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TOPOLOGICAL CENTERS OF THE $N-TH$ DUAL OF MODULE ACTIONS

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Abstract. We study the topological centers of nth dual of Banach $\mathcal{A}$-modules and we extend some propositions from Lau and Ülger into $n-th$ dual of Banach $\mathcal{A}-modules$ where $n \geq 0$ is even number. Let $\mathcal{B}$ be a Banach $\mathcal{A}-bimodule$. By using some new conditions, we show that $Z^{\ell}_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)}) = \mathcal{B}^{(n)}$ and $Z^{\ell}_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)}) = \mathcal{A}^{(n)}$. We get some conclusions on group algebras.

1. Introduction

Throughout this paper, $\mathcal{A}$ is a Banach algebra and $\mathcal{A}^*$, $\mathcal{A}^{**}$, respectively, are the first and second dual of $\mathcal{A}$. A bounded net $(e_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ is called a *bounded left approximate identity* (BLAI) [respectively *bounded right approximate identity* (BRAI)] if, for each $a \in \mathcal{A}$, $e_\alpha a \to a$ [respectively $ae_\alpha \to a$]. Moreover, (e_α) is called a (two sided) bounded approximate identity (BAI), if for every $a \in \mathcal{A}$, the conditions $e_\alpha a \to a$ and $ae_\alpha \to a$ both hold. For $a \in \mathcal{A}$ and $a' \in \mathcal{A}^*$, we denote by $a'a$ and aa' respectively, the functionals on $\mathcal{A}^*$ defined by $\langle a'a, b\rangle = \langle a', ab\rangle = a'(ab)$ and $\langle aa', b\rangle = \langle a', ba\rangle = a'(ba)$ for all $b \in \mathcal{A}$. The Banach algebra $\mathcal{A}$ is embedded in its second dual via the identification $\langle a, a'\rangle - \langle a', a\rangle$ for every $a \in \mathcal{A}$ and $a' \in \mathcal{A}^*$. We denote the set $\{a'a : \ a \in \mathcal{A} \ and \ a' \in \mathcal{A}^*\}$

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and $\{aa' : a \in \mathcal{A} \; and \; a' \in \mathcal{A}^*\}$ by $\mathcal{A}^*A$ and $A\mathcal{A}^*$, respectively. Clearly these two sets are subsets of $\mathcal{A}^*$.
Let $\mathcal{A}$ has a BAI. If the equality $\mathcal{A}^*\mathcal{A} = \mathcal{A}^*$, $(\mathcal{A}\mathcal{A}^* = \mathcal{A}^*)$ holds, then we say that $\mathcal{A}^*$ factors on the left (right). If both equalities $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = \mathcal{A}^*$ hold, then we say that $\mathcal{A}^*$ factors on both sides.
Let X, Y, Z be normed spaces and let $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as follows
1. $m^* : Z^* \times X \to Y^*$, given by $\langle m^*(z', x), y\rangle = \langle z', m(x, y)\rangle$ where $x \in X$, $y \in Y$, $z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \to X^*$, given by $\langle m^{**}(y'', z'), x\rangle = \langle y'', m^*(z', x)\rangle$ where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $\langle m^{***}(x'', y''), z'\rangle$ $= \langle x'', m^{**}(y'', z')\rangle$ where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.
The mapping m^{***} is the unique extension of m such that $x'' \to m^{***}(x'', y'')$ from X^{**} into Z^{**} is $weak^*$-to-$weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^*$-to-$weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as follows

$$Z_1(m) = \{x'' \in X^{**} : \; y'' \to m^{***}(x'', y'') \; is \; weak^*\text{-}to\text{-}weak^* \; continuous\}.$$

Now let $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1]. If $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^*$-to-$weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^*$-to-$weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : \; x'' \to m^{t***t}(x'', y'') \; is \; weak^*\text{-}to\text{-}weak^* \; continuous\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j)\rangle = \lim_j \lim_i \langle z', m(x_i, y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [18].
The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.
Now let $\mathcal{B}$ be a Banach $\mathcal{A}$-*bimodule*, and let

$$\pi_\ell : \ \mathcal{A} \times \mathcal{B} \to \mathcal{B} \ and \ \pi_r : \ \mathcal{B} \times \mathcal{A} \to \mathcal{B}.$$

be the left and right module actions of $\mathcal{A}$ on $\mathcal{B}$, respectively. Then $\mathcal{B}^{**}$ is a Banach $\mathcal{A}^{**}$-bimodule with the following module actions where $\mathcal{A}^{**}$ is equipped with the left Arens product

$$\pi_\ell^{***} : \ \mathcal{A}^{**} \times \mathcal{B}^{**} \to \mathcal{B}^{**} \ and \ \pi_r^{***} : \ \mathcal{B}^{**} \times \mathcal{A}^{**} \to \mathcal{B}^{**}.$$

Similarly, $\mathcal{B}^{**}$ is a Banach $\mathcal{A}^{**}$-bimodule with the following module actions where $\mathcal{A}^{**}$ is equipped with the right Arens product

$$\pi_\ell^{t***t} : \ \mathcal{A}^{**} \times \mathcal{B}^{**} \to \mathcal{B}^{**} \ and \ \pi_r^{t***t} : \ \mathcal{B}^{**} \times \mathcal{A}^{**} \to \mathcal{B}^{**}.$$

We may therefore define the topological centers of the left and right module actions of $\mathcal{A}^{**}$ on $\mathcal{B}^{**}$ as follows:

$$\begin{gathered}
Z_{\mathcal{B}^{**}}(\mathcal{A}^{**}) = Z(\pi_\ell) = \{a'' \in \mathcal{A}^{**} : \ the\ map\ b'' \to \pi_\ell^{***}(a'', b'') \ : \\
\mathcal{B}^{**} \to \mathcal{B}^{**}\ is\ weak^*\text{-}to\text{-}weak^*\ continuous\} \\
Z^t_{\mathcal{B}^{**}}(\mathcal{A}^{**}) = Z(\pi_r^t) = \{a'' \in \mathcal{A}^{**} : \ the\ map\ b'' \to \pi_r^{t***}(a'', b'') \ : \\
\mathcal{B}^{**} \to \mathcal{B}^{**}\ is\ weak^*\text{-}to\text{-}weak^*\ continuous\} \\
Z_{\mathcal{A}^{**}}(\mathcal{B}^{**}) = Z(\pi_r) = \{b'' \in \mathcal{B}^{**} : \ the\ map\ a'' \to \pi_r^{***}(b'', a'') \ : \\
\mathcal{A}^{**} \to \mathcal{B}^{**}\ is\ weak^*\text{-}to\text{-}weak^*\ continuous\} \\
Z^t_{\mathcal{A}^{**}}(\mathcal{B}^{**}) = Z(\pi_\ell^t) = \{b'' \in \mathcal{B}^{**} : \ the\ map\ a'' \to \pi_\ell^{t***}(b'', a'') \ : \\
\mathcal{A}^{**} \to \mathcal{B}^{**}\ is\ weak^*\text{-}to\text{-}weak^*\ continuous\}.
\end{gathered}$$

We note that if $\mathcal{B}$ is a left(respectively right) Banach $\mathcal{A}$-module and $\pi_\ell : \ \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ (respectively $\pi_r : \ \mathcal{B} \times \mathcal{A} \to \mathcal{B}$) is left (respectively right) module action of $\mathcal{A}$ on $\mathcal{B}$, then $\mathcal{B}^*$ is a right (respectively left) Banach $\mathcal{A}$-module.
We write $ab = \pi_\ell(a, b)$, $ba = \pi_r(b, a)$, $\pi_\ell(a_1a_2, b) = \pi_\ell(a_1, a_2b)$, $\pi_r(b, a_1a_2) = \pi_r(ba_1, a_2)$, $\pi_\ell^*(a_1b', a_2) = \pi_\ell^*(b', a_2a_1)$, $\pi_r^*(b'a, b) = \pi_r^*(b', ab)$, for all $a_1, a_2, a \in \mathcal{A}$, $b \in \mathcal{B}$ and $b' \in \mathcal{B}^*$ when there is no confusion.
Regarding $\mathcal{A}$ as a Banach $\mathcal{A}$-*bimodule*, the operation $\pi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ extends to π^{***} and π^{t***t} defined on $\mathcal{A}^{**} \times \mathcal{A}^{**}$. These extensions are

known as the first(left) and the second (right) Arens products, respectively and in both cases, the second dual space $\mathcal{A}^{**}$ becomes a Banach algebra. In this situation, we will also simplify our notations. So the first (left) Arens product of $a'', b'' \in \mathcal{A}^{**}$ will be simply denoted by $a''b''$ and will be defined by the following three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle,$$
$$\langle a''a', a\rangle = \langle a'', a'a\rangle,$$
$$\langle a''b'', a'\rangle = \langle a'', b''a'\rangle.$$

for every $a, b \in \mathcal{A}$ and $a' \in \mathcal{A}^*$. Similarly, the second (right) Arens product of $a'', b'' \in \mathcal{A}^{**}$ will be denoted by $a'' \circ b''$ and will be defined by :

$$\langle aoa', b\rangle = \langle a', ba\rangle,$$
$$\langle a' \circ a'', a\rangle = \langle a'', a \circ a'\rangle,$$
$$\langle a'' \circ b'', a'\rangle = \langle b'', a' \circ b''\rangle,$$

for all $a, b \in \mathcal{A}$ and $a' \in \mathcal{A}^*$.

The regularity of a normed algebra $\mathcal{A}$ is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of $\mathcal{A}^{**}$. By *Goldstine's* Theorem [4, p.425] there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\mathcal{A}$ such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in \mathcal{A}^*$, we have

$$\lim_\alpha \lim_\beta \langle a', \pi(a_\alpha, b_\beta)\rangle = \langle a''b'', a'\rangle,$$
$$\lim_\beta \lim_\alpha \langle a', \pi(a_\alpha, b_\beta)\rangle = \langle a''ob'', a'\rangle,$$

where $a''b''$ and $a'' \circ b''$ are the first and second Arens products of $\mathcal{A}^{**}$, respectively, see [14, 18].

We find the usual first and second topological center of $\mathcal{A}^{**}$, which are

$$Z_{\mathcal{A}^{**}}(\mathcal{A}^{**}) = Z(\pi) = \{a'' \in \mathcal{A}^{**} : b'' \to a''b'' \textit{ is weak}^*\textit{-to-weak}^* \textit{ continuous}\},$$
$$Z^t_{\mathcal{A}^{**}}(\mathcal{A}^{**}) = Z(\pi^t) = \{a'' \in \mathcal{A}^{**} : a'' \to a'' \circ b'' \textit{ is weak}^*\textit{-to-weak}^* \textit{ continuous}\}.$$

An element e'' of $\mathcal{A}^{**}$ is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if, for each $a'' \in \mathcal{A}^{**}$, $a''e'' = e'' \circ a'' = a''$. By [4, p.146] an element e'' of $\mathcal{A}^{**}$ is mixed unit if and only if it is a $weak^*$ cluster point of some BAI $(e_\alpha)_{\alpha \in I}$ in $\mathcal{A}$.

A functional a' in $\mathcal{A}^*$ is said to be *wap* (weakly almost periodic) on $\mathcal{A}$ if the mapping $a \to a'a$ from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact. Pym in [18] showed that this definition is equivalent to the following condition.
For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a \in \mathcal{A} : \parallel a \parallel \leq 1\}$, we have

$$\lim_\alpha \lim_\beta \langle a', a_\alpha b_\beta\rangle = \lim_\beta \lim_\alpha \langle a', a_\alpha b_\beta\rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on $\mathcal{A}$ is denoted by $wap(\mathcal{A})$. We also have $a' \in wap(A)$ if and only if $\langle a''b'', a'\rangle = \langle a'' \circ b'', a'\rangle$ for every $a'',\ b'' \in \mathcal{A}^{**}$.
This paper is organized as follows:
a) Let $\mathcal{B}$ be a Banach $\mathcal{A}$-*bimodule*. Let $n \geq 0$ be an even number and $0 \leq r \leq \frac{n}{2}$. Assume that $U_{n,r} = (\mathcal{A}^{(n-r)}\mathcal{A}^{(r)})^{(r)}$ or $U_{n,r} = (\mathcal{A}^{(n-r)}\mathcal{A}^{(r-1)})^{(r)}$ and $\phi \in U_{n,r}$. Then $\phi \in {Z^\ell}_{\mathcal{B}^{(n)}}(U_{n,r})$ if and only if $b^{(n-1)}\phi \in \mathcal{B}^{(n-1)}$ for all $b^{(n-1)} \in \mathcal{B}^{(n-1)}$.
b) Let $\mathcal{B}$ be a Banach $\mathcal{A}$-*bimodule*. Then

(1) $b^{(n)} \in {Z^\ell}_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$ if and only if $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$ for all $b^{(n-1)} \in \mathcal{B}^{(n-1)}$.

(2) If $\phi \in {Z^\ell}_{\mathcal{B}^{(n)}}(U_{n,r})$, then $a^{(n-2)}\phi \in {Z^\ell}_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)})$ for all $a^{(n-2)} \in \mathcal{A}^{(n-2)}$.

c) Let $\mathcal{B}$ be a Banach space such that $\mathcal{B}^{(n)}$ is weakly compact. Then for Banach $\mathcal{A}$-*bimodule* $\mathcal{B}$, we have the following assertions.

(1) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ is a $BLAI$ for $\mathcal{B}^{(n)}$ such that

$$e_\alpha^{(n)}\mathcal{B}^{(n+2)} \subseteq \mathcal{B}^{(n)},$$

for every α. Then $\mathcal{B}$ is reflexive.

(2) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ is a $BRAI$ for $\mathcal{B}^{(n)}$ and

$$Z^\ell_{e^{(n+2)}}(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)},$$

where $e_\alpha^{(n)} \overset{w^*}{\to} e^{(n+2)}$ in $\mathcal{A}^{(n)}$. If $\mathcal{B}^{(n+2)}e_\alpha^{(n)} \subseteq \mathcal{B}^{(n)}$ for every α, then $Z^\ell_{\mathcal{A}^{(n+2)}}(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)}$.

d) Assume that $\mathcal{B}$ is a Banach $\mathcal{A}$-*bimodule*. Then

(1) $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq wap_\ell(\mathcal{B}^{(n)})$ if and only if

$$\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq Z^\ell_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)}).$$

(2) If $\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq \mathcal{A}^{(n)}{Z^\ell}_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)})$, then

$$\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq Z^\ell_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)}).$$

e) Let $\mathcal{B}$ be a left Banach $\mathcal{A}$-*bimodule* and let $n \geq 0$ be an even number. Suppose that $b_0^{(n+1)} \in \mathcal{B}^{(n+1)}$. Then $b_0^{(n+1)} \in wap_\ell(\mathcal{B}^{(n)})$ if and only if the mapping $T : b^{(n+2)} \to b^{(n+2)}b_0^{(n+1)}$ form $\mathcal{B}^{(n+2)}$ into $\mathcal{A}^{(n+1)}$ is *weak**-*to*-*weak* continuous.

f) Let $\mathcal{B}$ be a left Banach $\mathcal{A}$-*bimodule*. Then for $n \geq 2$, we have the following assertions.

(1) If $\mathcal{A}^{(n)} = a_0^{(n-2)}\mathcal{A}^{(n)}$ [respectively, $\mathcal{A}^{(n)} = \mathcal{A}^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w- property [respectively Lw^*w- property] with respect to $\mathcal{B}^{(n)}$, then $Z_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)}) = \mathcal{A}^{(n)}$.

(2) If $\mathcal{B}^{(n)} = a_0^{(n-2)}\mathcal{B}^{(n)}$ [respectively, $\mathcal{B}^{(n)} = \mathcal{B}^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w- property [respectively Lw^*w- property] with respect to $\mathcal{B}^{(n)}$, then $Z_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)}) = \mathcal{B}^{(n)}$.

2. Topological Centers of Module Actions

Suppose that $\mathcal{A}$ is a Banach algebra and $\mathcal{B}$ is a Banach $\mathcal{A}$-*bimodule*. According to [5, p.27-28] $\mathcal{B}^{**}$ is a Banach $\mathcal{A}^{**}$-*bimodule*, where $\mathcal{A}^{**}$ is equipped with the first Arens product. We recall the topological centers of module actions of $\mathcal{A}^{**}$ on $\mathcal{B}^{**}$ as follows.

$$Z^\ell_{\mathcal{A}^{**}}(\mathcal{B}^{**}) = \{b'' \in \mathcal{B}^{**} :\ \textit{the map } a'' \to b''a''\ :\ \mathcal{A}^{**} \to \mathcal{B}^{**}$$
$$\textit{is weak}^*\textit{-to-weak}^*\textit{ continuous}\}$$
$$Z^\ell_{\mathcal{B}^{**}}(\mathcal{A}^{**}) = \{a'' \in \mathcal{A}^{**} :\ \textit{the map } b'' \to a''b''\ :\ \mathcal{B}^{**} \to \mathcal{B}^{**}$$
$$\textit{is weak}^*\textit{-to-weak}^*\textit{ continuous}\}.$$

Let $\mathcal{A}^{(n)}$ and $\mathcal{B}^{(n)}$ be *nth dual* of $\mathcal{A}$ and $\mathcal{B}$, respectively. By [25, p. 4132-4134] if $n \geq 0$ is an even number, then $\mathcal{B}^{(n)}$ is a Banach $\mathcal{A}^{(n)}$-*bimodule*. Then for $n \geq 2$, we define $\mathcal{B}^{(n)}\mathcal{B}^{(n-1)}$ as a subspace of $\mathcal{A}^{(n-1)}$, that is, for all $b^{(n)} \in \mathcal{B}^{(n)}$, $b^{(n-1)} \in \mathcal{B}^{(n-1)}$ and $a^{(n-2)} \in \mathcal{A}^{(n-2)}$ we define

$$\langle b^{(n)}b^{(n-1)}, a^{(n-2)}\rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-2)}\rangle.$$

If n is odd number, we define $\mathcal{B}^{(n)}\mathcal{B}^{(n-1)}$ as a subspace of $\mathcal{A}^{(n)}$, that is, for all $b^{(n)} \in \mathcal{B}^{(n)}$, $b^{(n-1)} \in \mathcal{B}^{(n-1)}$ and $a^{(n-1)} \in \mathcal{A}^{(n-1)}$, we define

$$\langle b^{(n)}b^{(n-1)}, a^{(n-1)}\rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-1)}\rangle.$$

If $n=0$, we take $\mathcal{A}^{(0)}=\mathcal{A}$ and $\mathcal{B}^{(0)}=\mathcal{B}$.
We also define the topological centers of module actions of $\mathcal{A}^{(n)}$ on $\mathcal{B}^{(n)}$ as follows

$$Z^\ell_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})=\{b^{(n)}\in\mathcal{B}^{(n)}:\ \textit{the map } a^{(n)}\to b^{(n)}a^{(n)}\ :\ \mathcal{A}^{(n)}\to\mathcal{B}^{(n)}$$
$$\textit{is weak}^*\textit{-to-weak}^*\textit{ continuous}\}$$
$$Z^\ell_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)})=\{a^{(n)}\in\mathcal{A}^{(n)}:\ \textit{the map } b^{(n)}\to a^{(n)}b^{(n)}\ :\ \mathcal{B}^{(n)}\to\mathcal{B}^{(n)}$$
$$\textit{is weak}^*\textit{-to-weak}^*\textit{ continuous}\}.$$

Let $\mathcal{A}$ be a Banach algebra with a (BAI) and suppose that $\mathcal{A}^{(n)}$ and $\mathcal{A}^{(m)}$ are the *nth dual* and *mth dual* of $\mathcal{A}$, respectively. Suppose that at least one of the integers n or m is an even number. Then we define the set $\mathcal{A}^{(n)}\mathcal{A}^{(m)}$ as a linear space that is generated by the following set

$$\{a^{(n)}a^{(m)}:\ a^{(n)}\in\mathcal{A}^{(n)}\ \textit{and}\ a^{(m)}\in\mathcal{A}^{(m)}\},$$

where the multiplication $a^{(n)}a^{(m)}$ is defined with respect to the first Arens product. If $n\geq m$, then $\mathcal{A}^{(n)}\mathcal{A}^{(m)}$ is a subspace of $\mathcal{A}^{(n)}$. Observe that $\mathcal{A}^{(n)}\mathcal{A}^{(m)}$ is Banach algebra whenever n and m are even numbers, but if one of them is an odd number, then $\mathcal{A}^{(n)}\mathcal{A}^{(m)}$ is not in general a Banach algebra.

Let $n\geq 0$ be an even number and $0\leq r\leq\frac{n}{2}$. For a Banach algebra $\mathcal{A}$, we define a new Banach algebra $U_{n,r}$ with respect to the first Arens product as follows.

If r is an even (respectively odd) number, then we write
$U_{n,r}=(\mathcal{A}^{(n-r)}\mathcal{A}^{(r)})^{(r)}$ (respectively $U_{n,r}=(\mathcal{A}^{(n-r)}\mathcal{A}^{(r-1)})^{(r)}$). It is clear that $U_{n,r}$ is a subalgebra of $\mathcal{A}^{(n)}$. For example, if we take $n=2$ and $r=1$, then $U_{2,1}=(\mathcal{A}^*\mathcal{A})^*$ is a subalgebra of $\mathcal{A}^{**}$ with respect to the first Arens product.

Now if $\mathcal{B}$ is a Banach $\mathcal{A}$-*bimodule*, then it is clear that $\mathcal{B}^{(n)}$ is a Banach $U_{n,r}-$ *bimodule* with respect to the first Arens product, for detail see [25]. Thus we can define the topological centers of module actions $U_{n,r}$ on $\mathcal{B}^{(n)}$ as $Z^\ell_{\mathcal{B}^{(n)}}(U_{n,r})$ and $Z^\ell_{U_{n,r}}(\mathcal{B}^{(n)})$ similar to the preceding definitions. In every parts of this paper, $n\geq 0$ is even number.

Theorem 2.1. *Let $\mathcal{B}$ be a Banach $\mathcal{A}$-bimodule and $\phi\in U_{n,r}$. Then $\phi\in {Z^\ell}_{\mathcal{B}^{(n)}}(U_{n,r})$ if and only if $b^{(n-1)}\phi\in\mathcal{B}^{(n-1)}$ for all $b^{(n-1)}\in\mathcal{B}^{(n-1)}$.*

Proof. Let $\phi\in {Z^\ell}_{\mathcal{B}^{(n)}}(U_{n,r})$. Suppose that $(b^{(n)}_\alpha)_\alpha\subseteq\mathcal{B}^{(n)}$ such that $b^{(n)}_\alpha\overset{w^*}{\to}b^{(n)}$ in $\mathcal{B}^{(n)}$. Then, for every $b^{(n-1)}\in\mathcal{B}^{(n-1)}$, we have

$$\langle b^{(n-1)}\phi,b^{(n)}_\alpha\rangle=\langle b^{(n-1)},\phi b^{(n)}_\alpha\rangle=\langle\phi b^{(n)}_\alpha,b^{(n-1)}\rangle\to\langle\phi b^{(n)},b^{(n-1)}\rangle$$

$$= \langle b^{(n-1)}\phi, b^{(n)}\rangle.$$

It follows that $b^{(n-1)}\phi \in (\mathcal{B}^{(n)}, weak^*)^* = \mathcal{B}^{(n-1)}$.
Conversely, let $b^{(n-1)}\phi \in \mathcal{B}^{(n-1)}$ for every $b^{(n-1)} \in \mathcal{B}^{(n-1)}$ and suppose that $(b_\alpha^{(n)})_\alpha \subseteq \mathcal{B}^{(n)}$ such that $b_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}$ in $\mathcal{B}^{(n)}$. Then

$$\langle \phi b_\alpha^{(n)}, b^{(n-1)}\rangle = \langle \phi, b_\alpha^{(n)} b^{(n-1)}\rangle = \langle b_\alpha^{(n)} b^{(n-1)}, \phi\rangle = \langle b_\alpha^{(n)}, b^{(n-1)}\phi\rangle$$
$$\to \langle b^{(n)}, b^{(n-1)}\phi\rangle = \langle \phi b^{(n)}, b^{(n-1)}\rangle.$$

It follows that $\phi b_\alpha^{(n)} \xrightarrow{w^*} \phi b^{(n)}$, and so $\phi \in {Z^\ell}_{\mathcal{B}^{(n)}}(U_{n,r})$. □

In the preceding theorem if we take $\mathcal{B} = \mathcal{A}$, $n = 2$ and $r = 1$, we obtain Lemma 3.1 (b) of [14].

Theorem 2.2. *Let $\mathcal{B}$ be a Banach $\mathcal{A}$-bimodule and $b^{(n)} \in \mathcal{B}^{(n)}$. Then we have the following assertions:*

(1) $b^{(n)} \in {Z^\ell}_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$ *if and only if* $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$ *for all* $b^{(n-1)} \in \mathcal{B}^{(n-1)}$.

(2) *If* $\phi \in {Z^\ell}_{\mathcal{B}^{(n)}}(U_{n,r})$, *then* $a^{(n-2)}\phi \in {Z^\ell}_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)})$ *for all* $a^{(n-2)} \in \mathcal{A}^{(n-2)}$.

Proof. (1) Let $b^{(n)} \in {Z^\ell}_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$. We show that $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$ where $b^{(n-1)} \in \mathcal{B}^{(n-1)}$. Suppose that $(a_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ and $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$ in $\mathcal{A}^{(n)}$. Then we have

$$\langle b^{(n-1)}b^{(n)}, a_\alpha^{(n)}\rangle = \langle b^{(n-1)}, b^{(n)}a_\alpha^{(n)}\rangle = \langle b^{(n)}a_\alpha^{(n)}, b^{(n-1)}\rangle$$
$$\to \langle b^{(n)}a^{(n)}, b^{(n-1)}\rangle = \langle b^{(n-1)}b^{(n)}, a^{(n)}\rangle.$$

Consequently, $b^{(n-1)}b^{(n)} \in (\mathcal{A}^{(n)}, weak^*)^* = \mathcal{A}^{(n-1)}$. It follows that $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$.
Conversely, let $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$ for each $b^{(n-1)} \in \mathcal{B}^{(n-1)}$. Suppose that $(a_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ and $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$ in $\mathcal{A}^{(n)}$. Then we have

$$\langle b^{(n)}a_\alpha^{(n)}, b^{(n-1)}\rangle = \langle b^{(n)}, a_\alpha^{(n)}b^{(n-1)}\rangle = \langle a_\alpha^{(n)}b^{(n-1)}, b^{(n)}\rangle$$
$$= \langle a_\alpha^{(n)}, b^{(n-1)}b^{(n)}\rangle \to \langle a^{(n)}, b^{(n-1)}b^{(n)}\rangle = \langle b^{(n)}a^{(n)}, b^{(n-1)}\rangle.$$

It follows that $b^{(n)}a_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}a^{(n)}$, and hence $b^{(n)} \in {Z^\ell}_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$.

(2) Let $\phi \in {Z^\ell}_{\mathcal{B}^{(n)}}(U_{n,r})$ and $a^{(n-2)} \in \mathcal{A}^{(n-2)}$. Assume that $(b_\alpha^{(n)})_\alpha \subseteq \mathcal{B}^{(n)}$ such that $b_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}$ in $\mathcal{B}^{(n)}$. Then for all $b^{(n-1)} \in \mathcal{B}^{(n-1)}$, we have

$$\langle (a^{(n-2)}\phi)b_\alpha^{(n)}, b^{(n-1)}\rangle = \langle \phi b_\alpha^{(n)}, b^{(n-1)}a^{(n-2)}\rangle \to$$

$$\langle \phi b^{(n)}, b^{(n-1)} a^{(n-2)} \rangle = \langle (a^{(n-2)}\phi) b^{(n)}, b^{(n-1)} \rangle.$$

It follows that $(a^{(n-2)}\phi) b^{(n)}_\alpha \xrightarrow{w^*} (a^{(n-2)}\phi) b^{(n)}$, and hence

$$a^{(n-2)}\phi \in Z^{\ell}{}_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)}).$$

$\square$

In the preceding theorem, part (1), if we take $\mathcal{B} = \mathcal{A}$ and $n = 2$, we conclude Lemma 3.1 (a) of [14]. In part (2) of this theorem, if we take $\mathcal{B} = \mathcal{A}$, $n = 2$ and $r = 1$, we also obtain Lemma 3.1 (c) from [14].

Definition 2.3. *Let $\mathcal{B}$ be a Banach $\mathcal{A}$-bimodule and suppose that $a'' \in \mathcal{A}^{**}$. Assume that $(a''_\alpha)_\alpha \subseteq \mathcal{A}^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$. If for every $b'' \in \mathcal{B}^{**}$, $b''a''_\alpha \xrightarrow{w^*} b''a''$, then we say that $a'' \to b''a''$ is weak*-to-weak* point continuous.*
Suppose that $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule. Assume that $a'' \in \mathcal{A}^{**}$. Then we define the locally topological center of a'' on $\mathcal{B}^{**}$ as follows

$$Z^{\ell}_{a''}(\mathcal{B}^{**}) = \{ b'' \in \mathcal{B}^{**} : \ a'' \to b''a'' \text{ is weak}^*\text{-to-weak}^* \text{ point continuous}\}.$$

The definition of $Z^{\ell}_{b''}(\mathcal{A}^{**})$ where $b'' \in \mathcal{B}^{**}$ is similar.
It is clear that

$$\bigcap_{a'' \in \mathcal{A}^{**}} Z^{\ell}_{a''}(\mathcal{B}^{**}) = Z^{\ell}_{\mathcal{A}^{**}}(\mathcal{B}^{**}),$$

$$\bigcap_{b'' \in \mathcal{B}^{**}} Z^{\ell}_{b''}(\mathcal{A}^{**}) = Z^{\ell}_{\mathcal{B}^{**}}(\mathcal{A}^{**}).$$

Let $\mathcal{B}$ be a Banach space. Then $K \subseteq B$ is called weakly compact, if K is compact with respect to weak topology on $\mathcal{B}$. By [7], we know that K is weakly compact if and only if K is weakly limit point compact.

Theorem 2.4. *Assume that $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule such that $\mathcal{B}^{(n)}$ is weakly compact. Then we have the following assertions:*

1. *Suppose that $(e^{(n)}_\alpha)_\alpha \subseteq \mathcal{A}^{(n)}$ is a BLAI for $\mathcal{B}^{(n)}$ such that*
$$e^{(n)}_\alpha \mathcal{B}^{(n+2)} \subseteq \mathcal{B}^{(n)},$$
for every α. Then $\mathcal{B}$ is reflexive.
2. *Suppose that $(e^{(n)}_\alpha)_\alpha \subseteq \mathcal{A}^{(n)}$ is a BRAI for $\mathcal{B}^{(n)}$ and*
$$Z^{\ell}_{e^{(n+2)}}(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)},$$

where $e_\alpha^{(n)} \xrightarrow{w^*} e^{(n+2)}$ *in* $\mathcal{A}^{(n)}$*. If* $\mathcal{B}^{(n+2)} e_\alpha^{(n)} \subseteq \mathcal{B}^{(n)}$ *for every* α*, then* $Z^\ell_{\mathcal{A}^{(n+2)}}(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)}$.

Proof. (1) Let $b^{n+2} \in \mathcal{B}^{(n+2)}$. Since $(e_\alpha^{(n)})_\alpha$ is a $BLAI$ for $\mathcal{B}^{(n)}$, without loss generality, there is a left unit $e^{(n+2)} \in \mathcal{A}^{n+2}$ for $\mathcal{B}^{(n+2)}$ such that $e_\alpha^{(n)} \xrightarrow{w^*} e^{(n+2)}$ in $\mathcal{A}^{(n+2)}$, see [10]. Then we have $e_\alpha^{(n)} b^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$ in $\mathcal{B}^{(n+2)}$. Since $e_\alpha^{(n)} b^{(n+2)} \in \mathcal{B}^{(n)}$, we have $e_\alpha^{(n)} b^{(n+2)} \xrightarrow{w} b^{(n+2)}$ in $\mathcal{B}^{(n)}$. We conclude that $b^{n+2} \in \mathcal{B}^{(n)}$, because $\mathcal{B}^{(n)}$ is weakly compact.

(2) Suppose that $b^{(n+2)} \in Z^\ell_{\mathcal{A}^{(n+2)}}(\mathcal{B}^{(n+2)})$ and $e_\alpha^{(n)} \xrightarrow{w^*} e^{(n+2)}$ in $\mathcal{A}^{(n)}$ such that $e^{(n+2)}$ is a right unit for $\mathcal{B}^{(n+2)}$, see [10]. Then we have $b^{(n+2)} e_\alpha^{(n)} \xrightarrow{w^*} b^{(n+2)}$ in $\mathcal{B}^{(n+2)}$. Since $\mathcal{B}^{(n+2)} e_\alpha^{(n)} \subseteq \mathcal{B}^{(n)}$ for every α, $b^{(n+2)} e_\alpha^{(n)} \xrightarrow{w} b^{(n+2)}$ in $\mathcal{B}^{(n)}$ and since $\mathcal{B}^{(n)}$ is weakly compact, $b^{(n+2)} \in \mathcal{B}^{(n)}$. It follows that $Z^\ell_{\mathcal{A}^{(n+2)}}(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)}$.

□

Definition 2.5. *Let* $\mathcal{B}$ *be a Banach* $\mathcal{A}$*-bimodule and* $n \geq 0$*. Then* $b^{(n+2)} \in \mathcal{B}^{(n+2)}$ *is said to be weakly left almost periodic functional if the set*

$$\{b^{(n+1)} a^{(n)} :\ a^{(n)} \in \mathcal{A}^{(n)},\ \parallel a^{(n)} \parallel \leq 1\},$$

is relatively weakly compact, and $b^{(n+2)} \in \mathcal{B}^{(n+2)}$ *is said to be weakly right almost periodic functional if the set*

$$\{a^{(n)} b^{(n+1)} :\ a^{(n)} \in \mathcal{A}^{(n)},\ \parallel a^{(n)} \parallel \leq 1\},$$

is relatively weakly compact. We denote by $wap_\ell(\mathcal{B}^{(n)})$ *[respectively* $wap_r(\mathcal{B}^{(n)})$*] the closed subspace of* $\mathcal{B}^{(n+1)}$ *consisting of all weakly left [respectively right] almost periodic functionals in* $\mathcal{B}^{(n+1)}$*.*

By [6, 14, 18] , the definition of $wap_\ell(\mathcal{B}^{(n)})$ and $wap_r(\mathcal{B}^{(n)})$, respectively, are equivalent to the following:

$$wap_\ell(\mathcal{B}^{(n)}) = \{b^{(n+1)} \in \mathcal{B}^{(n+1)} :\ \langle b^{(n+2)} a_\alpha^{(n+2)}, b^{(n+1)} \rangle \to \langle b^{(n+2)} a^{(n+2)}, b^{(n+1)} \rangle \text{ where } a_\alpha^{(n+2)} \xrightarrow{w^*} a^{(n+2)}\}.$$

and

$$wap_r(\mathcal{B}^{(n)}) = \{b^{(n+1)} \in \mathcal{B}^{(n+1)} :\ \langle a^{(n+2)} b_\alpha^{(n+2)}, b^{(n+1)} \rangle \to \langle a^{(n+2)} b^{(n+2)}, b^{(n+1)} \rangle \text{ where } b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}\}.$$

If we take $\mathcal{A} = \mathcal{B}$ and $n = 0$, then $wap_\ell(\mathcal{A}) = wap_r(\mathcal{A}) = wap(\mathcal{A})$.

Theorem 2.6. *Assume that $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule and $n \geq 0$. Then we have the following assertions:*

(1) $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq wap_\ell(\mathcal{B}^{(n)})$ *if and only if*
$$\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq Z^\ell_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)}).$$

(2) *If* $\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq \mathcal{A}^{(n)}{Z^\ell}_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)})$, *then*
$$\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq Z^\ell_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)}).$$

Proof. (1) Suppose that $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq wap_\ell(\mathcal{B}^{(n)})$. Let $a^{(n)} \in \mathcal{A}^{(n)}$, $a^{(n+2)} \in \mathcal{A}^{(n+2)}$ and let $(b_\alpha^{(n+2)})_\alpha \subseteq \mathcal{B}^{(n+2)}$ such that $b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$. Then for every $b^{(n+1)} \in \mathcal{B}^{(n+1)}$, we have
$$\langle (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)}\rangle = \langle a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)}\rangle$$
$$\to \langle a^{(n+2)}b^{(n+2)}, b^{(n+1)}a^{(n)}\rangle = \langle (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)}\rangle.$$
It follows that $a^{(n)}a^{(n+2)} \in Z^\ell_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)})$.
Conversely, let $a^{(n)}a^{(n+2)} \in Z^\ell_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)})$ for every $a^{(n)} \in \mathcal{A}^{(n)}$, $a^{(n+2)} \in \mathcal{A}^{(n+2)}$. Suppose that $(b_\alpha^{(n+2)})_\alpha \subseteq \mathcal{B}^{(n+2)}$ such that $b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$. Then for every $b^{(n+1)} \in \mathcal{B}^{(n+1)}$, we have
$$\langle a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)}\rangle = \langle (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)}\rangle$$
$$\to \langle (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)}\rangle = \langle a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)}\rangle.$$
It follows that $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq wap_\ell(\mathcal{B}^{(n)})$.

(2) Since $\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq \mathcal{A}^{(n)}{Z^\ell}_{\mathcal{B}^{(n)}}((\mathcal{A}^{(n+2)})$, for every $a^{(n)} \in \mathcal{A}^{(n)}$ and $a^{(n+2)} \in \mathcal{A}^{(n+2)}$, we have $a^{(n)}a^{(n+2)} \in \mathcal{A}^{(n)}{Z^\ell}_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)})$. Then there are $x^{(n)} \in \mathcal{A}^{(n)}$ and $\phi \in Z^\ell_{\mathcal{B}^{(n+2)}}(\mathcal{A}^{(n+2)})$ such that $a^{(n)}a^{(n+2)} = x^{(n)}\phi$. Suppose that $(b_\alpha^{(n+2)})_\alpha \subseteq \mathcal{B}^{(n+2)}$ such that $b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$. Then for every $b^{(n+1)} \in \mathcal{B}^{(n+1)}$, we have
$$\begin{aligned}\langle (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)}\rangle &= \langle (x^{(n)}\phi)b_\alpha^{(n+2)}, b^{(n+1)}\rangle \\ &= \langle \phi b_\alpha^{(n+2)}, b^{(n+1)}x^{(n)}\rangle \to \langle \phi b^{(n+2)}, b^{(n+1)}x^{(n)}\rangle \\ &= \langle (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)}\rangle.\end{aligned}$$
□

In the preceding theorem, if we take $\mathcal{B} = \mathcal{A}$ and $n = 0$, we conclude Theorem 3.6 (a) of [14].

Theorem 2.7. *Assume that $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule and $n \geq 0$. If $\mathcal{A}^{(n)}$ is a left ideal in $\mathcal{A}^{(n+2)}$, then $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq wap_\ell(\mathcal{B}^{(n)})$.*

Proof. The proof is clear. □

Theorem 2.8. *Let $\mathcal{B}$ be a left Banach $\mathcal{A}$-bimodule and $n \geq 0$ be an even number. Suppose that $b_0^{(n+1)} \in \mathcal{B}^{(n+1)}$. Then $b_0^{(n+1)} \in wap_\ell(\mathcal{B}^{(n)})$ if and only if the mapping $T : b^{(n+2)} \to b^{(n+2)}b_0^{(n+1)}$ form $\mathcal{B}^{(n+2)}$ into $\mathcal{A}^{(n+1)}$ is weak*-to-weak continuous.*

Proof. Let $b_0^{(n+1)} \in \mathcal{B}^{(n+1)}$ and suppose that $b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$ in $\mathcal{B}^{(n+2)}$. Then for every $a^{(n+2)} \in \mathcal{A}^{(n+2)}$, we have

$$\begin{aligned}\langle a^{(n+2)}, b_\alpha^{(n+2)}b_0^{(n+1)}\rangle &= \langle a^{(n+2)}b_\alpha^{(n+2)}, b_0^{(n+1)}\rangle \to \langle a^{(n+2)}b^{(n+2)}, b_0^{(n+1)}\rangle \\ &= \langle a^{(n+2)}, b^{(n+2)}b_0^{(n+1)}\rangle.\end{aligned}$$

It follows that $b_\alpha^{(n+2)}b_0^{(n+1)} \xrightarrow{w} b^{(n+2)}b_0^{(n+1)}$ in $\mathcal{A}^{(n+1)}$. The proof of the converse is similar to the preceding proof. □

Corollary 2.9. *Assume that $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule. Then $Z^\ell_{\mathcal{A}^{(n+2)}}(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)}$ if and only if the mapping $T : b^{(n+2)} \to b^{(n+2)}b_0^{(n+1)}$ form $\mathcal{B}^{(n+2)}$ into $\mathcal{A}^{(n+1)}$ is weak*-to-weak continuous for every $b_0^{(n+1)} \in \mathcal{B}^{(n+1)}$.*

Corollary 2.10. *Let $\mathcal{A}$ be a Banach algebra. Assume that $a' \in \mathcal{A}^*$ and $T_{a'}$ is a linear operator from $\mathcal{A}$ into $\mathcal{A}^*$ defined by $T_{a'}a = a'a$. Then, $a' \in wap(\mathcal{A})$ if and only if the adjoint of $T_{a'}$ is weak*-to-weak continuous. So $\mathcal{A}$ is Arens regular if and only if the adjoint of the mapping $T_{a'}a = a'a$ is weak*-to-weak continuous for every $a' \in \mathcal{A}^*$.*

Definition 2.11. *Let $\mathcal{B}$ be a left Banach $\mathcal{A}$-bimodule and $a^{(n)} \in \mathcal{A}^{(n)}$. Let $(b_\alpha^{(n+1)})_\alpha \subseteq \mathcal{B}^{(n+1)}$ such that $a^{(n)}b_\alpha^{(n+1)} \xrightarrow{w^*} 0$. We say that $a^{(n)}$ has $Left-weak^*$-weak property ($= Lw^*w-$ property) with respect to $\mathcal{B}^{(n)}$ when $a^{(n)}b_\alpha^{(n+1)} \xrightarrow{w} 0$. If every $a^{(n)} \in \mathcal{A}$ has Lw^*w- property with respect to $\mathcal{B}^{(n)}$, then we say that $\mathcal{A}^{(n)}$ has Lw^*w- property with respect to $\mathcal{B}^{(n)}$. The definition of the Right$-$weak*-weak property ($= Rw^*w-$ property) is the same.*
We say that $a^{(n)} \in \mathcal{A}^{(n)}$ has weak*-weak property ($= w^*w-$ property) with respect to $\mathcal{B}^{(n)}$ if it has Lw^*w- property and Rw^*w- property with respect to $\mathcal{B}^{(n)}$.
If $a^{(n)} \in \mathcal{A}^{(n)}$ has Lw^*w- property with respect to itself, then we say that $a^{(n)} \in \mathcal{A}^{(n)}$ has Lw^*w- property.

Example 2.12. (1) *If $\mathcal{B}$ is Banach $\mathcal{A}$-bimodule and reflexive, then $\mathcal{A}$ has w^*w-property with respect to $\mathcal{B}$.*

(2) *$L^1(G)$, $M(G)$ and $A(G)$ have w^*w-property when G is finite.*
(3) *Let G be locally compact group. Then $L^1(G)$ [respectively $M(G)$] has w^*w- property [respectively Lw^*w- property] with respect to $L^p(G)$ whenever $p > 1$.*
(4) *Suppose that $\mathcal{B}$ is a left Banach $\mathcal{A}$-module and e is a left unit element of $\mathcal{A}$ such that $eb = b$ for all $b \in B$. If e has Lw^*w-property, then $\mathcal{B}$ is reflexive.*
(5) *If S is a compact semigroup, then $C^+(S) = \{f \in C(S) : \ f > 0\}$ has w^*w-property.*

Theorem 2.13. *Let $\mathcal{B}$ be a left Banach $\mathcal{A}$-bimodule and $n \geq 2$. Then we have the following assertions:*

(1) *If $\mathcal{A}^{(n)} = a_0^{(n-2)}\mathcal{A}^{(n)}$ [respectively $\mathcal{A}^{(n)} = \mathcal{A}^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w- property [respectively Lw^*w- property] with respect to $\mathcal{B}^{(n)}$, then $Z_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)}) = \mathcal{A}^{(n)}$.*

(2) *If $\mathcal{B}^{(n)} = a_0^{(n-2)}\mathcal{B}^{(n)}$ [respectively $\mathcal{B}^{(n)} = \mathcal{B}^{(n)}a_0^{(n-2)}$]‘ for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w- property [respectively Lw^*w- property] with respect to $\mathcal{B}^{(n)}$, then $Z_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)}) = \mathcal{B}^{(n)}$.*

Proof. (1) Suppose that $\mathcal{A}^{(n)} = a_0^{(n-2)}\mathcal{A}^{(n)}$ for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w- property. Let $(b_\alpha^{(n)})_\alpha \subseteq \mathcal{B}^{(n)}$ such that $b_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}$. Then for every $a^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $b^{(n-1)} \in \mathcal{B}^{(n-1)}$, we have

$$\begin{aligned}\langle b_\alpha^{(n)}b^{(n-1)}, a^{(n-2)}\rangle &= \langle b_\alpha^{(n)}, b^{(n-1)}a^{(n-2)}\rangle \to \langle b^{(n)}, b^{(n-1)}a^{(n-2)}\rangle \\ &= \langle b^{(n)}b^{(n-1)}, a^{(n-2)}\rangle.\end{aligned}$$

It follows that $b_\alpha^{(n)}b^{(n-1)} \xrightarrow{w^*} b^{(n)}b^{(n-1)}$. It also is clear that $(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \xrightarrow{w^*} (b^{(n)}b^{(n-1)})a_0^{(n-2)}$. Since $a_0^{(n-2)}$ has Rw^*w-property, $(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \xrightarrow{w} (b^{(n)}b^{(n-1)})a_0^{(n-2)}$. Now, let $a^{(n)} \in \mathcal{A}^{(n)}$. Since $\mathcal{A}^{(n)} = a_0^{(n-2)}\mathcal{A}^{(n)}$, there is $x^{(n)} \in \mathcal{A}^{(n)}$ such that $a^{(n)} = a_0^{(n-2)}x^{(n)}$. Thus we have

$$\begin{aligned}\langle a^{(n)}b_\alpha^{(n)}, b^{(n-1)}\rangle &= \langle a^{(n)}, b_\alpha^{(n)}b^{(n-1)}\rangle = \langle a_0^{(n-2)}x^{(n)}, b_\alpha^{(n)}b^{(n-1)}\rangle \\ &= \langle x^{(n)}, (b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\rangle \to \langle x^{(n)}, (b^{(n)}b^{(n-1)})a_0^{(n-2)}\rangle \\ &= \langle a^{(n)}b, b^{(n-1)}\rangle.\end{aligned}$$

It follows that $a^{(n)} \in Z_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$.
The proof of the next part is similar to the preceding proof.

(2) Let $\mathcal{B}^{(n)} = a_0^{(n-2)}\mathcal{B}^{(n)}$ for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and let $a_0^{(n-2)}$ has Rw^*w-property with respect to $\mathcal{B}^{(n)}$. Assume that $(a_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ such that $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$. Then for every $b^{(n-1)} \in \mathcal{B}^{(n-1)}$, we have

$$\langle a_\alpha^{(n)} b^{(n-1)}, b^{(n-2)}\rangle = \langle a_\alpha^{(n)}, b^{(n-1)}b^{(n-2)}\rangle \to \langle a^{(n)}, b^{(n-1)}b^{(n-2)}\rangle$$
$$= \langle a^{(n)}b^{(n-1)}, b^{(n-2)}\rangle.$$

We conclude that $a_\alpha^{(n)}b^{(n-1)} \xrightarrow{w^*} a^{(n)}b^{(n-1)}$. It is clear that

$$(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \xrightarrow{w^*} (a^{(n)}b^{(n-1)})a_0^{(n-2)}.$$

Since $a_0^{(n-2)}$ has Rw^*w- property,

$$(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \xrightarrow{w} (a^{(n)}b^{(n-1)})a_0^{(n-2)}.$$

Suppose that $b^{(n)} \in \mathcal{B}^{(n)}$. Since $\mathcal{B}^{(n)} = a_0^{(n-2)}\mathcal{B}^{(n)}$, there is $y^{(n)} \in \mathcal{B}^{(n)}$ such that $b^{(n)} = a_0^{(n-2)}y^{(n)}$. Consequently, we have

$$\langle b^{(n)}a_\alpha^{(n)}, b^{(n-1)}\rangle = \langle b^{(n)}, a_\alpha^{(n)}b^{(n-1)}\rangle = \langle a_0^{(n-2)}y^{(n)}, a_\alpha^{(n)}b^{(n-1)}\rangle$$
$$= \langle y^{(n)}, (a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\rangle \to \langle y^{(n)}, (a^{(n)}b^{(n-1)})a_0^{(n-2)}\rangle$$
$$= \langle a_0^{(n-2)}y^{(n)}, (a^{(n)}b^{(n-1)})\rangle = \langle b^{(n)}a^{(n)}, b^{(n-1)}\rangle.$$

Thus $b^{(n)}a_\alpha^{(n)} \xrightarrow{w} b^{(n)}a^{(n)}$. It follows that $b^{(n)} \in Z_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$.
The proof of the next part is similar to the preceding proof.

□

Example 2.14. *i) Let G be a locally compact group. Since $M(G)$ is a Banach $L^1(G)$-bimodule and the unit element of $M(G)^{(n)}$ does not have Lw^*w- property or Rw^*w- property, by using the preceding theorem, we have*

$$Z_{L^1(G)^{(n)}}(M(G)^{(n)}) \neq M(G)^{(n)}.$$

ii) If G is finite, then by using the preceding theorem, we conclude that

$$Z_{M(G)^{(n)}}(L^1(G)^{(n)}) = L^1(G)^{(n)},$$
$$Z_{L^1(G)^{(n)}}(M(G)^{(n)}) = M(G)^{(n)}.$$

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