

ON n -COHERENT RINGS, n -HEREDITARY RINGS AND n -REGULAR RINGS

Z. ZHU

Communicated by Fariborz Azarpanah

ABSTRACT. We observe some new characterizations of n -presented modules. Using the concepts of $(n, 0)$ -injectivity and $(n, 0)$ -flatness of modules, we also present some characterizations of right n -coherent rings, right n -hereditary rings, and right n -regular rings.

1. Introduction

Throughout this paper, n is a positive integer unless a special note, R denotes an associative ring with identity and all modules considered are unitary. For any R -module M , $M^+ = \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$ will be the character module of M .

B. Stenström [10] defined and studied *FP-injective modules*. Following [10], a right R -module M is said to be *FP-injective*, if $\text{Ext}_R^1(A, M) = 0$, for every finitely presented right R -module A . A right R -module A is said to be finitely presented, if there is an exact sequence $F_1 \rightarrow F_0 \rightarrow A \rightarrow 0$ in which F_1, F_0 are finitely generated free right R -modules, or equivalently, if there is an exact sequence $P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$ in which P_1, P_0 are finitely generated projective right R -modules. *FP-injective* modules are also called absolutely pure modules in [8], these modules

MSC(2000): Primary: 16D10; Secondary: 16E40.

Keywords: $(n, 0)$ -injective modules, $(n, 0)$ -flat modules, n -coherent rings, n -hereditary rings n -regular rings.

Received: 13 February 2009, Accepted: 16 August 2010.

© 2011 Iranian Mathematical Society.

have been studied by many authors. In papers [8] and [10], right Noetherian rings, right coherent rings, right semihereditary rings and regular rings are characterized by FP -injective right R -modules. It is well known that a left R -module M is flat if and only if $\text{Tor}_1^R(A, M) = 0$, for every finitely presented right R -module A . Costa [2] introduced the concept of n -presented modules. Let n be a non-negative integer. According to [2], a right R -module M is called n -presented in case there is an exact sequence of right R -modules $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ in which every F_i is a finitely generated free, equivalently projective right R -module; And a ring R is called right n -coherent [2] in case every n -presented right R -module is $(n+1)$ -presented. Clearly, a ring R is right coherent if and only if it is right 1-coherent. We remark that the terminology of “ n -coherence” in this paper is Costa’s “ n -coherence” but is not the same as that of [3]. Let n, d be non-negative integers. According to [12], a right R -module M is called (n, d) -injective, if $\text{Ext}_R^{d+1}(A, M) = 0$, for every n -presented right R -module A ; A left R -module M is called (n, d) -flat, if $\text{Tor}_{d+1}^R(A, M) = 0$, for every n -presented right R -module A ; A ring R is called a right (n, d) -ring, if every n -presented right R -module has the projective dimension at most d . Recall that a commutative right (n, d) -ring is called an (n, d) -ring [2], (n, d) -rings have been studied by several authors [2, 5, 6, 7, 12]. An $(n, 0)$ -ring is called an n -von Neumann regular ring in papers [6] and [7].

In this paper, We will give Some characterizations and properties of n -presented modules and $(n, 0)$ -injective modules as well as $(n, 0)$ -flat modules. Moreover, we will generalize the concept of right semihereditary rings to right n -hereditary rings, and then we will generalize the concepts of regular rings and n -von Neumann regular rings to right n -regular rings. Right n -coherent rings, right n -hereditary rings and right n -regular rings will be characterized by $(n, 0)$ -injective right R -modules and $(n, 0)$ -flat left R -modules. $(n, 0)$ -injective dimensions of right R -modules over right n -coherent rings and $(n, 0)$ -flat dimensions of right R -modules over left n -coherent rings will be discussed.

First of all, we give some characterizations of n -presented modules.

Proposition 1.1. *The following statements are equivalent for a right R -module M :*

- (1) M is n -presented.
- (2) There exists an exact sequence of right R -modules

$$0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

such that F_0, \dots, F_{n-1} are finitely generated free right R -modules and K_n is finitely generated.

- (3) There exists an exact sequence of right R -modules

$$0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

such that P_0, \dots, P_{n-1} are finitely generated projective right R -modules and K_n is finitely generated.

- (4) There exists an exact sequence of right R -modules

$$P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

such that P_0, \dots, P_{n-1}, P_n are finitely generated projective right R -modules.

- (5) There exists an exact sequence of right R -modules

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$$

such that F is finitely generated free right R -module and K is $(n-1)$ -presented.

- (6) There exists an exact sequence of right R -modules

$$0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$$

such that P is finitely generated projective right R -module and K is $(n-1)$ -presented.

- (7) M is finitely generated and, if the sequence of right R -modules $0 \rightarrow L \rightarrow F \rightarrow M \rightarrow 0$ is exact with F finitely generated free, then L is $(n-1)$ -presented.

- (8) M is finitely generated and, if the sequence of right R -modules $0 \rightarrow L \rightarrow P \rightarrow M \rightarrow 0$ is exact with P finitely generated projective, then L is $(n-1)$ -presented.

Proof. (1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4), (7) \Rightarrow (5), (8) \Rightarrow (6) are obvious.

(3) \Rightarrow (2). Use induction on n . In case of $n = 1$, suppose there exists an exact sequence of right R -modules $0 \rightarrow K_1 \rightarrow P_0 \rightarrow M \rightarrow 0$, where P_0 is finitely generated projective module and K_1 is finitely generated. Then, there exists an exact sequence of right R -modules $0 \rightarrow K \rightarrow F_0 \rightarrow M \rightarrow 0$ with F_0 finitely generated free module. By Schanuel's Lemma, $K_1 \oplus F_0 \cong K \oplus P_0$, so K is finitely generated and the result follows. Now, suppose (3) implies (2), for $n - 1$. Then, if there exists an exact sequence of right R -modules

$$0 \rightarrow K_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

such that P_0, \dots, P_{n-1} are finitely generated projective right R -modules and K_n is finitely generated. Then, we have an exact sequence

$$0 \rightarrow \text{im}(d_{n-1}) \rightarrow P_{n-2} \rightarrow \dots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0.$$

By induction hypothesis, There exists an exact sequence of right R -modules

$$0 \rightarrow K_{n-1} \rightarrow F_{n-2} \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$$

such that F_0, \dots, F_{n-2} are finitely generated free right R -modules and K_{n-1} is finitely generated. Let $F_{n-1} \xrightarrow{\pi} K_{n-1}$ be epic with F_{n-1} finitely generated free module, then we obtain an exact sequence

$$0 \rightarrow \text{Ker}(\pi) \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0.$$

By the generalization of Schanuel's Lemma [9, Exercise 3.37], $\text{Ker}(\pi)$ is finitely generated, and (2) follows.

(2) \Rightarrow (5). Suppose there exists an exact sequence of right R -modules

$$0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

such that F_0, \dots, F_{n-1} are finitely generated free right R -modules and K_n is finitely generated. Take $K = \text{im}(d_1)$, then K is $(n-1)$ -presented and the sequence $0 \rightarrow K \rightarrow F_0 \rightarrow M \rightarrow 0$ is exact.

(5) \Rightarrow (2). Let $0 \rightarrow K \xrightarrow{\alpha} F \xrightarrow{\beta} M \rightarrow 0$ be exact, where K is $(n-1)$ -presented. Then, there exists an exact sequence

$$0 \rightarrow K_n \rightarrow F_{n-1} \xrightarrow{d_{n-1}} \dots \rightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} K \rightarrow 0$$

such that F_1, \dots, F_{n-1} are finitely generated free right R -modules and K_n is finitely generated, and thus we have an exact sequence of right R -modules

$$0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_2 \rightarrow F_1 \xrightarrow{\alpha d_1} F \xrightarrow{\beta} M \rightarrow 0$$

and (2) follows.

(5) \Rightarrow (7). Assume (5), then there exists an exact sequence of right R -modules $0 \rightarrow K \rightarrow F' \rightarrow M \rightarrow 0$. Clearly, M is finitely generated. If the sequence of right R -modules $0 \rightarrow L \rightarrow F \rightarrow M \rightarrow 0$ is exact with F finitely generated free, then by Schanuel's Lemma, $K \oplus F \cong L \oplus F'$, and so L is $(n-1)$ -presented by [11, Theorem 1].

(3) \Rightarrow (6) is similar to (2) \Rightarrow (5), (6) \Rightarrow (8) is similar to (5) \Rightarrow (7). \square

From Proposition 1.1(5), it is easy to see that right n -coherent ring is right $(n+1)$ -coherent.

2. n -coherent rings

We begin this section with some characterizations of right n -coherent rings.

Theorem 2.1. *The following statements are equivalent for a ring R :*

- (1) R is right n -coherent.
- (2) If the sequence

$$(*) \quad F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

is exact, where each F_i is a finitely generated free right R -module, then there exists an exact sequence of right R -modules

$$(**) \quad F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

where each F_i is a finitely generated free right R -module.

- (3) Every $(n-1)$ -presented submodule of a projective right R -module is n -presented.

Proof. (1) \Rightarrow (2). By the exactness of (*), we have an exact sequence

$$0 \rightarrow \text{Ker}(d_n) \rightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

Since R is right n -coherent, M is $(n+1)$ -presented, so there exists an exact sequence of right R -modules

$$0 \rightarrow L_{n+1} \rightarrow F'_n \rightarrow F'_{n-1} \rightarrow \cdots \rightarrow F'_1 \rightarrow F'_0 \rightarrow M \rightarrow 0$$

where each F'_i is finitely generated free, L_{n+1} is finitely generated. By the generalization of Schanuel's Lemma [9, Exercise 3.37], $\text{Ker}(d_n)$ is finitely generated, and then there exists a finitely generated free module F_{n+1} such that (**) holds.

(2) \Rightarrow (1) is clear.

(1) \Leftrightarrow (3) by Proposition 1.1. □

Recall that a right R -module M is FP -injective if and only if it is pure in every module containing it as a submodule. A submodule A of the right R -module B is said to be a pure submodule if for all left R -module M , the induced map $A \otimes_R M \rightarrow B \otimes_R M$ is monic, or equivalently, every finitely presented module is projective with respect to the exact sequence $0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$. In this case, the exact sequence $0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$ is called pure. We call a short exact sequence

of right R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ n -pure, if every n -presented right R -module is projective with respect to this sequence.

Next, we give some characterizations of $(n, 0)$ -injective modules.

Theorem 2.2. *Let M be a right R -module, then the following statements are equivalent:*

- (1) M is $(n, 0)$ -injective.
- (2) M is injective with respect to every exact sequence $0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0$ of right R -modules with A n -presented.
- (3) If K is an $(n - 1)$ -presented submodule of a projective right R -module P , then every right R -homomorphism f from K to M extends to a homomorphism from P to M .
- (4) Every exact sequence $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ is n -pure.
- (5) There exists an n -pure exact sequence $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ of right R -modules with M' injective.
- (6) There exists an n -pure exact sequence $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ of right R -modules with M' $(n, 0)$ -injective.

Proof. (1) \Rightarrow (2). By the exact sequence $Hom(B, M) \rightarrow Hom(C, M) \rightarrow Ext_R^1(A, M) = 0$.

(2) \Rightarrow (3). Let $F = P \oplus P'$, where F is a free right R -module. Since K is finitely generated, there exists a finitely generated free module F_1 such that $K \leq F_1 \leq^\oplus F$. But, K is $(n - 1)$ -presented, so F_1/K is n -presented, and thus the induced map $Hom(F_1, M) \rightarrow Hom(K, M)$ is surjective by (2), and (3) follows.

(3) \Rightarrow (1). For any n -presented module A , there exists an exact sequence $0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$, where P is finitely generated projective, K is $(n - 1)$ -presented. Hence, we get an exact sequence $Hom(P, M) \rightarrow Hom(K, M) \rightarrow Ext_R^1(A, M) \rightarrow Ext_R^1(P, M) = 0$, and thus $Ext_R^1(A, M) = 0$ by (3). Therefore, M is $(n, 0)$ -injective.

(1) \Rightarrow (4). Assume (1). Then, we have an exact sequence $Hom(A, M') \rightarrow Hom(A, M'') \rightarrow Ext_R^1(A, M) = 0$, for every n -presented module A , and so (4) follows.

(4) \Rightarrow (5) \Rightarrow (6) are obvious.

(6) \Rightarrow (1). By (6), we have an n -pure exact sequence $0 \rightarrow M \rightarrow M' \xrightarrow{f} M'' \rightarrow 0$ of right R -modules where M' is $(n, 0)$ -injective, and so, for each n -presented module A , we have an exact sequence $Hom(A, M') \xrightarrow{f_*} Hom(A, M'') \rightarrow Ext_R^1(A, M) \rightarrow Ext_R^1(A, M') = 0$ with f_* epic. Which implies that $Ext_R^1(A, M) = 0$, and (1) follows. \square

Lemma 2.9(2) in [1] and Theorem 2.2(3) immediately yield the next two results.

Proposition 2.3. *Let $n \geq 2$, then every direct limit of $(n, 0)$ -injective right R -modules is $(n, 0)$ -injective.*

Proposition 2.4. *Let $\{M_i \mid i \in I\}$ be a family of right R -modules, then the following statements are equivalent:*

- (1) *Each M_i is $(n, 0)$ -injective.*
- (2) *$\prod_{i \in I} M_i$ is $(n, 0)$ -injective.*
- (3) *$\bigoplus_{i \in I} M_i$ is $(n, 0)$ -injective.*

Lemma 2.5. *Let E be an injective right R -module and N its $(k, 0)$ -injective submodule, then E/N is $(k + 1, 0)$ -injective.*

Proof. Let A be any $(k + 1)$ -presented right R -module. Then, there exists an exact sequence $0 \rightarrow B \rightarrow P \rightarrow A \rightarrow 0$, where P is a finitely generated projective module and B is k -presented. So we get two exact sequences

$$0 = \text{Ext}_R^1(A, E) \rightarrow \text{Ext}_R^1(A, E/N) \rightarrow \text{Ext}_R^2(A, N) \rightarrow \text{Ext}_R^2(A, E) = 0$$

and

$$0 = \text{Ext}_R^1(P, N) \rightarrow \text{Ext}_R^1(B, N) \rightarrow \text{Ext}_R^2(A, N) \rightarrow \text{Ext}_R^2(P, N) = 0$$

Hence, $\text{Ext}_R^1(A, E/N) \cong \text{Ext}_R^1(B, N) = 0$, this follows that E/N is $(k + 1, 0)$ -injective. \square

Theorem 2.6. *Let A be an $(n - 1)$ -presented right R -module. Then, A is n -presented if and only if $\text{Ext}_R^1(A, M) = 0$, for any $(n, 0)$ -injective module M .*

Proof. \Rightarrow . It is obvious.

\Leftarrow . Use induction on n . In case $n = 1$, then the implication holds by [4]. Suppose the implication holds when $n = k$. Then, when $n = k + 1$, assume A is an k -presented right R -module and $\text{Ext}_R^1(A, M) = 0$, for every $(k + 1, 0)$ -injective module M . Since A is k -presented, there exists an exact sequence $0 \rightarrow L \rightarrow F \rightarrow A \rightarrow 0$ with F finitely generated free and L $(k - 1)$ -presented. So, for any $(k, 0)$ -injective module N , we have $\text{Ext}_R^1(L, N) \cong \text{Ext}_R^2(A, N) \cong \text{Ext}_R^1(A, E(N)/N)$. By Lemma 2.5, $E(N)/N$ is $(k + 1, 0)$ -injective, so $\text{Ext}_R^1(A, E(N)/N) = 0$ by conditions,

and whence $\text{Ext}_R^1(L, N) = 0$. Therefore, L is k -presented by hypothesis, which shows that A is $(k + 1)$ -presented. \square

Theorem 2.7. *The following statements are equivalent for a ring R .*

- (1) R is right n -coherent.
- (2) $\text{Ext}_R^1(A, N) = 0$, for any n -presented right R -module A and any $(n + 1, 0)$ -injective right R -module N .
- (3) $\text{Ext}_R^2(A, N) = 0$, for any n -presented right R -module A and any $(n, 0)$ -injective right R -module N .
- (4) If N is an $(n, 0)$ -injective right R -module, N_1 is an $(n, 0)$ -injective submodule of N , then N/N_1 is $(n, 0)$ -injective.
- (5) For any $(n, 0)$ -injective right R -module N , $E(N)/N$ is $(n, 0)$ -injective.

Proof. (1) \Rightarrow (2) and (4) \Rightarrow (5) are obvious.

(2) \Rightarrow (1) by Theorem 2.6.

(1) \Rightarrow (3). Since A is n -presented, by Proposition 1.1(5), there exists an exact sequence of right R -modules $0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$, where F is finitely generated free, K is $(n - 1)$ -presented, and we get an induced exact sequence

$$0 = \text{Ext}_R^1(F, N) \rightarrow \text{Ext}_R^1(K, N) \rightarrow \text{Ext}_R^2(A, N) \rightarrow \text{Ext}_R^2(F, N) = 0.$$

Hence, $\text{Ext}_R^2(A, N) \cong \text{Ext}_R^1(K, N)$. Since R is right n -coherent, by Theorem 2.1, K is n -presented, so $\text{Ext}_R^1(K, N) = 0$, and thus $\text{Ext}_R^2(A, N) = 0$.

(3) \Rightarrow (4). For any n -presented right R -module A . The exact sequence $0 \rightarrow N_1 \rightarrow N \rightarrow N/N_1 \rightarrow 0$ induces the exactness of the sequence

$$0 = \text{Ext}^1(A, N) \rightarrow \text{Ext}^1(A, N/N_1) \rightarrow \text{Ext}^2(A, N_1) = 0.$$

Therefore, $\text{Ext}^1(A, N/N_1) = 0$, as desired.

(5) \Rightarrow (1). Let A be any n -presented right R -module. Then, by Proposition 1.1(5), there is an exact sequence of right R -modules $0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$, where F is finitely generated free, K is $(n - 1)$ -presented. Then, for any $(n, 0)$ -injective module N , $E(N)/N$ is $(n, 0)$ -injective by (5). From the exactness of the two sequences

$$0 = \text{Ext}^1(F, N) \rightarrow \text{Ext}^1(K, N) \rightarrow \text{Ext}^2(A, N) \rightarrow \text{Ext}^2(F, N) = 0$$

and

$$0 = \text{Ext}^1(A, E(N)) \rightarrow \text{Ext}^1(A, E(N)/N) \rightarrow \text{Ext}^2(A, N) \rightarrow \text{Ext}^2(A, E(N)) = 0,$$

we have $\text{Ext}^1(K, N) \cong \text{Ext}^2(A, N) \cong \text{Ext}^1(A, E(N)/N) = 0$, so $\text{Ext}^1(K, N) = 0$. By Theorem 2.6, K is n -presented, hence A is $(n + 1)$ -presented. Therefore, R is right n -coherent. \square

Definition 2.8.

- (1). The $(n, 0)$ -injective dimension of a module M_R is defined by

$$(n, 0)\text{-id}(M_R) = \inf\{k : \text{Ext}_R^{k+1}(A, M) = 0, \text{ for every } n\text{-presented module } A\}$$
- (2). The right $(n, 0)$ -injective global dimension of a ring R is defined by

$$r.(n, 0)\text{-ID}(R) = \sup\{(n, 0)\text{-id}(M) : M \text{ is a right } R\text{-module}\}$$

Lemma 2.9. *Let R be a right n -coherent ring and let M be a right R -module, then the following statements are equivalent:*

- (1) $(n, 0)\text{-id}(M) \leq k$.
- (2) $\text{Ext}_R^{k+1}(A, M) = 0$, for every n -presented right R -module A .

Proof. (1) \Rightarrow (2). Use induction on k . Clearly, if $(n, 0)\text{-id}(M) = k$. If $(n, 0)\text{-id}(M) \leq k - 1$. Since A is n -presented, there exists an exact sequence $0 \rightarrow N \rightarrow P \rightarrow A \rightarrow 0$, where P is a finitely generated projective module and N is $(n - 1)$ -presented. But, R is right n -coherent, N is n -presented by Theorem 2.1, and so $\text{Ext}_R^{k+1}(A, M) \cong \text{Ext}_R^k(N, M) = 0$ by induction hypothesis.

(2) \Rightarrow (1) is clear. \square

Corollary 2.10. *Let R be a right n -coherent ring and let M_R be $(n, 0)$ -injective, then $\text{Ext}_R^k(A, M) = 0$, for all n -presented modules A and all positive integers k .*

Corollary 2.11. *Let R be a right n -coherent ring and let M be a right R -module. If the sequence $0 \rightarrow M \xrightarrow{\varepsilon} E_0 \xrightarrow{d_0} \cdots \rightarrow E_{k-1} \xrightarrow{d_{k-1}} E_k \rightarrow 0$ is exact with E_0, \dots, E_{k-1} $(n, 0)$ -injective, then $\text{Ext}_R^{k+1}(A, M) \cong \text{Ext}_R^1(A, E_k)$, for any n -presented right R -module A .*

Proof. Since R is right n -coherent and E_0, E_1, \dots, E_{k-1} are $(n, 0)$ -injective, by Corollary 2.10, we have $\text{Ext}_R^{k+1}(A, M) \cong \text{Ext}_R^k(A, \text{im}(d_0)) \cong \text{Ext}_R^{k-1}(A, \text{im}(d_1)) \cong \dots \cong \text{Ext}_R^1(A, \text{im}(d_{k-1})) = \text{Ext}_R^1(A, E_k)$. \square

Theorem 2.12. *Let R be a right n -coherent ring, M a right R -module and k a non-negative integer, then the following statements are equivalent:*

- (1) $(n, 0)\text{-id}(M_R) \leq k$.
- (2) $\text{Ext}_R^{k+l}(A, M) = 0$, for all n -presented modules A and all positive integers l .
- (3) $\text{Ext}_R^{k+1}(A, M) = 0$, for all n -presented modules A .
- (4) If the sequence $0 \rightarrow M \rightarrow E_0 \rightarrow \dots \rightarrow E_{k-1} \rightarrow E_k \rightarrow 0$ is exact with E_0, \dots, E_{k-1} $(n, 0)$ -injective, then E_k is also $(n, 0)$ -injective.
- (5) There exists an exact sequence $0 \rightarrow M \rightarrow E_0 \rightarrow \dots \rightarrow E_{k-1} \rightarrow E_k \rightarrow 0$ of right R -modules with E_0, \dots, E_{k-1}, E_k $(n, 0)$ -injective.

Proof. (1) \Rightarrow (2). Assume (1), then $(n, 0)\text{-id}(M_R) \leq k + l - 1$, and so (2) follows from Lemma 2.9.

(2) \Rightarrow (3) and (4) \Rightarrow (5) are obvious. (3) \Rightarrow (4) and (5) \Rightarrow (1) by Corollary 2.11. \square

Theorem 2.13. *A right R -module M is $(n, 0)$ -flat if and only if the canonical map $M \otimes K \rightarrow M \otimes P$ is monic for every finitely generated projective left R -module P and any $(n - 1)$ -presented submodule K of P .*

Proof. It follows from the exact sequence

$$0 = \text{Tor}_1^R(M, P) \rightarrow \text{Tor}_1^R(M, P/K) \rightarrow M \otimes K \rightarrow M \otimes P.$$

\square

Theorem 2.14. *Let $\{M_i \mid i \in I\}$ be a family of right R -modules, consider the following conditions:*

- (1) Each M_i is $(n, 0)$ -flat.
- (2) $\bigoplus_{i \in I} M_i$ is $(n, 0)$ -flat.
- (3) $\prod_{i \in I} M_i$ is $(n, 0)$ -flat.

Then, we always have (3) \Rightarrow (1) \Leftrightarrow (2). If $n \geq 2$, then these conditions are equivalent.

Proof. (1) \Leftrightarrow (2) by the isomorphism $Tor_1^R(\prod_{i \in I} M_i, A) \cong \prod_{i \in I} Tor_1^R(M_i, A)$. (3) \Rightarrow (1) is obvious. If $n \geq 2$, then by [1, Lemma 2.10], there is an isomorphism $Tor_1^R(\prod_{i \in I} M_i, A) \cong \prod_{i \in I} Tor_1^R(M_i, A)$, for every n -presented left R -module A , so in this case, the conditions (1), (2) and (3) are equivalent. \square

Theorem 2.15. *Let M be a right R -module, then*

- (1) M is $(n, 0)$ -flat if and only if M^+ is $(n, 0)$ -injective.
- (2) If $n \geq 2$, then M is $(n, 0)$ -injective if and only if M^+ is $(n, 0)$ -flat.

Proof. (1) follows from the isomorphism $Tor_1^R(M, A)^+ \cong Ext_R^1(A, M^+)$. (2). Since $n \geq 2$, we have an isomorphism $Tor_1^R(A, M^+) \cong Ext_R^1(A, M)^+$, for every n -presented right R -module A by [1, Lemma 2.7(2)], and so (2) holds. \square

Corollary 2.16. *If R is right coherent, then a right R -module M is FP-injective if and only if M^+ is flat.*

Proof. Since R is right coherent, a right R -module is finitely presented if and only if it is 2-presented. And so the result follows from Theorem 2.15(2). \square

Corollary 2.17. *Pure submodules of $(n, 0)$ -flat modules is $(n, 0)$ -flat.*

Proof. Let M be an $(n, 0)$ -flat module and M_1 a pure submodule of M , then the pure exact sequence $0 \rightarrow M_1 \rightarrow M \rightarrow M/M_1 \rightarrow 0$ induces a split exact sequence $0 \rightarrow (M/M_1)^+ \rightarrow M^+ \rightarrow M_1^+ \rightarrow 0$. By Theorem 2.15(1), M^+ is $(n, 0)$ -injective, so M_1^+ is $(n, 0)$ -injective by Theorem 2.4, and hence M_1 is $(n, 0)$ -flat by Theorem 2.15(1). \square

Definition 2.18. *The $(n, 0)$ -flat dimension of a module M_R is defined by*

$$(n, 0)\text{-fd}(M_R) = \inf\{k : Tor_{k+1}^R(M, A) = 0, \text{ for all } n\text{-presented left } R\text{-modules } A.\}$$

Lemma 2.19. *Let R be a left n -coherent ring and let M be a right R -module, then the following statements are equivalent:*

- (1) $(n, 0)\text{-fd}(M_R) \leq k$.
- (2) $\text{Tor}_{k+1}^R(M, A) = 0$, for every n -presented left R -module A .

Proof. (1) \Rightarrow (2). Use induction on k . Clear, if $(n, 0)\text{-fd}(M) = k$. If $(n, 0)\text{-fd}(M) \leq k - 1$. Since A is n -presented, there exists an exact sequence $0 \rightarrow N \rightarrow P \rightarrow A \rightarrow 0$, where P is a finitely generated projective module and N is $(n - 1)$ -presented. But, R is left n -coherent, N is n -presented, and hence $\text{Tor}_{k+1}^R(M, A) \cong \text{Tor}_k^R(M, N) = 0$ by induction hypothesis.

(2) \Rightarrow (1) is clear. □

Corollary 2.20. *Let R be a left n -coherent ring and M_R be $(n, 0)$ -flat, then $\text{Tor}_k^R(M, A) = 0$, for all n -presented left R -modules A and all positive integers k .*

Corollary 2.21. *Let R be a left n -coherent ring and M be a right R -module. If the sequence of right R -modules $0 \rightarrow F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$ is exact with F_0, \dots, F_{k-1} $(n, 0)$ -flat, then $\text{Tor}_1^R(F_k, A) \cong \text{Tor}_{k+1}^R(M, A)$, for any n -presented left R module A .*

Proof. Since R is left n -coherent and F_0, F_1, \dots, F_{k-1} are $(n, 0)$ -flat, by Corollary 2.20, we have

$$\begin{aligned} \text{Tor}_{k+1}^R(M, A) &\cong \text{Tor}_k^R(\text{Ker}(d_0), A) \cong \text{Tor}_{k-1}^R(\text{Ker}(d_1), A) \cong \cdots \\ &\cong \text{Tor}_1^R(\text{Ker}(d_{k-1}), A) \cong \text{Tor}_1^R(F_k, A). \end{aligned}$$

□

Theorem 2.22. *Let R be a left n -coherent ring, M be a right R -module and $k \geq 0$, then the following statements are equivalent:*

- (1) $(n, 0)\text{-fd}(M_R) \leq k$.
- (2) $\text{Tor}_{k+l}^R(M, A) = 0$, for all n -presented left R -modules A and all positive integers l .
- (3) $\text{Tor}_{k+1}^R(M, A) = 0$, for all n -presented left R -modules A .
- (4) If the sequence $0 \rightarrow F_k \rightarrow F_{k-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$ is exact with F_0, \dots, F_{k-1} $(n, 0)$ -flat, then also F_k is $(n, 0)$ -flat.

- (5) *There exists an exact sequence $0 \rightarrow F_k \rightarrow F_{k-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$ of right R -modules with F_0, \dots, F_{k-1}, F_k $(n, 0)$ -flat.*

Proof. (1) \Rightarrow (2). Assume (1), then $(n, 0)\text{-fd}(M_R) \leq k + l - 1$, and so (2) follows from Lemma 2.19.

(2) \Rightarrow (3) and (4) \Rightarrow (5) are obvious. (3) \Rightarrow (4) and (5) \Rightarrow (1) by Corollary 2.21 and Lemma 2.19. \square

3. n -hereditary rings and n -regular rings

Recall that a ring R is called right semihereditary, if every finitely generated right ideal of R is projective, or equivalently, if every finitely generated submodule of a projective right R -modules is projective. Next, we define n -hereditary rings as follows.

Definition 3.1. *A ring R is called right n -hereditary, if every $(n - 1)$ -presented submodule of projective right R -module is projective.*

Clearly, a ring R is right semihereditary if and only if it is right 1-hereditary. Right n -hereditary ring is right $(n + 1)$ -hereditary.

Theorem 3.2. *The following statements are equivalent for a ring R :*

- (1) *R is right n -hereditary.*
- (2) *R is right n -coherent and $r.(n, 0)\text{-ID}(R) \leq 1$.*
- (3) *Factor module of $(n, 0)$ -injective right R -module is $(n, 0)$ -injective.*
- (4) *Factor module of injective right R -module is $(n, 0)$ -injective.*
- (5) *R is a right $(n, 1)$ -ring.*

Proof. (1) \Rightarrow (2). Since R is right n -hereditary, every $(n - 1)$ -presented submodule of a projective right R -module is finitely generated projective, and hence n -presented, so R is right n -coherent. Now, let M be any right R -module. Then, for any n -presented right R -module A , we have an exact sequence $0 \rightarrow N \rightarrow P \rightarrow A \rightarrow 0$ of right R -modules, where P is finitely generated and projective, N is $(n - 1)$ -presented and projective. Thus, the exact sequence $0 = \text{Ext}_R^1(N, M) \rightarrow \text{Ext}_R^2(A, M) \rightarrow \text{Ext}_R^2(P, M) = 0$ implies that $\text{Ext}_R^2(A, M) = 0$. This follows that $r.(n, 0)\text{-ID}(R) \leq 1$ by Definition 2.8.

(2) \Rightarrow (3). Let M be an $(n, 0)$ -injective right R -module and K its submodule. Then, for any n -presented module A , we have an exact

sequence $0 = \text{Ext}_R^1(A, M) \rightarrow \text{Ext}_R^1(A, M/K) \rightarrow \text{Ext}_R^2(A, K) = 0$ by (2) and Lemma 2.9, and so $\text{Ext}_R^1(A, M/K) = 0$, as required.

(3) \Rightarrow (4). It is obvious.

(4) \Rightarrow (5). Since $\text{Ext}_R^2(A, B) \cong \text{Ext}_R^1(A, E(B)/B)$ holds for any right R -modules A and B , so (5) follows from (4).

(5) \Rightarrow (1). Let N be an $(n-1)$ -presented submodule of a projective right R -module P . Then, there exists a finitely generated free module F such that N is a submodule of F . Now, for any injective right R -module E and every submodule K of E , since F/N is n -presented, $\text{Ext}_R^2(F/N, K) = 0$ by (5), and so $\text{Ext}_R^1(N, K) = 0$ as the sequence $0 = \text{Ext}_R^1(F, K) \rightarrow \text{Ext}_R^1(N, K) \rightarrow \text{Ext}_R^2(F/N, K) = 0$ is exact. This shows that N is E -projective because of the exact sequence $\text{Hom}(N, E) \rightarrow \text{Hom}(N, E/K) \rightarrow \text{Ext}_R^1(N, K) = 0$. Therefore, N is projective. \square

Example 3.3. *Let R be a non-coherent commutative ring of weak dimension one, then R is a $(2,1)$ -ring but not a $(1,1)$ -ring by [2, Example (6.5)], and so R is a 2-hereditary ring which is not 1-hereditary by Theorem 3.2.*

Theorem 3.4. *A domain R is n -hereditary if and only if every $(n-1)$ -presented torsion-free R -module is projective.*

Proof. Since R is a domain, every finitely generated torsion-free R -module may be imbedded in a free module and every submodule of a free R -module is torsion-free. Hence, the results follows. \square

Theorem 3.5. *If $n \geq 2$, then the following statements are equivalent for a ring R :*

- (1) *R is a right n -hereditary ring.*
- (2) *Every submodule of an $(n, 0)$ -flat left R -module is $(n, 0)$ -flat.*

Proof. (1) \Rightarrow (2). Let M be an $(n, 0)$ -flat left R -module and let K be its submodule. Then, for any n -presented right R -module A , there exists an exact sequence $0 \rightarrow N \rightarrow P \rightarrow A \rightarrow 0$, where P is a finitely generated projective module and N is $(n-1)$ -presented. Since R is a right n -hereditary ring, N is projective, hence we have an exact sequence $0 = \text{Tor}_2^R(P, M/K) \rightarrow \text{Tor}_2^R(A, M/K) \rightarrow \text{Tor}_1^R(N, M/K) = 0$, it shows that $\text{Tor}_2^R(A, M/K) = 0$. Therefore, by the exact sequence $0 = \text{Tor}_2^R(A, M/K) \rightarrow \text{Tor}_1^R(A, K) \rightarrow \text{Tor}_1^R(A, M) = 0$, we get $\text{Tor}_1^R(A, K) = 0$, i.e., K is $(n, 0)$ -flat.

(2) \Rightarrow (1). Suppose B is an $(n, 0)$ -injective right R -module with an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. Then, B^+ is an $(n, 0)$ -flat left R -module by Theorem 2.15(2), and the sequence $0 \rightarrow C^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0$ is exact. By (2), C^+ is $(n, 0)$ -flat, so C is $(n, 0)$ -injective again by Theorem 2.15(2). Hence, R is right n -hereditary by Theorem 3.2(3). \square

Corollary 3.6. *If $n \geq 2$ and the weak dimension of R $wD(R) \leq 1$, then R is left and right n -hereditary.*

Proof. Assume M is an $(n, 0)$ -flat right R -module and K is a submodule of M . Then, for any n -presented left R -module A , since $wD(R) \leq 1$, $Tor_2^R(M/K, A) = 0$, this follows that $Tor_1^R(K, A) = 0$ because M is $(n, 0)$ -flat, and thus K is $(n, 0)$ -flat. By Theorem 3.5, R is left n -hereditary. Similarly, one can prove that R is right n -hereditary. \square

Next, we generalize the concepts of regular rings and n -von Neumann rings to right n -regular rings.

Definition 3.7. *A ring R is called right n -regular, if it is a right $(n, 0)$ -ring.*

Clearly, R is regular if and only if it is right 1-regular, R is n -von Neumann ring, if it is a commutative right n -regular ring. Right n -regular ring is right $(n + 1)$ -regular.

Example 3.8. *Let K be a field and E be a K -vector space with infinite rank. Set $B = K \times E$ the trivial extension of K by E . Then, by [6, Theorem 3.4], R is a commutative 2-regular rings which is not regular. So, in general, right 2-regular ring need not be regular.*

Theorem 3.9. *The following conditions are equivalent for a ring R .*

- (1) R is a right n -regular ring.
- (2) Every right R -module is $(n, 0)$ -injective.
- (3) Every finitely generated right R -module is $(n, 0)$ -injective.
- (4) R is right n -hereditary and R_R is $(n, 0)$ -injective.
- (5) R is right n -coherent and every n -presented right R -module is $(n, 0)$ -injective.
- (6) Every $(n - 1)$ -presented submodule of a projective right R -module is a direct summand.

- (7) *Every n -presented right R -module is flat.*
 (8) *Every left R -module is $(n, 0)$ -flat.*

Proof. (1) \Rightarrow (2) \Rightarrow (3) are obvious.

(3) \Rightarrow (4). Assume (3). Then, clearly R_R is $(n, 0)$ -injective. Let P be a projective module and let K be an $(n - 1)$ -presented submodule of P . By (3), K is $(n, 0)$ -injective, so by Theorem 2.2(3), we have that K is a direct summand of P and hence K is projective. Therefore, R is right n -hereditary .

(4) \Rightarrow (5). Assume (4), then every $(n - 1)$ -presented submodule of a projective module is projective and finitely generated, and then it is n -presented, so R is right n -coherent by Theorem 2.1(3). Now, let M be an n -presented right R -module, then there exists an exact sequence of right R -modules $F \rightarrow M \rightarrow 0$, where F is finitely generated free. Since R_R is $(n, 0)$ -injective, by Proposition 2.4, F is $(n, 0)$ -injective. Observing that R is right n -hereditary, by Theorem 3.2(3), M is $(n, 0)$ -injective.

(5) \Rightarrow (6). Let M be an $(n - 1)$ -presented submodule of a projective right R -module P . Then, M is a submodule of a finitely generated free right R -module F . By Proposition 1.1(5), F/M is n -presented. Since R is right n -coherent, F/M is $(n + 1)$ -presented. So, M is n -presented by Proposition 1.1(7), and hence M is $(n, 0)$ -injective by (5). This follows that M is a direct summand of P by Theorem 2.2(3).

(6) \Rightarrow (1). Let M be an n -presented right R -module, then there exists an exact sequence of right R -modules $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$, where P is finitely generated projective and K is $(n - 1)$ -presented. By hypothesis, K is a direct summand of P . Hence, M is isomorphic to a direct summand of P , and so M is projective.

(1) \Leftrightarrow (7) and (7) \Leftrightarrow (8) are obvious. □

Acknowledgments

The author would like to thank the referee for a careful reading of the article and giving a detailed report.

REFERENCES

- [1] J. Chen and N. Ding, On n -coherent rings, *Comm. Algebra.* **24** (1996), no. 10, 3211-3216.

- [2] D. L. Costa, Parameterizing families of non-Noetherian rings, *Comm. Algebra*. **22** (1994), no. 10, 3997-4011.
- [3] D. E. Dobbs, S. Kabbaj and N. Mahdou, n -coherent rings and modules, Commutative ring theory, Lecture Notes in *Pure and Appl. Math.*, Dekker. **185** (1997) 269-281.
- [4] E. Enochs, A note on absolutely pure modules, *Canad. Math. Bull.* **19** (1976), no. 3, 361-362.
- [5] S. Kabbaj and N. Mahdou, Trivial extensions of local rings and a conjecture of Costa, *Commutative ring theory and applications*, (Fez, 2001), 301-311, Lecture Notes in Pure and Appl. Math., 231, Dekker, New York, 2003.
- [6] N. Mahdou, On Costa's conjecture, *Comm. Algebra*. **29** (2001), no. 7, 2775-2785.
- [7] N. Mahdou, On n -flat modules and n -von Neumann regular rings, *Int. J. Math. Math. Sci.* 2006, Art. ID 90868.
- [8] C. Megibben, Absolutely pure modules, *Proc. Amer. Math. Soc.* **26** (1970) 561-566.
- [9] J. J. Rotman, An Introduction to Homological Algebra, Pure and Applied Mathematics, 85, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1979.
- [10] B. Stenström, Coherent rings and FP-injective modules, *J. London Math. Soc.* **2** (1970), no. 2, 323-329.
- [11] W. M. Xue, On n -presented modules and almost excellent extensions, *Comm. Algebra*. **27** (1999), no. 3, 1091-1102.
- [12] D. X. Zhou, On n -coherent rings and (n, d) -rings, *Comm. Algebra*. **32** (2004), no. 6, 2425-2441.

Zhanmin Zhu

Department of Mathematics, Jiaying University, 314001, Jiaying,
Zhejiang Province, P. R. China

Email: zhanmin_zhu@hotmail.com