps in Hilbert spaces.

**Theorem 2.2.** Let \( H \) be a Hilbert space. Let \( A \) be a nonempty bounded closed convex subset of \( H \), and let \( B \) be a nonempty closed convex subset of \( H \). Let \( F : A \to 2^B \) be a nonexpansive set-valued map with nonempty compact values. Let \( G : A_0 \to 2^{A_0} \) be an onto set-valued map for which \( G^{-1}(C) \) is compact for each compact set \( C \subseteq A_0 \). Assume that for each compact subsets \( C \) and \( D \) of \( A_0 \)
\[
d_H(G^{-1}(C), G^{-1}(D)) \leq d_H(C, D).
\]
Assume that \( F(A_0) \subseteq B_0 \). Then, there exists a \( x_0 \in A_0 \) such that
\[
d(G(x_0), F(x_0)) = d(A, B).
\]

**Proof.** By Lemma 1.2, \( A_0 \) and \( B_0 \) are nonempty. Let us show that \( A_0 \) is closed. To this end, let \( x_n \in A_0 \) be a convergent sequence, say, \( x_n \to x_0 \in A \). Then, for each \( n \in \mathbb{N} \), there exists \( y_n \in B \) such that \( d(x_n, y_n) = d(A, B) \). Thus, \( \{y_n\} \) is a bounded sequence in \( B \) (note that \( \{x_n\} \) is bounded). Since bounded subsets of a reflexive Banach space are weakly sequentially compact [4, Theorem 28, page 68], then passing to a subsequence, if necessary, we may assume that \( (y_n) \) converges weakly, say to \( y_0 \in B \). Since \( \|\cdot\| \) is weakly lower semicontinuous, then we get
\[
\|x_0 - y_0\| \leq \lim_{n \to \infty} \|x_n - y_n\| = d(A, B).
\]
Therefore, \( \|x_0 - y_0\| = d(A, B) \), and so \( x_0 \in A_0 \). From Lemma 1.2, \( P_A(B_0) \subseteq A_0 \) and by Lemma 1.1,
\[
d_H(P_A(F(x)), P_A(F(y))) \leq d_H(F(x), F(y)) \leq \|x - y\|.
\]
Then, the map \( P_A(F(.)) : A_0 \to A_0 \) is a nonexpansive set-valued map. Moreover, \( A_0 \) is a nonempty closed bounded convex subsets of \( H \), and for each \( x \in A_0 \), \( P_A(F(x)) \) is a compact subset of \( A_0 \) (note \( F(x) \) is compact and \( P_A \) is continuous). Hence, by Theorem 2.1 there exists a \( x_0 \in A_0 \) such that
\[
P_A(F(x_0)) \cap G(x_0) \neq \emptyset.
Let \( z_0 \in P_A(F(x_0)) \cap G(x_0) \), then there exists \( y_0 \in F(x_0) \) so that \( z_0 = P_{A_0}(y_0) \). Since \( x_0 \in A_0 \) and \( y_0 \in F(x_0) \subseteq B_0 \), there exists \( a_0 \in A_0 \) such that \( d(a_0, y_0) = d(A, B) \). Therefore,
\[
d(A, B) \leq d(G(x_0), F(x_0)) \leq d(z_0, F(x_0)) \leq d(P_{A_0}(y_0), y_0) \leq d(a_0, y_0) = d(A, B)
\]
Thus,
\[
d(G(x_0), F(x_0)) = d(A, B).
\]

**Remark 2.3.** Let \( A \) be a nonempty bounded, closed convex subset of \( H \). Let \( G : A_0 \rightarrow A_0 \) be an onto isometry. We show that \( G \) satisfies all the conditions of Theorem 2.2. Let \( C \) be a compact subset of \( A_0 \). Since \( G \) is an isometry, then \( G^{-1}(C) = G^{-1}(C) \) is compact (note \( G^{-1} \) is isometry and so is continuous). Since \( G : A_0 \rightarrow A_0 \) is isometry, then \( d_H(G^{-1}(C), G^{-1}(D)) = d_H(C, D) \), for compact subsets \( C \) and \( D \) of \( A_0 \).

If we take \( G = I \), Theorem 2.2 reduces to Theorem 3.3 of Kirk, Reich and Veeramani [5].

**Theorem 2.4.** Let \( H \) be a Hilbert space. Let \( A \) be a nonempty bounded closed convex subset of \( H \), and let \( B \) be a nonempty closed convex subset of \( H \). Let \( F : A \rightarrow 2^B \) be a nonexpansive set-valued map with nonempty compact values. Assume that \( F(A_0) \subseteq B_0 \). Then, there exists a \( x_0 \in A_0 \) such that
\[
d(x_0, F(x_0)) = d(A, B).
\]

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