

TOPOLOGICAL CENTERS OF THE $N - TH$ DUAL OF MODULE ACTIONS

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ABSTRACT. We study the topological centers of n th dual of Banach \mathcal{A} -modules and we extend some propositions from Lau and Ülger into n -th dual of Banach \mathcal{A} -modules where $n \geq 0$ is even number. Let \mathcal{B} be a Banach \mathcal{A} -bimodule. By using some new conditions, we show that $Z_{\mathcal{A}^{(n)}}^\ell(\mathcal{B}^{(n)}) = \mathcal{B}^{(n)}$ and $Z_{\mathcal{B}^{(n)}}^\ell(\mathcal{A}^{(n)}) = \mathcal{A}^{(n)}$. We get some conclusions on group algebras.

1. Introduction

Throughout this paper, \mathcal{A} is a Banach algebra and \mathcal{A}^* , \mathcal{A}^{**} , respectively, are the first and second dual of \mathcal{A} . A bounded net $(e_\alpha)_{\alpha \in I}$ in \mathcal{A} is called a *bounded left approximate identity* (BLAI) [respectively *bounded right approximate identity* (BRAI)] if, for each $a \in \mathcal{A}$, $e_\alpha a \rightarrow a$ [respectively $ae_\alpha \rightarrow a$]. Moreover, (e_α) is called a (two sided) bounded approximate identity (BAI), if for every $a \in \mathcal{A}$, the conditions $e_\alpha a \rightarrow a$ and $ae_\alpha \rightarrow a$ both hold. For $a \in \mathcal{A}$ and $a' \in \mathcal{A}^*$, we denote by $a'a$ and aa' respectively, the functionals on \mathcal{A}^* defined by $\langle a'a, b \rangle = \langle a', ab \rangle = a'(ab)$ and $\langle aa', b \rangle = \langle a', ba \rangle = a'(ba)$ for all $b \in \mathcal{A}$. The Banach algebra \mathcal{A} is embedded in its second dual via the identification $\langle a, a' \rangle = \langle a', a \rangle$ for every $a \in \mathcal{A}$ and $a' \in \mathcal{A}^*$. We denote the set $\{a'a : a \in \mathcal{A} \text{ and } a' \in \mathcal{A}^*\}$

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and $\{aa' : a \in \mathcal{A} \text{ and } a' \in \mathcal{A}^*\}$ by \mathcal{A}^*A and $A\mathcal{A}^*$, respectively. Clearly these two sets are subsets of \mathcal{A}^* .

Let \mathcal{A} has a *BAI*. If the equality $\mathcal{A}^*\mathcal{A} = \mathcal{A}^*$, ($\mathcal{A}\mathcal{A}^* = \mathcal{A}^*$) holds, then we say that \mathcal{A}^* factors on the left (right). If both equalities $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = \mathcal{A}^*$ hold, then we say that \mathcal{A}^* factors on both sides.

Let X, Y, Z be normed spaces and let $m : X \times Y \rightarrow Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as follows

1. $m^* : Z^* \times X \rightarrow Y^*$, given by $\langle m^*(z', x), y \rangle = \langle z', m(x, y) \rangle$ where $x \in X, y \in Y, z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \rightarrow X^*$, given by $\langle m^{**}(y'', z'), x \rangle = \langle y'', m^*(z', x) \rangle$ where $x \in X, y'' \in Y^{**}, z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \rightarrow Z^{**}$, given by $\langle m^{***}(x'', y''), z' \rangle = \langle x'', m^{**}(y'', z') \rangle$ where $x'' \in X^{**}, y'' \in Y^{**}, z' \in Z^*$.

The mapping m^{***} is the unique extension of m such that $x'' \rightarrow m^{***}(x'', y'')$ from X^{**} into Z^{**} is *weak*-to-weak** continuous for every $y'' \in Y^{**}$, but the mapping $y'' \rightarrow m^{***}(x'', y'')$ is not in general *weak*-to-weak** continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as follows

$$Z_1(m) = \{x'' \in X^{**} : y'' \rightarrow m^{***}(x'', y'') \text{ is } \textit{weak}^* \textit{-to-weak}^* \textit{ continuous}\}.$$

Now let $m^t : Y \times X \rightarrow Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z , and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \rightarrow Z^{**}$. The mapping $m^{t***} : X^{**} \times Y^{**} \rightarrow Z^{**}$ in general is not equal to m^{***} , see [1]. If $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \rightarrow m^{t***t}(x'', y'')$ is *weak*-to-weak** continuous for every $y'' \in Y^{**}$, but the mapping $x'' \rightarrow m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general *weak*-to-weak** continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : x'' \rightarrow m^{t***t}(x'', y'') \text{ is } \textit{weak}^* \textit{-to-weak}^* \textit{ continuous}\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j) \rangle = \lim_j \lim_i \langle z', m(x_i, y_j) \rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [18].

The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.

Now let \mathcal{B} be a Banach \mathcal{A} -bimodule, and let

$$\pi_\ell : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{B} \text{ and } \pi_r : \mathcal{B} \times \mathcal{A} \rightarrow \mathcal{B}.$$

be the left and right module actions of \mathcal{A} on \mathcal{B} , respectively. Then \mathcal{B}^{**} is a Banach \mathcal{A}^{**} -bimodule with the following module actions where \mathcal{A}^{**} is equipped with the left Arens product

$$\pi_\ell^{***} : \mathcal{A}^{**} \times \mathcal{B}^{**} \rightarrow \mathcal{B}^{**} \text{ and } \pi_r^{***} : \mathcal{B}^{**} \times \mathcal{A}^{**} \rightarrow \mathcal{B}^{**}.$$

Similarly, \mathcal{B}^{**} is a Banach \mathcal{A}^{**} -bimodule with the following module actions where \mathcal{A}^{**} is equipped with the right Arens product

$$\pi_\ell^{t****} : \mathcal{A}^{**} \times \mathcal{B}^{**} \rightarrow \mathcal{B}^{**} \text{ and } \pi_r^{t****} : \mathcal{B}^{**} \times \mathcal{A}^{**} \rightarrow \mathcal{B}^{**}.$$

We may therefore define the topological centers of the left and right module actions of \mathcal{A}^{**} on \mathcal{B}^{**} as follows:

$$\begin{aligned} Z_{\mathcal{B}^{**}}(\mathcal{A}^{**}) &= Z(\pi_\ell) = \{a'' \in \mathcal{A}^{**} : \text{the map } b'' \rightarrow \pi_\ell^{***}(a'', b'') : \\ &\quad \mathcal{B}^{**} \rightarrow \mathcal{B}^{**} \text{ is weak}^* \text{-to-weak}^* \text{ continuous}\} \\ Z_{\mathcal{B}^{**}}^t(\mathcal{A}^{**}) &= Z(\pi_r^t) = \{a'' \in \mathcal{A}^{**} : \text{the map } b'' \rightarrow \pi_r^{t****}(a'', b'') : \\ &\quad \mathcal{B}^{**} \rightarrow \mathcal{B}^{**} \text{ is weak}^* \text{-to-weak}^* \text{ continuous}\} \\ Z_{\mathcal{A}^{**}}(\mathcal{B}^{**}) &= Z(\pi_r) = \{b'' \in \mathcal{B}^{**} : \text{the map } a'' \rightarrow \pi_r^{***}(b'', a'') : \\ &\quad \mathcal{A}^{**} \rightarrow \mathcal{B}^{**} \text{ is weak}^* \text{-to-weak}^* \text{ continuous}\} \\ Z_{\mathcal{A}^{**}}^t(\mathcal{B}^{**}) &= Z(\pi_\ell^t) = \{b'' \in \mathcal{B}^{**} : \text{the map } a'' \rightarrow \pi_\ell^{t****}(b'', a'') : \\ &\quad \mathcal{A}^{**} \rightarrow \mathcal{B}^{**} \text{ is weak}^* \text{-to-weak}^* \text{ continuous}\}. \end{aligned}$$

We note that if \mathcal{B} is a left (respectively right) Banach \mathcal{A} -module and $\pi_\ell : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{B}$ (respectively $\pi_r : \mathcal{B} \times \mathcal{A} \rightarrow \mathcal{B}$) is left (respectively right) module action of \mathcal{A} on \mathcal{B} , then \mathcal{B}^* is a right (respectively left) Banach \mathcal{A} -module.

We write $ab = \pi_\ell(a, b)$, $ba = \pi_r(b, a)$, $\pi_\ell(a_1 a_2, b) = \pi_\ell(a_1, a_2 b)$, $\pi_r(b, a_1 a_2) = \pi_r(b a_1, a_2)$, $\pi_\ell^*(a_1 b', a_2) = \pi_\ell^*(b', a_2 a_1)$, $\pi_r^*(b' a, b) = \pi_r^*(b', ab)$, for all $a_1, a_2, a \in \mathcal{A}$, $b \in \mathcal{B}$ and $b' \in \mathcal{B}^*$ when there is no confusion.

Regarding \mathcal{A} as a Banach \mathcal{A} -bimodule, the operation $\pi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ extends to π^{***} and π^{t****} defined on $\mathcal{A}^{**} \times \mathcal{A}^{**}$. These extensions are

known as the first(left) and the second (right) Arens products, respectively and in both cases, the second dual space \mathcal{A}^{**} becomes a Banach algebra. In this situation, we will also simplify our notations. So the first (left) Arens product of $a'', b'' \in \mathcal{A}^{**}$ will be simply denoted by $a''b''$ and will be defined by the following three steps:

$$\begin{aligned}\langle a'a, b \rangle &= \langle a', ab \rangle, \\ \langle a''a', a \rangle &= \langle a'', a'a \rangle, \\ \langle a''b'', a' \rangle &= \langle a'', b'a' \rangle.\end{aligned}$$

for every $a, b \in \mathcal{A}$ and $a' \in \mathcal{A}^*$. Similarly, the second (right) Arens product of $a'', b'' \in \mathcal{A}^{**}$ will be denoted by $a'' \circ b''$ and will be defined by :

$$\begin{aligned}\langle aoa', b \rangle &= \langle a', ba \rangle, \\ \langle a' \circ a'', a \rangle &= \langle a'', a \circ a' \rangle, \\ \langle a'' \circ b'', a' \rangle &= \langle b'', a' \circ b'' \rangle,\end{aligned}$$

for all $a, b \in \mathcal{A}$ and $a' \in \mathcal{A}^*$.

The regularity of a normed algebra \mathcal{A} is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of \mathcal{A}^{**} . By *Goldstine's* Theorem [4, p.425] there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in \mathcal{A} such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in \mathcal{A}^*$, we have

$$\begin{aligned}\lim_\alpha \lim_\beta \langle a', \pi(a_\alpha, b_\beta) \rangle &= \langle a''b'', a' \rangle, \\ \lim_\beta \lim_\alpha \langle a', \pi(a_\alpha, b_\beta) \rangle &= \langle a'' \circ b'', a' \rangle,\end{aligned}$$

where $a''b''$ and $a'' \circ b''$ are the first and second Arens products of \mathcal{A}^{**} , respectively, see [14, 18].

We find the usual first and second topological center of \mathcal{A}^{**} , which are

$$\begin{aligned}Z_{\mathcal{A}^{**}}(\mathcal{A}^{**}) &= Z(\pi) = \{a'' \in \mathcal{A}^{**} : b'' \rightarrow a''b'' \text{ is } weak^* \text{-to-} weak^* \\ &\quad \text{continuous}\}, \\ Z_{\mathcal{A}^{**}}^t(\mathcal{A}^{**}) &= Z(\pi^t) = \{a'' \in \mathcal{A}^{**} : a'' \rightarrow a'' \circ b'' \text{ is } weak^* \text{-to-} weak^* \\ &\quad \text{continuous}\}.\end{aligned}$$

An element e'' of \mathcal{A}^{**} is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if, for each $a'' \in \mathcal{A}^{**}$, $a''e'' = e'' \circ a'' = a''$. By [4, p.146] an element e'' of \mathcal{A}^{**} is mixed unit if and only if it is a *weak** cluster point of some BAI $(e_\alpha)_{\alpha \in I}$ in \mathcal{A} .

A functional a' in \mathcal{A}^* is said to be *wap* (weakly almost periodic) on \mathcal{A} if the mapping $a \rightarrow a'a$ from \mathcal{A} into \mathcal{A}^* is weakly compact. Pym in [18] showed that this definition is equivalent to the following condition. For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a \in \mathcal{A} : \|a\| \leq 1\}$, we have

$$\lim_{\alpha} \lim_{\beta} \langle a', a_\alpha b_\beta \rangle = \lim_{\beta} \lim_{\alpha} \langle a', a_\alpha b_\beta \rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on \mathcal{A} is denoted by $wap(\mathcal{A})$. We also have $a' \in wap(\mathcal{A})$ if and only if $\langle a''b'', a' \rangle = \langle a'' \circ b'', a' \rangle$ for every $a'', b'' \in \mathcal{A}^{**}$.

This paper is organized as follows:

a) Let \mathcal{B} be a Banach \mathcal{A} -bimodule. Let $n \geq 0$ be an even number and $0 \leq r \leq \frac{n}{2}$. Assume that $U_{n,r} = (\mathcal{A}^{(n-r)}\mathcal{A}^{(r)})^{(r)}$ or $U_{n,r} = (\mathcal{A}^{(n-r)}\mathcal{A}^{(r-1)})^{(r)}$ and $\phi \in U_{n,r}$. Then $\phi \in Z_{\mathcal{B}^{(n)}}^{\ell}(U_{n,r})$ if and only if $b^{(n-1)}\phi \in \mathcal{B}^{(n-1)}$ for all $b^{(n-1)} \in \mathcal{B}^{(n-1)}$.

b) Let \mathcal{B} be a Banach \mathcal{A} -bimodule. Then

- (1) $b^{(n)} \in Z_{\mathcal{A}^{(n)}}^{\ell}(\mathcal{B}^{(n)})$ if and only if $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$ for all $b^{(n-1)} \in \mathcal{B}^{(n-1)}$.
- (2) If $\phi \in Z_{\mathcal{B}^{(n)}}^{\ell}(U_{n,r})$, then $a^{(n-2)}\phi \in Z_{\mathcal{B}^{(n)}}^{\ell}(\mathcal{A}^{(n)})$ for all $a^{(n-2)} \in \mathcal{A}^{(n-2)}$.

c) Let \mathcal{B} be a Banach space such that $\mathcal{B}^{(n)}$ is weakly compact. Then for Banach \mathcal{A} -bimodule \mathcal{B} , we have the following assertions.

- (1) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ is a *BLAI* for $\mathcal{B}^{(n)}$ such that

$$e_\alpha^{(n)}\mathcal{B}^{(n+2)} \subseteq \mathcal{B}^{(n)},$$

for every α . Then \mathcal{B} is reflexive.

- (2) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ is a *BRAI* for $\mathcal{B}^{(n)}$ and

$$Z_{e^{(n+2)}}^{\ell}(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)},$$

where $e_\alpha^{(n)} \xrightarrow{w^*} e^{(n+2)}$ in $\mathcal{A}^{(n)}$. If $\mathcal{B}^{(n+2)}e_\alpha^{(n)} \subseteq \mathcal{B}^{(n)}$ for every α , then $Z_{\mathcal{A}^{(n+2)}}^{\ell}(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)}$.

d) Assume that \mathcal{B} is a Banach \mathcal{A} -bimodule. Then

- (1) $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq wap_{\ell}(\mathcal{B}^{(n)})$ if and only if

$$\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq Z_{\mathcal{B}^{(n+2)}}^{\ell}(\mathcal{A}^{(n+2)}).$$

- (2) If $\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq \mathcal{A}^{(n)}Z_{\mathcal{B}^{(n+2)}}^{\ell}(\mathcal{A}^{(n+2)})$, then

$$\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq Z_{\mathcal{B}^{(n+2)}}^{\ell}(\mathcal{A}^{(n+2)}).$$

e) Let \mathcal{B} be a left Banach \mathcal{A} -bimodule and let $n \geq 0$ be an even number. Suppose that $b_0^{(n+1)} \in \mathcal{B}^{(n+1)}$. Then $b_0^{(n+1)} \in \text{wap}_\ell(\mathcal{B}^{(n)})$ if and only if the mapping $T : b^{(n+2)} \rightarrow b^{(n+2)}b_0^{(n+1)}$ form $\mathcal{B}^{(n+2)}$ into $\mathcal{A}^{(n+1)}$ is *weak*-to-weak* continuous.

f) Let \mathcal{B} be a left Banach \mathcal{A} -bimodule. Then for $n \geq 2$, we have the following assertions.

- (1) If $\mathcal{A}^{(n)} = a_0^{(n-2)}\mathcal{A}^{(n)}$ [respectively, $\mathcal{A}^{(n)} = \mathcal{A}^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has *Rw*w-* property [respectively *Lw*w-* property] with respect to $\mathcal{B}^{(n)}$, then $Z_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)}) = \mathcal{A}^{(n)}$.
- (2) If $\mathcal{B}^{(n)} = a_0^{(n-2)}\mathcal{B}^{(n)}$ [respectively, $\mathcal{B}^{(n)} = \mathcal{B}^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has *Rw*w-* property [respectively *Lw*w-* property] with respect to $\mathcal{B}^{(n)}$, then $Z_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)}) = \mathcal{B}^{(n)}$.

2. Topological Centers of Module Actions

Suppose that \mathcal{A} is a Banach algebra and \mathcal{B} is a Banach \mathcal{A} -bimodule. According to [5, p.27-28] \mathcal{B}^{**} is a Banach \mathcal{A}^{**} -bimodule, where \mathcal{A}^{**} is equipped with the first Arens product. We recall the topological centers of module actions of \mathcal{A}^{**} on \mathcal{B}^{**} as follows.

$$\begin{aligned} Z_{\mathcal{A}^{**}}^\ell(\mathcal{B}^{**}) &= \{b'' \in \mathcal{B}^{**} : \text{the map } a'' \rightarrow b''a'' : \mathcal{A}^{**} \rightarrow \mathcal{B}^{**} \\ &\quad \text{is weak* -to-weak* continuous}\} \\ Z_{\mathcal{B}^{**}}^\ell(\mathcal{A}^{**}) &= \{a'' \in \mathcal{A}^{**} : \text{the map } b'' \rightarrow a''b'' : \mathcal{B}^{**} \rightarrow \mathcal{B}^{**} \\ &\quad \text{is weak* -to-weak* continuous}\}. \end{aligned}$$

Let $\mathcal{A}^{(n)}$ and $\mathcal{B}^{(n)}$ be n th dual of \mathcal{A} and \mathcal{B} , respectively. By [25, p. 4132-4134] if $n \geq 0$ is an even number, then $\mathcal{B}^{(n)}$ is a Banach $\mathcal{A}^{(n)}$ -bimodule. Then for $n \geq 2$, we define $\mathcal{B}^{(n)}\mathcal{B}^{(n-1)}$ as a subspace of $\mathcal{A}^{(n-1)}$, that is, for all $b^{(n)} \in \mathcal{B}^{(n)}$, $b^{(n-1)} \in \mathcal{B}^{(n-1)}$ and $a^{(n-2)} \in \mathcal{A}^{(n-2)}$ we define

$$\langle b^{(n)}b^{(n-1)}, a^{(n-2)} \rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-2)} \rangle.$$

If n is odd number, we define $\mathcal{B}^{(n)}\mathcal{B}^{(n-1)}$ as a subspace of $\mathcal{A}^{(n)}$, that is, for all $b^{(n)} \in \mathcal{B}^{(n)}$, $b^{(n-1)} \in \mathcal{B}^{(n-1)}$ and $a^{(n-1)} \in \mathcal{A}^{(n-1)}$, we define

$$\langle b^{(n)}b^{(n-1)}, a^{(n-1)} \rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-1)} \rangle.$$

If $n = 0$, we take $\mathcal{A}^{(0)} = \mathcal{A}$ and $\mathcal{B}^{(0)} = \mathcal{B}$.

We also define the topological centers of module actions of $\mathcal{A}^{(n)}$ on $\mathcal{B}^{(n)}$ as follows

$$\begin{aligned} Z_{\mathcal{A}^{(n)}}^{\ell}(\mathcal{B}^{(n)}) &= \{b^{(n)} \in \mathcal{B}^{(n)} : \text{the map } a^{(n)} \rightarrow b^{(n)}a^{(n)} : \mathcal{A}^{(n)} \rightarrow \mathcal{B}^{(n)} \\ &\quad \text{is weak}^*\text{-to-weak}^* \text{ continuous}\} \\ Z_{\mathcal{B}^{(n)}}^{\ell}(\mathcal{A}^{(n)}) &= \{a^{(n)} \in \mathcal{A}^{(n)} : \text{the map } b^{(n)} \rightarrow a^{(n)}b^{(n)} : \mathcal{B}^{(n)} \rightarrow \mathcal{B}^{(n)} \\ &\quad \text{is weak}^*\text{-to-weak}^* \text{ continuous}\}. \end{aligned}$$

Let \mathcal{A} be a Banach algebra with a (BAI) and suppose that $\mathcal{A}^{(n)}$ and $\mathcal{A}^{(m)}$ are the n th dual and m th dual of \mathcal{A} , respectively. Suppose that at least one of the integers n or m is an even number. Then we define the set $\mathcal{A}^{(n)}\mathcal{A}^{(m)}$ as a linear space that is generated by the following set

$$\{a^{(n)}a^{(m)} : a^{(n)} \in \mathcal{A}^{(n)} \text{ and } a^{(m)} \in \mathcal{A}^{(m)}\},$$

where the multiplication $a^{(n)}a^{(m)}$ is defined with respect to the first Arens product. If $n \geq m$, then $\mathcal{A}^{(n)}\mathcal{A}^{(m)}$ is a subspace of $\mathcal{A}^{(n)}$. Observe that $\mathcal{A}^{(n)}\mathcal{A}^{(m)}$ is Banach algebra whenever n and m are even numbers, but if one of them is an odd number, then $\mathcal{A}^{(n)}\mathcal{A}^{(m)}$ is not in general a Banach algebra.

Let $n \geq 0$ be an even number and $0 \leq r \leq \frac{n}{2}$. For a Banach algebra \mathcal{A} , we define a new Banach algebra $U_{n,r}$ with respect to the first Arens product as follows.

If r is an even (respectively odd) number, then we write $U_{n,r} = (\mathcal{A}^{(n-r)}\mathcal{A}^{(r)})^{(r)}$ (respectively $U_{n,r} = (\mathcal{A}^{(n-r)}\mathcal{A}^{(r-1)})^{(r)}$). It is clear that $U_{n,r}$ is a subalgebra of $\mathcal{A}^{(n)}$. For example, if we take $n = 2$ and $r = 1$, then $U_{2,1} = (\mathcal{A}^*\mathcal{A})^*$ is a subalgebra of \mathcal{A}^{**} with respect to the first Arens product.

Now if \mathcal{B} is a Banach \mathcal{A} -bimodule, then it is clear that $\mathcal{B}^{(n)}$ is a Banach $U_{n,r}$ -bimodule with respect to the first Arens product, for detail see [25]. Thus we can define the topological centers of module actions $U_{n,r}$ on $\mathcal{B}^{(n)}$ as $Z_{\mathcal{B}^{(n)}}^{\ell}(U_{n,r})$ and $Z_{U_{n,r}}^{\ell}(\mathcal{B}^{(n)})$ similar to the preceding definitions. In every parts of this paper, $n \geq 0$ is even number.

Theorem 2.1. *Let \mathcal{B} be a Banach \mathcal{A} -bimodule and $\phi \in U_{n,r}$. Then $\phi \in Z_{\mathcal{B}^{(n)}}^{\ell}(U_{n,r})$ if and only if $b^{(n-1)}\phi \in \mathcal{B}^{(n-1)}$ for all $b^{(n-1)} \in \mathcal{B}^{(n-1)}$.*

Proof. Let $\phi \in Z_{\mathcal{B}^{(n)}}^{\ell}(U_{n,r})$. Suppose that $(b_{\alpha}^{(n)})_{\alpha} \subseteq \mathcal{B}^{(n)}$ such that $b_{\alpha}^{(n)} \xrightarrow{w^*} b^{(n)}$ in $\mathcal{B}^{(n)}$. Then, for every $b^{(n-1)} \in \mathcal{B}^{(n-1)}$, we have

$$\langle b^{(n-1)}\phi, b_{\alpha}^{(n)} \rangle = \langle b^{(n-1)}, \phi b_{\alpha}^{(n)} \rangle = \langle \phi b_{\alpha}^{(n)}, b^{(n-1)} \rangle \rightarrow \langle \phi b^{(n)}, b^{(n-1)} \rangle$$

$$= \langle b^{(n-1)}\phi, b^{(n)} \rangle.$$

It follows that $b^{(n-1)}\phi \in (\mathcal{B}^{(n)}, weak^*)^* = \mathcal{B}^{(n-1)}$.

Conversely, let $b^{(n-1)}\phi \in \mathcal{B}^{(n-1)}$ for every $b^{(n-1)} \in \mathcal{B}^{(n-1)}$ and suppose that $(b_\alpha^{(n)})_\alpha \subseteq \mathcal{B}^{(n)}$ such that $b_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}$ in $\mathcal{B}^{(n)}$. Then

$$\begin{aligned} \langle \phi b_\alpha^{(n)}, b^{(n-1)} \rangle &= \langle \phi, b_\alpha^{(n)} b^{(n-1)} \rangle = \langle b_\alpha^{(n)} b^{(n-1)}, \phi \rangle = \langle b_\alpha^{(n)}, b^{(n-1)} \phi \rangle \\ &\rightarrow \langle b^{(n)}, b^{(n-1)} \phi \rangle = \langle \phi b^{(n)}, b^{(n-1)} \rangle. \end{aligned}$$

It follows that $\phi b_\alpha^{(n)} \xrightarrow{w^*} \phi b^{(n)}$, and so $\phi \in Z^\ell_{\mathcal{B}^{(n)}}(U_{n,r})$. \square

In the preceding theorem if we take $\mathcal{B} = \mathcal{A}$, $n = 2$ and $r = 1$, we obtain Lemma 3.1 (b) of [14].

Theorem 2.2. *Let \mathcal{B} be a Banach \mathcal{A} -bimodule and $b^{(n)} \in \mathcal{B}^{(n)}$. Then we have the following assertions:*

- (1) $b^{(n)} \in Z^\ell_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$ if and only if $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$ for all $b^{(n-1)} \in \mathcal{B}^{(n-1)}$.
- (2) If $\phi \in Z^\ell_{\mathcal{B}^{(n)}}(U_{n,r})$, then $a^{(n-2)}\phi \in Z^\ell_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)})$ for all $a^{(n-2)} \in \mathcal{A}^{(n-2)}$.

Proof. (1) Let $b^{(n)} \in Z^\ell_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$. We show that $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$ where $b^{(n-1)} \in \mathcal{B}^{(n-1)}$. Suppose that $(a_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ and $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$ in $\mathcal{A}^{(n)}$. Then we have

$$\begin{aligned} \langle b^{(n-1)}b^{(n)}, a_\alpha^{(n)} \rangle &= \langle b^{(n-1)}, b^{(n)} a_\alpha^{(n)} \rangle = \langle b^{(n)} a_\alpha^{(n)}, b^{(n-1)} \rangle \\ &\rightarrow \langle b^{(n)} a^{(n)}, b^{(n-1)} \rangle = \langle b^{(n-1)}b^{(n)}, a^{(n)} \rangle. \end{aligned}$$

Consequently, $b^{(n-1)}b^{(n)} \in (\mathcal{A}^{(n)}, weak^*)^* = \mathcal{A}^{(n-1)}$. It follows that $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$.

Conversely, let $b^{(n-1)}b^{(n)} \in \mathcal{A}^{(n-1)}$ for each $b^{(n-1)} \in \mathcal{B}^{(n-1)}$. Suppose that $(a_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ and $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$ in $\mathcal{A}^{(n)}$. Then we have

$$\begin{aligned} \langle b^{(n)} a_\alpha^{(n)}, b^{(n-1)} \rangle &= \langle b^{(n)}, a_\alpha^{(n)} b^{(n-1)} \rangle = \langle a_\alpha^{(n)} b^{(n-1)}, b^{(n)} \rangle \\ &= \langle a_\alpha^{(n)}, b^{(n-1)}b^{(n)} \rangle \rightarrow \langle a^{(n)}, b^{(n-1)}b^{(n)} \rangle = \langle b^{(n)} a^{(n)}, b^{(n-1)} \rangle. \end{aligned}$$

It follows that $b^{(n)} a_\alpha^{(n)} \xrightarrow{w^*} b^{(n)} a^{(n)}$, and hence $b^{(n)} \in Z^\ell_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$.

- (2) Let $\phi \in Z^\ell_{\mathcal{B}^{(n)}}(U_{n,r})$ and $a^{(n-2)} \in \mathcal{A}^{(n-2)}$. Assume that $(b_\alpha^{(n)})_\alpha \subseteq \mathcal{B}^{(n)}$ such that $b_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}$ in $\mathcal{B}^{(n)}$. Then for all $b^{(n-1)} \in \mathcal{B}^{(n-1)}$, we have

$$\langle (a^{(n-2)}\phi)b_\alpha^{(n)}, b^{(n-1)} \rangle = \langle \phi b_\alpha^{(n)}, b^{(n-1)} a^{(n-2)} \rangle \rightarrow$$

$$\langle \phi b^{(n)}, b^{(n-1)} a^{(n-2)} \rangle = \langle (a^{(n-2)} \phi) b^{(n)}, b^{(n-1)} \rangle.$$

It follows that $(a^{(n-2)} \phi) b^{(n)} \xrightarrow{w^*} (a^{(n-2)} \phi) b^{(n)}$, and hence

$$a^{(n-2)} \phi \in Z_{\mathcal{B}^{(n)}}^\ell(\mathcal{A}^{(n)}).$$

□

In the preceding theorem, part (1), if we take $\mathcal{B} = \mathcal{A}$ and $n = 2$, we conclude Lemma 3.1 (a) of [14]. In part (2) of this theorem, if we take $\mathcal{B} = \mathcal{A}$, $n = 2$ and $r = 1$, we also obtain Lemma 3.1 (c) from [14].

Definition 2.3. Let \mathcal{B} be a Banach \mathcal{A} -bimodule and suppose that $a'' \in \mathcal{A}^{**}$. Assume that $(a''_\alpha)_\alpha \subseteq \mathcal{A}^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$. If for every $b'' \in \mathcal{B}^{**}$, $b'' a''_\alpha \xrightarrow{w^*} b'' a''$, then we say that $a'' \rightarrow b'' a''$ is weak*-to-weak* point continuous.

Suppose that \mathcal{B} is a Banach \mathcal{A} -bimodule. Assume that $a'' \in \mathcal{A}^{**}$. Then we define the locally topological center of a'' on \mathcal{B}^{**} as follows

$$Z_{a''}^\ell(\mathcal{B}^{**}) = \{b'' \in \mathcal{B}^{**} : a'' \rightarrow b'' a'' \text{ is weak*-to-weak* point continuous}\}.$$

The definition of $Z_{b''}^\ell(\mathcal{A}^{**})$ where $b'' \in \mathcal{B}^{**}$ is similar.

It is clear that

$$\bigcap_{a'' \in \mathcal{A}^{**}} Z_{a''}^\ell(\mathcal{B}^{**}) = Z_{\mathcal{A}^{**}}^\ell(\mathcal{B}^{**}),$$

$$\bigcap_{b'' \in \mathcal{B}^{**}} Z_{b''}^\ell(\mathcal{A}^{**}) = Z_{\mathcal{B}^{**}}^\ell(\mathcal{A}^{**}).$$

Let \mathcal{B} be a Banach space. Then $K \subseteq \mathcal{B}$ is called weakly compact, if K is compact with respect to weak topology on \mathcal{B} . By [7], we know that K is weakly compact if and only if K is weakly limit point compact.

Theorem 2.4. Assume that \mathcal{B} is a Banach \mathcal{A} -bimodule such that $\mathcal{B}^{(n)}$ is weakly compact. Then we have the following assertions:

(1) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ is a BLAI for $\mathcal{B}^{(n)}$ such that

$$e_\alpha^{(n)} \mathcal{B}^{(n+2)} \subseteq \mathcal{B}^{(n)},$$

for every α . Then \mathcal{B} is reflexive.

(2) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ is a BRAI for $\mathcal{B}^{(n)}$ and

$$Z_{e^{(n+2)}}^\ell(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)},$$

where $e_\alpha^{(n)} \xrightarrow{w^*} e^{(n+2)}$ in $\mathcal{A}^{(n)}$. If $\mathcal{B}^{(n+2)} e_\alpha^{(n)} \subseteq \mathcal{B}^{(n)}$ for every α , then $Z_{\mathcal{A}^{(n+2)}}^\ell(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)}$.

Proof. (1) Let $b^{n+2} \in \mathcal{B}^{(n+2)}$. Since $(e_\alpha^{(n)})_\alpha$ is a BLAI for $\mathcal{B}^{(n)}$, without loss generality, there is a left unit $e^{(n+2)} \in \mathcal{A}^{n+2}$ for $\mathcal{B}^{(n+2)}$ such that $e_\alpha^{(n)} \xrightarrow{w^*} e^{(n+2)}$ in $\mathcal{A}^{(n+2)}$, see [10]. Then we have $e_\alpha^{(n)} b^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$ in $\mathcal{B}^{(n+2)}$. Since $e_\alpha^{(n)} b^{(n+2)} \in \mathcal{B}^{(n)}$, we have $e_\alpha^{(n)} b^{(n+2)} \xrightarrow{w} b^{(n+2)}$ in $\mathcal{B}^{(n)}$. We conclude that $b^{n+2} \in \mathcal{B}^{(n)}$, because $\mathcal{B}^{(n)}$ is weakly compact.

(2) Suppose that $b^{(n+2)} \in Z_{\mathcal{A}^{(n+2)}}^\ell(\mathcal{B}^{(n+2)})$ and $e_\alpha^{(n)} \xrightarrow{w^*} e^{(n+2)}$ in $\mathcal{A}^{(n)}$ such that $e^{(n+2)}$ is a right unit for $\mathcal{B}^{(n+2)}$, see [10]. Then we have $b^{(n+2)} e_\alpha^{(n)} \xrightarrow{w^*} b^{(n+2)}$ in $\mathcal{B}^{(n+2)}$. Since $\mathcal{B}^{(n+2)} e_\alpha^{(n)} \subseteq \mathcal{B}^{(n)}$ for every α , $b^{(n+2)} e_\alpha^{(n)} \xrightarrow{w} b^{(n+2)}$ in $\mathcal{B}^{(n)}$ and since $\mathcal{B}^{(n)}$ is weakly compact, $b^{(n+2)} \in \mathcal{B}^{(n)}$. It follows that $Z_{\mathcal{A}^{(n+2)}}^\ell(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)}$. \square

Definition 2.5. Let \mathcal{B} be a Banach \mathcal{A} -bimodule and $n \geq 0$. Then $b^{(n+2)} \in \mathcal{B}^{(n+2)}$ is said to be weakly left almost periodic functional if the set

$$\{b^{(n+1)} a^{(n)} : a^{(n)} \in \mathcal{A}^{(n)}, \|a^{(n)}\| \leq 1\},$$

is relatively weakly compact, and $b^{(n+2)} \in \mathcal{B}^{(n+2)}$ is said to be weakly right almost periodic functional if the set

$$\{a^{(n)} b^{(n+1)} : a^{(n)} \in \mathcal{A}^{(n)}, \|a^{(n)}\| \leq 1\},$$

is relatively weakly compact. We denote by $wap_\ell(\mathcal{B}^{(n)})$ [respectively $wap_r(\mathcal{B}^{(n)})$] the closed subspace of $\mathcal{B}^{(n+1)}$ consisting of all weakly left [respectively right] almost periodic functionals in $\mathcal{B}^{(n+1)}$.

By [6, 14, 18], the definition of $wap_\ell(\mathcal{B}^{(n)})$ and $wap_r(\mathcal{B}^{(n)})$, respectively, are equivalent to the following:

$$\begin{aligned} wap_\ell(\mathcal{B}^{(n)}) &= \{b^{(n+1)} \in \mathcal{B}^{(n+1)} : \langle b^{(n+2)} a_\alpha^{(n+2)}, b^{(n+1)} \rangle \rightarrow \\ &\langle b^{(n+2)} a^{(n+2)}, b^{(n+1)} \rangle \text{ where } a_\alpha^{(n+2)} \xrightarrow{w^*} a^{(n+2)}\}. \end{aligned}$$

and

$$\begin{aligned} wap_r(\mathcal{B}^{(n)}) &= \{b^{(n+1)} \in \mathcal{B}^{(n+1)} : \langle a^{(n+2)} b_\alpha^{(n+2)}, b^{(n+1)} \rangle \rightarrow \\ &\langle a^{(n+2)} b^{(n+2)}, b^{(n+1)} \rangle \text{ where } b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}\}. \end{aligned}$$

If we take $\mathcal{A} = \mathcal{B}$ and $n = 0$, then $wap_\ell(\mathcal{A}) = wap_r(\mathcal{A}) = wap(\mathcal{A})$.

Theorem 2.6. *Assume that \mathcal{B} is a Banach \mathcal{A} -bimodule and $n \geq 0$. Then we have the following assertions:*

(1) $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq \text{wap}_\ell(\mathcal{B}^{(n)})$ if and only if

$$\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq Z_{\mathcal{B}^{(n+2)}}^\ell(\mathcal{A}^{(n+2)}).$$

(2) If $\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq \mathcal{A}^{(n)}Z_{\mathcal{B}^{(n+2)}}^\ell(\mathcal{A}^{(n+2)})$, then

$$\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq Z_{\mathcal{B}^{(n+2)}}^\ell(\mathcal{A}^{(n+2)}).$$

Proof. (1) Suppose that $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq \text{wap}_\ell(\mathcal{B}^{(n)})$. Let $a^{(n)} \in \mathcal{A}^{(n)}$, $a^{(n+2)} \in \mathcal{A}^{(n+2)}$ and let $(b_\alpha^{(n+2)})_\alpha \subseteq \mathcal{B}^{(n+2)}$ such that $b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$. Then for every $b^{(n+1)} \in \mathcal{B}^{(n+1)}$, we have

$$\begin{aligned} \langle (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)} \rangle &= \langle a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)} \rangle \\ \rightarrow \langle a^{(n+2)}b^{(n+2)}, b^{(n+1)}a^{(n)} \rangle &= \langle (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)} \rangle. \end{aligned}$$

It follows that $a^{(n)}a^{(n+2)} \in Z_{\mathcal{B}^{(n+2)}}^\ell(\mathcal{A}^{(n+2)})$.

Conversely, let $a^{(n)}a^{(n+2)} \in Z_{\mathcal{B}^{(n+2)}}^\ell(\mathcal{A}^{(n+2)})$ for every $a^{(n)} \in \mathcal{A}^{(n)}$, $a^{(n+2)} \in \mathcal{A}^{(n+2)}$. Suppose that $(b_\alpha^{(n+2)})_\alpha \subseteq \mathcal{B}^{(n+2)}$ such that $b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$. Then for every $b^{(n+1)} \in \mathcal{B}^{(n+1)}$, we have

$$\begin{aligned} \langle a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)} \rangle &= \langle (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)} \rangle \\ \rightarrow \langle (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)} \rangle &= \langle a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)} \rangle. \end{aligned}$$

It follows that $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq \text{wap}_\ell(\mathcal{B}^{(n)})$.

(2) Since $\mathcal{A}^{(n)}\mathcal{A}^{(n+2)} \subseteq \mathcal{A}^{(n)}Z_{\mathcal{B}^{(n+2)}}^\ell(\mathcal{A}^{(n+2)})$, for every $a^{(n)} \in \mathcal{A}^{(n)}$ and $a^{(n+2)} \in \mathcal{A}^{(n+2)}$, we have $a^{(n)}a^{(n+2)} \in \mathcal{A}^{(n)}Z_{\mathcal{B}^{(n+2)}}^\ell(\mathcal{A}^{(n+2)})$. Then there are $x^{(n)} \in \mathcal{A}^{(n)}$ and $\phi \in Z_{\mathcal{B}^{(n+2)}}^\ell(\mathcal{A}^{(n+2)})$ such that $a^{(n)}a^{(n+2)} = x^{(n)}\phi$. Suppose that $(b_\alpha^{(n+2)})_\alpha \subseteq \mathcal{B}^{(n+2)}$ such that $b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$. Then for every $b^{(n+1)} \in \mathcal{B}^{(n+1)}$, we have

$$\begin{aligned} \langle (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)} \rangle &= \langle (x^{(n)}\phi)b_\alpha^{(n+2)}, b^{(n+1)} \rangle \\ &= \langle \phi b_\alpha^{(n+2)}, b^{(n+1)}x^{(n)} \rangle \rightarrow \langle \phi b^{(n+2)}, b^{(n+1)}x^{(n)} \rangle \\ &= \langle (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)} \rangle. \end{aligned}$$

□

In the preceding theorem, if we take $\mathcal{B} = \mathcal{A}$ and $n = 0$, we conclude Theorem 3.6 (a) of [14].

Theorem 2.7. *Assume that \mathcal{B} is a Banach \mathcal{A} -bimodule and $n \geq 0$. If $\mathcal{A}^{(n)}$ is a left ideal in $\mathcal{A}^{(n+2)}$, then $\mathcal{B}^{(n+1)}\mathcal{A}^{(n)} \subseteq \text{wap}_\ell(\mathcal{B}^{(n)})$.*

Proof. The proof is clear. \square

Theorem 2.8. *Let \mathcal{B} be a left Banach \mathcal{A} -bimodule and $n \geq 0$ be an even number. Suppose that $b_0^{(n+1)} \in \mathcal{B}^{(n+1)}$. Then $b_0^{(n+1)} \in \text{wap}_\ell(\mathcal{B}^{(n)})$ if and only if the mapping $T : b^{(n+2)} \rightarrow b^{(n+2)}b_0^{(n+1)}$ form $\mathcal{B}^{(n+2)}$ into $\mathcal{A}^{(n+1)}$ is weak*-to-weak continuous.*

Proof. Let $b_0^{(n+1)} \in \mathcal{B}^{(n+1)}$ and suppose that $b_\alpha^{(n+2)} \xrightarrow{w^*} b^{(n+2)}$ in $\mathcal{B}^{(n+2)}$. Then for every $a^{(n+2)} \in \mathcal{A}^{(n+2)}$, we have

$$\begin{aligned} \langle a^{(n+2)}, b_\alpha^{(n+2)}b_0^{(n+1)} \rangle &= \langle a^{(n+2)}b_\alpha^{(n+2)}, b_0^{(n+1)} \rangle \rightarrow \langle a^{(n+2)}b^{(n+2)}, b_0^{(n+1)} \rangle \\ &= \langle a^{(n+2)}, b^{(n+2)}b_0^{(n+1)} \rangle. \end{aligned}$$

It follows that $b_\alpha^{(n+2)}b_0^{(n+1)} \xrightarrow{w} b^{(n+2)}b_0^{(n+1)}$ in $\mathcal{A}^{(n+1)}$. The proof of the converse is similar to the preceding proof. \square

Corollary 2.9. *Assume that \mathcal{B} is a Banach \mathcal{A} -bimodule. Then $Z_{\mathcal{A}^{(n+2)}}^\ell(\mathcal{B}^{(n+2)}) = \mathcal{B}^{(n+2)}$ if and only if the mapping $T : b^{(n+2)} \rightarrow b^{(n+2)}b_0^{(n+1)}$ form $\mathcal{B}^{(n+2)}$ into $\mathcal{A}^{(n+1)}$ is weak*-to-weak continuous for every $b_0^{(n+1)} \in \mathcal{B}^{(n+1)}$.*

Corollary 2.10. *Let \mathcal{A} be a Banach algebra. Assume that $a' \in \mathcal{A}^*$ and $T_{a'}$ is a linear operator from \mathcal{A} into \mathcal{A}^* defined by $T_{a'}a = a'a$. Then, $a' \in \text{wap}(\mathcal{A})$ if and only if the adjoint of $T_{a'}$ is weak*-to-weak continuous. So \mathcal{A} is Arens regular if and only if the adjoint of the mapping $T_{a'}a = a'a$ is weak*-to-weak continuous for every $a' \in \mathcal{A}^*$.*

Definition 2.11. *Let \mathcal{B} be a left Banach \mathcal{A} -bimodule and $a^{(n)} \in \mathcal{A}^{(n)}$. Let $(b_\alpha^{(n+1)})_\alpha \subseteq \mathcal{B}^{(n+1)}$ such that $a^{(n)}b_\alpha^{(n+1)} \xrightarrow{w^*} 0$. We say that $a^{(n)}$ has Left – weak*-weak property (= Lw^*w- property) with respect to $\mathcal{B}^{(n)}$ when $a^{(n)}b_\alpha^{(n+1)} \xrightarrow{w} 0$. If every $a^{(n)} \in \mathcal{A}$ has Lw^*w- property with respect to $\mathcal{B}^{(n)}$, then we say that $\mathcal{A}^{(n)}$ has Lw^*w- property with respect to $\mathcal{B}^{(n)}$. The definition of the Right – weak*-weak property (= Rw^*w- property) is the same.*

We say that $a^{(n)} \in \mathcal{A}^{(n)}$ has weak-weak property (= w^*w- property) with respect to $\mathcal{B}^{(n)}$ if it has Lw^*w- property and Rw^*w- property with respect to $\mathcal{B}^{(n)}$.*

*If $a^{(n)} \in \mathcal{A}^{(n)}$ has Lw^*w- property with respect to itself, then we say that $a^{(n)} \in \mathcal{A}^{(n)}$ has Lw^*w- property.*

Example 2.12. (1) *If \mathcal{B} is Banach \mathcal{A} -bimodule and reflexive, then \mathcal{A} has w^*w -property with respect to \mathcal{B} .*

- (2) $L^1(G)$, $M(G)$ and $A(G)$ have w^*w -property when G is finite.
- (3) Let G be locally compact group. Then $L^1(G)$ [respectively $M(G)$] has w^*w - property [respectively Lw^*w - property] with respect to $L^p(G)$ whenever $p > 1$.
- (4) Suppose that \mathcal{B} is a left Banach \mathcal{A} -module and e is a left unit element of \mathcal{A} such that $eb = b$ for all $b \in B$. If e has Lw^*w -property, then \mathcal{B} is reflexive.
- (5) If S is a compact semigroup, then $C^+(S) = \{f \in C(S) : f > 0\}$ has w^*w -property.

Theorem 2.13. Let \mathcal{B} be a left Banach \mathcal{A} -bimodule and $n \geq 2$. Then we have the following assertions:

- (1) If $\mathcal{A}^{(n)} = a_0^{(n-2)}\mathcal{A}^{(n)}$ [respectively $\mathcal{A}^{(n)} = \mathcal{A}^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w - property [respectively Lw^*w - property] with respect to $\mathcal{B}^{(n)}$, then $Z_{\mathcal{B}^{(n)}}(\mathcal{A}^{(n)}) = \mathcal{A}^{(n)}$.
- (2) If $\mathcal{B}^{(n)} = a_0^{(n-2)}\mathcal{B}^{(n)}$ [respectively $\mathcal{B}^{(n)} = \mathcal{B}^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w - property [respectively Lw^*w - property] with respect to $\mathcal{B}^{(n)}$, then $Z_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)}) = \mathcal{B}^{(n)}$.

Proof. (1) Suppose that $\mathcal{A}^{(n)} = a_0^{(n-2)}\mathcal{A}^{(n)}$ for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $a_0^{(n-2)}$ has Rw^*w - property. Let $(b_\alpha^{(n)})_\alpha \subseteq \mathcal{B}^{(n)}$ such that $b_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}$. Then for every $a^{(n-2)} \in \mathcal{A}^{(n-2)}$ and $b^{(n-1)} \in \mathcal{B}^{(n-1)}$, we have

$$\begin{aligned} \langle b_\alpha^{(n)}b^{(n-1)}, a^{(n-2)} \rangle &= \langle b_\alpha^{(n)}, b^{(n-1)}a^{(n-2)} \rangle \rightarrow \langle b^{(n)}, b^{(n-1)}a^{(n-2)} \rangle \\ &= \langle b^{(n)}b^{(n-1)}, a^{(n-2)} \rangle. \end{aligned}$$

It follows that $b_\alpha^{(n)}b^{(n-1)} \xrightarrow{w^*} b^{(n)}b^{(n-1)}$. It also is clear that $(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \xrightarrow{w^*} (b^{(n)}b^{(n-1)})a_0^{(n-2)}$. Since $a_0^{(n-2)}$ has Rw^*w -property, $(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \xrightarrow{w} (b^{(n)}b^{(n-1)})a_0^{(n-2)}$. Now, let $a^{(n)} \in \mathcal{A}^{(n)}$. Since $\mathcal{A}^{(n)} = a_0^{(n-2)}\mathcal{A}^{(n)}$, there is $x^{(n)} \in \mathcal{A}^{(n)}$ such that $a^{(n)} = a_0^{(n-2)}x^{(n)}$. Thus we have

$$\begin{aligned} \langle a^{(n)}b_\alpha^{(n)}, b^{(n-1)} \rangle &= \langle a^{(n)}, b_\alpha^{(n)}b^{(n-1)} \rangle = \langle a_0^{(n-2)}x^{(n)}, b_\alpha^{(n)}b^{(n-1)} \rangle \\ &= \langle x^{(n)}, (b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \rangle \rightarrow \langle x^{(n)}, (b^{(n)}b^{(n-1)})a_0^{(n-2)} \rangle \\ &= \langle a^{(n)}b, b^{(n-1)} \rangle. \end{aligned}$$

It follows that $a^{(n)} \in Z_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$.

The proof of the next part is similar to the preceding proof.

- (2) Let $\mathcal{B}^{(n)} = a_0^{(n-2)}\mathcal{B}^{(n)}$ for some $a_0^{(n-2)} \in \mathcal{A}^{(n-2)}$ and let $a_0^{(n-2)}$ has Rw^*w -property with respect to $\mathcal{B}^{(n)}$. Assume that $(a_\alpha^{(n)})_\alpha \subseteq \mathcal{A}^{(n)}$ such that $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$. Then for every $b^{(n-1)} \in \mathcal{B}^{(n-1)}$, we have

$$\begin{aligned} \langle a_\alpha^{(n)}b^{(n-1)}, b^{(n-2)} \rangle &= \langle a_\alpha^{(n)}, b^{(n-1)}b^{(n-2)} \rangle \rightarrow \langle a^{(n)}, b^{(n-1)}b^{(n-2)} \rangle \\ &= \langle a^{(n)}b^{(n-1)}, b^{(n-2)} \rangle. \end{aligned}$$

We conclude that $a_\alpha^{(n)}b^{(n-1)} \xrightarrow{w^*} a^{(n)}b^{(n-1)}$. It is clear that

$$(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \xrightarrow{w^*} (a^{(n)}b^{(n-1)})a_0^{(n-2)}.$$

Since $a_0^{(n-2)}$ has Rw^*w -property,

$$(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \xrightarrow{w} (a^{(n)}b^{(n-1)})a_0^{(n-2)}.$$

Suppose that $b^{(n)} \in \mathcal{B}^{(n)}$. Since $\mathcal{B}^{(n)} = a_0^{(n-2)}\mathcal{B}^{(n)}$, there is $y^{(n)} \in \mathcal{B}^{(n)}$ such that $b^{(n)} = a_0^{(n-2)}y^{(n)}$. Consequently, we have

$$\begin{aligned} \langle b^{(n)}a_\alpha^{(n)}, b^{(n-1)} \rangle &= \langle b^{(n)}, a_\alpha^{(n)}b^{(n-1)} \rangle = \langle a_0^{(n-2)}y^{(n)}, a_\alpha^{(n)}b^{(n-1)} \rangle \\ &= \langle y^{(n)}, (a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)} \rangle \rightarrow \langle y^{(n)}, (a^{(n)}b^{(n-1)})a_0^{(n-2)} \rangle \\ &= \langle a_0^{(n-2)}y^{(n)}, (a^{(n)}b^{(n-1)}) \rangle = \langle b^{(n)}a^{(n)}, b^{(n-1)} \rangle. \end{aligned}$$

Thus $b^{(n)}a_\alpha^{(n)} \xrightarrow{w} b^{(n)}a^{(n)}$. It follows that $b^{(n)} \in Z_{\mathcal{A}^{(n)}}(\mathcal{B}^{(n)})$.

The proof of the next part is similar to the preceding proof. \square

Example 2.14. *i) Let G be a locally compact group. Since $M(G)$ is a Banach $L^1(G)$ -bimodule and the unit element of $M(G)^{(n)}$ does not have Lw^*w -property or Rw^*w -property, by using the preceding theorem, we have*

$$Z_{L^1(G)^{(n)}}(M(G)^{(n)}) \neq M(G)^{(n)}.$$

ii) If G is finite, then by using the preceding theorem, we conclude that

$$Z_{M(G)^{(n)}}(L^1(G)^{(n)}) = L^1(G)^{(n)},$$

$$Z_{L^1(G)^{(n)}}(M(G)^{(n)}) = M(G)^{(n)}.$$

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REFERENCES

- [1] R. E. Arens, The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.* **2** (1951), 839–848.
- [2] N. Arikan, A simple condition ensuring the Arens regularity of bilinear mappings, *Proc. Amer. Math. Soc.* **84** (1982), no.4, 525–532.
- [3] J. Baker, A. T. Lau, J. S. Pym, Module homomorphisms and topological centres associated with weakly sequentially complete Banach algebras, *J. Funct. Anal.* **158** (1998), no. 1, 186–208.
- [4] F. F. Bonsall, J. Duncan, Complete Normed Algebras, Springer-Verlag, New York-Heidelberg, 1973.
- [5] H. G. Dales, A. Rodrigues-Palacios, M. V. Velasco, The second transpose of a derivation, *J. London Math. Soc.* **64** (2001), no. 2, 707–721.
- [6] H. G. Dales, Banach Algebra and Automatic Continuity, London Math. Soc. Monographs, New Ser. 24, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 2000.
- [7] N. Dunford, J. T. Schwartz, Linear Operators I, John Wiley and Sons, Inc., New York, 1958.
- [8] M. Eshaghi Gordji, M. Filali, Arens regularity of module actions, *Studia Math.* **181** (2007), no. 3, 237–254.
- [9] M. Eshaghi Gordji, M. Filali, Weak amenability of the second dual of a Banach algebra, *Studia Math.* **182** (2007), no. 3, 205–213.
- [10] K. Haghnejad Azar, A. Riazi, Arens regularity of bilinear forms and unital Banach module space, submitted.
- [11] E. Hewitt, K. A. Ross, Abstract Harmonic Analysis Vol I, Springer-Verlag, New York-Berlin, 1979.
- [12] E. Hewitt, K. A. Ross, Abstract Harmonic Analysis Vol II, Springer-Verlag, New York-Berlin, 1970.
- [13] A. T. Lau, V. Losert, On the second conjugate algebra of $L_1(G)$ of a locally compact group, *J. London Math. Soc.* **37** (1988), no. 2, 464–480.
- [14] A. T. Lau, A. Ülger, Topological center of certain dual algebras, *Trans. Amer. Math. Soc.* **348** (1996), no. 3, 1191–1212.
- [15] S. Mohamadzadieh, H. R. E. Vishki, Arens regularity of module actions and the second adjoint of a derivation, *Bull. Aust. Math. Soc.* **77** (2008), no. 3, 465–476.
- [16] M. Neufang, Solution to a conjecture by Hofmeier-Wittstock, *J. Funct. Anal.* **217** (2004), no. 1, 171–180.
- [17] M. Neufang, On a conjecture by Ghahramani-Lau and related problems concerning topological centres, *J. Funct. Anal.* **224** (2005), no. 1, 217–229.
- [18] J. S. Pym, The convolution of functionals on spaces of bounded functions, *Proc. London Math. Soc.* **15** (1965), no. 3, 84–104.
- [19] A. Ülger, Arens regularity of the algebra $A\hat{\otimes}B$, *Trans. Amer. Math. Soc.* **305** (1988), no. 2, 623–639.

- [20] A. Ülger, Arens regularity sometimes implies the RNP, *Pacific J. Math.* **143** (1990), no. 2, 377–399.
- [21] A. Ülger, Some stability properties of Arens regular bilinear operators, *Proc. Amer. Math. Soc.* **34** (1991), no. 3, 443–454.
- [22] A. Ülger, Arens regularity of weakly sequentially complete Banach algebras, *Proc. Amer. Math. Soc.* **127** (1999), no. 11, 3221–3227.
- [23] P. K. Wong, The second conjugate algebras of Banach algebras, *J. Math. Sci.* **17** (1994), no. 1, 15–18.
- [24] N. J. Young, The irregularity of multiplication in group algebra, *Quart. J. Math. Oxford Ser.* **24** (2) (1973) 59–62.
- [25] Y. Zhang, Weak amenability of module extensions of Banach algebras, *Trans. Amer. Math. Soc.* **354** (2002), no. 10, 4131–4151.

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