THE UNIT SUM NUMBER OF DISCRETE MODULES

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Abstract. In this paper, we show that every element of a discrete module is a sum of two units if and only if its endomorphism ring has no factor ring isomorphic to \( \mathbb{Z}_2 \). We also characterize unit sum number equal to two for the endomorphism ring of quasi-discrete modules with finite exchange property.

1. Introduction

The study of rings generated additively by their units seems to have its beginning in 1954 with the paper by Zelinsky [14] when he showed that if \( V \) is any (finite or infinite-dimensional) vector space over a division ring \( D \), then every linear transformation is the sum of two automorphisms unless \( \dim V = 1 \) and \( D = \mathbb{Z}_2 \) is the field of two elements. Interest in this topic increased recently after Goldsmith, Pabst and Scott defined the unit sum number in [4]. Zelinsky’s result motivated Skornjakov to ask in [11, Problem 31, p. 167], if every element in a (von Neumann) regular ring \( R \) can be expressed as sum of fixed (and finite) number of units. Of course one needs to add some conditions ensuring that \( \mathbb{Z}_2 \) is not a factor ring (for example, \( 1/2 \in R \)) to exclude the exceptional case already noted in the result of Zelinsky. Vámos in [13] showed that if \( R \) is such a regular ring, then \( R \) is 2 – good (for the definition see


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(13), if it is strongly regular and every element can indeed be written as a sum of a finite number of units, if $R$ is (right) regular. Vámos also proved that every element of a regular right self-injective ring is a sum of two units, if the ring has no nonzero corner ring which is Boolean. Recently, Ashish and Dinesh in [6] proved that every element of a right self-injective ring is a sum of two units if and only if it has no factor ring isomorphic to $\mathbb{Z}_2$. They extended this result to endomorphism rings of right quasi-continuous modules with finite exchange property. We investigate whether these results are true for discrete modules. Discrete modules were first studied by Takeuchi [12] and he called them codirect modules with condition (I) [2, see Remark 27.18]. Mohamed and Singh [9] studied direct projective and lifting modules and called them dual-continuous modules. They studied the basic properties and the endomorphism ring of a discrete module. Also, a decomposition theorem for discrete modules was obtained by Mohamed and Singh [9] and later improved by Mohamed and Müller [8]. In this paper we prove that every element of a discrete module is a sum of two units if and only if no factor ring of the endomorphism ring is isomorphic to $\mathbb{Z}_2$. Then, we extend this result to the endomorphism ring of quasi-discrete modules with finite exchange property.

2. Definitions

All rings $R$ in this paper are assumed to be associative and will have an identity element. We say that $R$ has the $n$-sum property, for a positive integer $n$, if every element of $R$ can be written as a sum of exactly $n$ units of $R$. The unit sum number of a ring, denoted by $usn(R)$, is the least integer $n$, if any such integer exists, such that $R$ has the $n$-sum property. If $R$ has an element that is not a sum of units, then we set $usn(R)$ to be $\infty$, and if every element of $R$ is a sum of units but $R$ does not have $n$-sum property, for any $n$, then we set $usn(R) = 0$. Clearly, $usn(R) = 1$ if and only if $R$ is the trivial ring with $0 = 1$. The unit sum number of a module $M$, denoted by $usn(M)$, is the unit sum number of its endomorphism ring.

A submodule $A$ of a module $M$ is called small in $M$ (denoted by $A \ll M$), if $A + B \neq M$ for any proper submodule $B$ of $M$. A module $H$ is called hollow, if every proper submodule of $H$ is small. Let $A$ and $B$ be submodules of $M$. $B$ is called a supplement of $A$, if it is minimal with the property $A + B = M$. $L$ is called a supplement submodule,
if \( L \) is a supplement of some submodule of \( M \). A module \( M \) is called supplemented, if for any two submodules \( A \) and \( B \) with \( A + B = M \), \( B \) contains a supplement of \( A \). A supplemented module \( M \) is called strongly discrete, if it is self-projective.

**Definition 2.1.** For a module \( M \), consider the following conditions:

(\( D_1 \)) For every submodule \( A \) of \( M \), there exists a decomposition \( M = M_1 \oplus M_2 \) such that \( M_1 \leq A \) and \( M_2 \cap A \) is small in \( M \).

(\( D_2 \)) If \( A \leq M \) such that \( M/A \) is isomorphic to a summand of \( M \), then \( A \) is a summand of \( M \).

(\( D_3 \)) If \( M_1 \) and \( M_2 \) are summands of \( M \) with \( M_1 + M_2 = M \), then \( M_1 \cap M_2 \) is a summand of \( M \).

\( M \) is called discrete, if it has (\( D_1 \)) and (\( D_2 \)); \( M \) is called quasi-discrete, if it has (\( D_1 \)) and (\( D_3 \)).

**Definition 2.2.** A module \( M \) is said to have the (finite) exchange property, if for any (finite) index set \( I \), whenever \( M \oplus N = \bigoplus_{i \in I} A_i \), for modules \( N \) and \( A_i \), then \( M \oplus N = M \oplus \left( \bigoplus_{i \in I} B_i \right) \), for submodules \( B_i \leq A_i \). A module \( M \) is said to have the lifting property, if for any index set \( I \) and any submodule \( X \) of \( M \), if \( M/X = \bigoplus_{i \in I} A_i \), then there exists a decomposition \( M = M_0 \oplus \left( \bigoplus_{i \in I} M_i \right) \) such that:

(i) \( M_0 \leq X \),
(ii) \( M_i = M/M_i = A_i \),
(iii) \( X \cap \left( \bigoplus_{i \in I} M_i \right) \ll M \).

3. The unit sum number of discrete modules

For a ring \( R \), \( J(R) \) will denote the Jacobson radical of \( R \). Before discussing the main results we need some properties of the unit sum number of rings and modules.

**Lemma 3.1.** Let \( D \) be a division ring. If \( |D| \geq 3 \), then usn\((D) = 2\), whereas, if \( |D| = 2 \), that is, \( D = \mathbb{Z}_2 \) the field of two elements, then usn\((\mathbb{Z}_2) = \omega \).

*Proof.* See [13, Lemma 2]. \( \square \)

**Lemma 3.2.** Let \( R \) be a ring and let \( I \) be an ideal of \( R \). Then, usn\((R/I) \leq \text{usn}(R) \) with equality, if \( I \) is contained in the Jacobson radical of \( R \).

*Proof.* See [13, Lemma 2]. \( \square \)
Remark 3.3. From Lemma 1 and Lemma 2 it is clear that if \( R \) is a local ring which has no factor ring isomorphic to \( \mathbb{Z}_2 \), then \( \text{usn}(R) = 2 \).

Lemma 3.4. If the ring \( R_i \), for every \( i \in I \), has the \( n \)-sum property, then so has the ring direct product \( \prod_{i \in I} R_i \).

Proof. See [4, 1.2]. \( \square \)

Lemma 3.5. Let \( R \) be a nonzero Boolean ring with more than two elements. Then, \( \text{usn}(R) = \infty \).

Proof. Since a nonzero Boolean ring with more than two elements has a factor ring isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), the result follows. \( \square \)

Theorem 3.6. Let \( M \) be a discrete \( R \)-module and \( S = \text{End}_R(M) \). The following conditions are equivalent:

(1) Every element of \( S \) is a sum of two units.
(2) The identity element of \( S \) is a sum of two units.
(3) \( S \) has no factor ring isomorphic to \( \mathbb{Z}_2 \).

Proof. The results (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are obvious. Now, we show (3) \( \Rightarrow \) (1).

We know that a discrete module \( M \) has a decomposition, unique up to isomorphism, \( M = \oplus M_i \), where the \( M_i \)'s, have local endomorphism rings. Since \( S \) has no factor ring isomorphic to \( \mathbb{Z}_2 \) and \( \text{End}_R(M) = \text{End}(\oplus_{i \in I} M_i) \cong \prod_{i \in I} \text{End}(M_i) \), none of \( \text{End}(M_i) \) has a factor ring isomorphic to \( \mathbb{Z}_2 \). Let, for \( i \in I \), \( T_i = \text{End}(M_i) \). Thus, \( T_i \) is a local ring which has no factor ring isomorphic to \( \mathbb{Z}_2 \). Therefore, by Remark 3.3, for each \( i \), \( \text{usn}(T_i) = 2 \). Now, by Lemma 3.4, it is clear that \( \text{usn}(M) = 2 \).

Let \( M \) be a projective \( R \)-module with lifting property, then [1] gives that \( M \) is supplemented. Hence, by [10, Lemma 2.3], \( M \) is a semi-perfect \( R \)-module and so by [7, Corollary 4.43] it is a discrete module. Also, if \( R \) is a perfect ring and \( M \) is a quasi-projective \( R \)-module, then by [5, Proposition 2.5] we know that \( M \) is again a discrete module. Therefore, we have:

Corollary 3.7. Let \( M \) be an \( R \)-module and \( S = \text{End}_R(M) \). If \( M \) is a strongly discrete module or a projective module with lifting property
or if $R$ is a perfect ring and $M$ is a quasi-projective $R$-module, then $\text{usn}(M) = 2$ if and only if $S$ has a factor ring isomorphic to $\mathbb{Z}_2$.

**Proof.** As mentioned above, in all cases $M$ is a discrete module and therefore the result follows at once from Theorem 8. $\square$

**Theorem 3.8.** Let $M$ be a nonzero discrete $R$-module and $S = \text{End}_R(M)$. Then, the unit sum number of $M$ is 2, $\omega$ or $\infty$. Moreover,

1. $\text{usn}(M) = 2$ if and only if $S$ has no factor ring isomorphic to a nonzero Boolean ring.
2. $\text{usn}(M) \geq \omega$, if $S$ has a factor ring isomorphic to $\mathbb{Z}_2$. Further, if $S$ has a factor ring isomorphic to a nonzero Boolean ring with more than two elements, then $\text{usn}(M) = \infty$.

**Proof.** (1) Since $\mathbb{Z}_2$ is a homomorphic image of every nonzero Boolean ring, the result follows from Theorem 3.6.

(2) Let $S$ have a factor ring isomorphic to $\mathbb{Z}_2$, i.e., $S/I \cong \mathbb{Z}_2$. But then, since $\text{usn}(S/I) \leq \text{usn}(S)$, it follows that $\text{usn}(S) \geq \omega$. Now, if $S$ has a factor isomorphic to a nonzero Boolean ring with more than two elements, then Lemma 3.5 implies $\text{usn}(M) = \infty$. $\square$

**Remark 3.9.** Note that in Theorem 8, if $S$ has a factor ring isomorphic to $\mathbb{Z}_2$, then $\text{usn}(M) \geq \omega$. Further, if $S$ has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $\text{usn}(M) = \infty$.

**Theorem 3.10.** Let $M$ be a quasi-discrete $R$-module with finite exchange property and $S = \text{End}_R(M)$. Then, every element of $S$ is a sum of two units if and only if no factor ring of $S$ is isomorphic to $\mathbb{Z}_2$.

**Proof.** Suppose that no factor ring of $S$ is isomorphic to $\mathbb{Z}_2$. Let $\nabla = \{f \in S : \text{Im}f \ll M\}$. It is easy to check that $\nabla$ is an ideal of $S$. By [7, 5.7] $\overline{S} = S/\nabla \cong S_1 \oplus S_2$, where $S_1$ is a regular ring and $S_2$ is a reduced ring. Moreover, $\overline{S}$ has no non-zero nilpotent element and every idempotent is central. Since $S/\nabla$ has no nontrivial idempotent, $S_1$ has no nontrivial idempotent. Therefore, it is a division ring which has no factor ring isomorphic to $\mathbb{Z}_2$, so each element of $S_1$ is a sum of two units. Now, it is enough to show that every element of $S_2$, which has no factor ring isomorphic to $\mathbb{Z}_2$, is a sum of two units. Let $a \in S_2$ and suppose to
the contrary that \(a\) is not a sum of two units.

Let \(\Omega = \{I \mid I \text{ is an ideal of } S_2 \text{ and } a + I \text{ is not a sum of two units in } S_2/I\}\).

Clearly, \(\Omega\) is non-empty and it can be easily checked that \(\Omega\) is inductive. So, by Zorn’s Lemma, \(\Omega\) has a maximal element, say, \(I\). Clearly, \(S_2/I\) is an indecomposable ring and hence has no central idempotent. But, \(S_2\) is an exchange ring, so \(S_2/I\) is an exchange ring too. Since it has no central idempotent, it is clean and therefore \(S_2/I\) is a local ring. Let \(T_2 = S_2/I\). Since \(x = a + I\) is not a sum of two units in \(S_2/I\), \(x + J(T_2)\) is not a sum of two units in \(T_2/J(T_2)\), which is a division ring. Therefore, \(T_2/J(T_2) \cong \mathbb{Z}_2\), a contradiction. Hence, each element of \(S_2\) is also a sum of two units. Therefore, every element of \(\mathcal{S}\) is a sum of two units. Since \(\forall \subseteq J(S)\), we may conclude that every element of \(S\) is a sum of two units.

The converse is obvious. \(\square\)

**Corollary 3.11.** Let \(M\) be a \(R\)-module with indecomposable decomposition and with finite exchange property and \(S = \text{End}_R(M)\). If no factor ring of \(S\) is isomorphic to \(\mathbb{Z}_2\), then \(\text{usn}(M) = 2\).

*Proof.\* By [3, Theorem 2.8] the endomorphism ring of \(M\) is a local ring. Thus, \(S/J(S)\) is a division ring which has no factor ring isomorphic to \(\mathbb{Z}_2\), so \(\text{usn}(M) = 2\). \(\square\)

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**References**


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