LIE TRIPLE DERIVATION ALGEBRA OF VIRASORO-LIKE ALGEBRA

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Abstract. Let $L$ be the Virasoro-like algebra and $g$ its derived algebra, respectively. We investigate the structure of the Lie triple derivation algebra of $L$ and $g$. We prove that they are both isomorphic to $L$, which provides two examples of invariance under triple derivation.

1. Introduction

Let $A$ be a Lie algebra over the complex number field $\mathbb{C}$. Recall that a $\mathbb{C}$-linear mapping $\phi : A \rightarrow A$ is called a Lie derivation if

$$\phi([a, b]) = [\phi(a), b] + [a, \phi(b)] \quad (\forall \ a, b \in A).$$

If $\phi$ satisfies

$$\phi([[a, b], c]) = [[[\phi(a), b], c] + [[a, \phi(b)], c] + [[a, b], \phi(c)]$$

for all $a, b, c \in A$, it is called a Lie triple derivation.

A derivation is a triple derivation. Like derivation algebra, the set of triple derivations is a Lie algebra under the usual bracket. It is denoted by $\text{Der}^{(3)}A$. For some algebra, the derivation algebra $\text{Der}A$ is a proper subalgebra of $\text{Der}^{(3)}A$, see [7]. Lie triple derivation algebra of some kinds of operator algebras have been studied in [1, 4].
Recently, Virasoro algebra, as a universal central extension of Witt algebra, has played an important role in mathematics and physics. The \(q\)-analog Virasoro algebra was introduced by Kirkman et al as a derivation subalgebra of quantum torus (see [6]). On one hand, Virasoro-like algebra is the generalization of Virasoro algebra. On the other hand, it can be regarded as a special case of \(q\)-analog Virasoro algebra when \(q = 1\). Lin and Tan [8] and Wang and Zhao[11] studied the nonzero level Harish-Chandra modules and Verma module of Virasoro-like algebra, respectively. Jiang and Meng[5] gave the structure of its derivation algebra and the automorphism group. Song et al. got the quantization of generalized Virsoro-like algebra in [10]. As early as in 1957, Herstein in [3] proved that any Jordan triple derivation is an ordinary derivation on a prime ring of characteristic different from 2. In this paper, we will prove that the Lie triple derivation of Virasoro-like algebra \(\mathfrak{L}\) is isomorphic to \(\mathfrak{L}\). Meanwhile we obtain that any Lie triple derivation of Virasoro-like algebra is a Lie derivation, i.e.,

\[
\text{Der}\mathfrak{L} = \text{Der}^{(3)}\mathfrak{L}.
\]

(1.2)

The paper is arranged as follows. In Section 2, we give some notations and recall some basic facts on \(\mathfrak{L}\). In Section 3, we study the graded version of \(\text{Der}^{(3)}\mathfrak{L}\), which is a generalization of [2]. In Section 4, we give the main result of this paper.

2. Preliminaries

Throughout this paper we use \(\mathbb{C}, \mathbb{Z}\) to denote the set of complex numbers, and integers, respectively. We assume that \(\mathcal{U}\) is a 2-dimensional vector space over \(\mathbb{C}\). We choose

\[
e_1 = (1, 0), \quad e_2 = (0, 1)
\]

as the basis of \(\mathcal{U}\) and let \((\cdot, \cdot)\) be the standard bilinear form on \(\mathcal{U}\), i.e.,

\[
(e_i, e_j) = \delta_{ij}, \quad 1 \leq i, j \leq 2.
\]

Let \(A = \mathbb{C}[x_1^\pm, x_2^\pm]\) be a Laurent polynomial with 2 commuting variables \(x_1, x_2\) over \(\mathbb{C}\).

The elements of \(A\) are denoted by \(x^r = x_1^{r_1} x_2^{r_2}\) for \(r = (r_1, r_2)\). Set \(D_i(r) = x^r x_i (\partial/\partial x_1)\) for \(r \in \mathbb{Z}^2\). For all \(u = (u_1, u_2) \in \mathcal{U}\), we define

\[
D(u, r) = u_1 D_1(r) + u_2 D_2(r).
\]
The space $\text{Der}A$ is a Lie algebra under the following bracket:

$$[D(u, r), D(u', r')] = D(u'', r + r'),$$

where $u'' = (u, r')u' - (u', r)u$.

A subalgebra $\mathfrak{L}$ of $\text{Der}A$ is called (centerless) Virasoro-like algebra if it satisfies

$$\mathfrak{L} = \bigoplus_{r \in \mathbb{Z}^2} \mathfrak{L}_r,$$

where $\mathfrak{L}_r = \{D(u, r)|(u, r) = 0\}$. Let $\mathfrak{g} = [\mathfrak{L}, \mathfrak{L}]$ be the derived algebra of $\mathfrak{L}$, then $\mathfrak{L} = \mathfrak{g} \bigoplus \mathfrak{h}$, where

$$\mathfrak{g}_r = \begin{cases} \mathfrak{L}_r, & r \neq 0 \\ 0, & r = 0 \end{cases}$$

and $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{L}$.

In this paper we study the centerless Virasoro-like algebra.

3. Gradation of $\mathfrak{L}$ and $\text{Der}^{(3)}\mathfrak{L}$

In this section, we prove that $\text{Der}^{(3)}\mathfrak{L}$ and $\text{Der}^{(3)}\mathfrak{g}$ are both graded algebras.

Let $G$ be an abelian group, $L = \bigoplus_{g \in G} L_g$ a $G$-graded Lie algebra over an algebraically closed field $\mathbb{F}$. Let $V$ be a $G$-graded $L$-module. A linear mapping $\varphi : L \rightarrow V$ is called a triple derivation if

$$(3.1) \quad \varphi([x, [y, z]]) = z.y.\varphi(x) - z.x.\varphi(y) + x.y.\varphi(z) - y.x.\varphi(z)$$

for all $x, y, z \in L$. We say that a triple derivation $\varphi$ has degree $g$ if $\varphi(L_h) \subseteq V_{g+h}$.

This definition generalizes (1.1): Take the regular module $V = L$, then (3.1) will be (1.1).

**Theorem 3.1.** Let $L$ be a finitely generated Lie algebra and $V$ a $G$-graded $L$-module. Then

$$\text{Der}^{(3)}_{\mathbb{F}}(L, V) = \bigoplus_{g \in G} \text{Der}^{(3)}_{\mathbb{F}}(L, V)_g.$$ 

This theorem can be proved using a similar argument as in [2].
Proof. Let $S$ be a finite generating set of $L$. For each element $g \in G$ we define two canonical projections $p_g : L \rightarrow L_g$ and $\pi_g : V \rightarrow V_g$. Suppose $\varphi : L \rightarrow V$ is a triple derivation and there are finite subsets $Q, R \subset G$ such that $S \subset \sum_{g \in Q} L_g$ and $\varphi(S) \subset \sum_{g \in R} V_g$.

For $g \in G$ let $\varphi_g := \sum_{h \in G} \pi_{g+h} \circ \varphi \circ p_h$. Now for $x_h \in L_h$, $x_k \in L_k$ and $x_l \in L_l$, we have

$$
\varphi_g([x_h, x_k], x_l] = \pi_{g+h+k+l} \circ \varphi([x_h, x_k], x_l]) = x_l \cdot x_k \cdot \pi_{g+h} \cdot \varphi(x_h) - x_l \cdot x_h \cdot \pi_{g+h} \cdot \varphi(x_k) - x_k \cdot x_h \cdot \pi_{g+h} \cdot \varphi(x_l) + x_l \cdot x_h \cdot \pi_{g+h} \cdot \varphi(x_k) + x_k \cdot x_h \cdot \pi_{g+h} \cdot \varphi(x_l) - x_l \cdot x_k \cdot \pi_{g+h} \cdot \varphi(x_l),
$$

which follows that $\varphi_g$ is contained in $\text{Der}^{(3)}_{F}(L, V)_g$.

Let $T = \{g - h | g \in R, h \in Q\}$. For any $y \in S$ we have

$$
\varphi(y) = \sum_{g \in R} \sum_{h \in Q} \pi_g \circ \varphi \circ p_h(y) = \sum_{g \in R} \sum_{h \in Q} \pi_{g-h} \circ \varphi \circ p_h(y) = \sum_{g \in T} \varphi_g(y).
$$

This shows that $\varphi = \sum_{q \in T} \varphi_q$ on $S$ and subsequently $L$. \hfill \Box

Lemma 3.2. The Lie algebra $\mathfrak{g}$ is generated by $D(e_1, \pm e_2), D(e_2, \pm e_1)$, where $e_1, e_2$ are defined above.

Note that $\mathfrak{L}$ is generated by $\mathfrak{g}$ and $D(e_1, 0), D(e_2, 0)$, so $\mathfrak{L}$ is also finitely generated with the generators $D(e_1, \pm e_2), D(e_2, \pm e_1), D(e_1, 0), D(e_2, 0)$.
Theorem 3.1 and Lemma 3.2 imply that \( \text{Der}^{(3)}(L) \) and \( \text{Der}^{(3)}(g) \) are \( \mathbb{Z}^2 \)-graded Lie algebras.

4. Structure of \( \text{Der}^{(3)}g \) and \( \text{Der}^{(3)}L \)

4.1. The Structure of \( \text{Der}^{(3)}g \). In this section we will characterize the structure \( \text{Der}^{(3)}g \) and we will show that \( \text{Der}^{(3)}g \cong L \).

Lemma 4.1. For any nonzero \( r \in \mathbb{Z}^2 \), \( (\text{Der}^{(3)}g)_r = \text{ad}g_r \).

Proof. We only need to show that there exists an element \( y \in g_r \) such that \( \text{ad}y = \varphi \) for any \( \varphi \in (\text{Der}^{(3)}g)_r \) and \( r \in \mathbb{Z}^2 \).

By Lemma 3.2, it is enough to prove \( \text{ad}y = \varphi \) holds for the generators of \( g \).

Suppose \( \varphi(D(e_1, e_2)) = D(w, e_2 + r) \), where \( w = (w_1, w_2) \in U \).

Case 1: \( r_1 \neq 0 \). By choosing

\[
u = \left(-\frac{w_2}{r_1}, -\frac{r_1 w_1 + w_2}{r_1^2}\right),
\]

we get

\[
\varphi(D(e_1, e_2)) = \text{ad}D(u, r)(D(e_1, e_2)).
\]

It is convenient to replace \( \varphi \) by \( \varphi - \text{ad}D(u, r) \), then \( \varphi(D(e_1, e_2)) = 0 \).

We assume \( \varphi(D(e_1, -e_2)) = D(v, -e_2 + r) \) with \( v = (v_1, v_2) \in U \).

Observe that \( [D(e_1, e_2), D(e_1, -e_2)] = 0 \), so

\[
\varphi([D(e_1, e_2), D(e_1, -e_2)], D(e_1, e_2)) = 0.
\]

However, \( \varphi \) is a triple derivation, so we have

\[
\varphi([D(e_1, e_2), D(e_1, -e_2)], D(e_1, e_2)) = [D(e_1, e_2), D(v, -e_2 + r)], D(e_1, e_2)] = D(v', e_2 + r)
\]

for some \( v' \in \mathbb{C}^2 \). Subsequently, \( \varphi([D(e_1, e_2), D(e_1, -e_2)], D(e_1, e_2)) = D(v', e_2 + r) = 0 \), which forces \( v' = 0 \).

On the other hand, direction computation of \( v' \) gives that

\[
\begin{align*}
2r_1 v_2 - r_1^2 v_1 &= 0 \\
r_1^2 v_2 &= 0,
\end{align*}
\]

which implies that \( \varphi(D(e_1, -e_2)) = 0 \).

We prove \( \varphi(D(e_2, e_1)) = \varphi(D(e_2, -e_1)) = 0 \) in several steps.
We first assume that \( \varphi(D(e_2, e_1)) = D(\pi, e_1 + r) \) with \( \pi = (\pi_1, \pi_2) \).

If \( r \neq (-2, r_2) \), by considering the image of \( \varphi \) on the following equality
\[
[[D(e_1, e_2), D(e_2, e_1)], D(e_1, -e_2)] = D(e_2, e_1),
\]
we know \((r_1 + 1)^2\pi = \pi\). Hence \( \pi = 0 \).

Similarly if \( r \neq (2, r_2) \), by applying \( \varphi \) on
\[
[[D(e_1, e_2), D(e_2, -e_1)], D(e_1, -e_2)] = -D(e_2, -e_1),
\]
we get \( \varphi(D(e_2, -e_1)) = 0 \).

So we can conclude that if:

- if \( r = (-2, r_2) \),
  \( \varphi(D(e_1, \pm e_2)) = \varphi(D(e_2, -e_1)) = 0 \);
- if \( r = (2, r_2) \),
  \( \varphi(D(e_1, \pm e_2)) = \varphi(D(e_2, e_1)) = 0 \).

Next we consider the case \( r = (-2, r_2) \).

If \( r_2 \neq 0 \),
\[
\varphi([[D(e_2, e_1), D(e_2, -e_1)], D(e_1, e_2)])
= [[D(\pi, e_1 + r), D(e_2, -e_1)], D(e_1, e_2)]
= D(\overline{\pi}', e_2 + r)
\]
where \( \overline{\pi}' = -(2r_2 + 1)\pi_1 + r_2\pi_2, 2\pi_1 + 2r_2\pi_2 \), which tells us \( \pi = 0 \). i.e. \( \varphi(D(e_2, e_1)) = 0 \).

If \( r = (-2, 0) \), then
\[
[[D(e_1, e_2), D(e_2, e_1)], D(e_2, -e_1)] = D(e_1, e_2)
\]
also implies \( \varphi(D(e_2, e_1)) = 0 \).

Similar discussion will lead to \( \varphi(D(e_2, -e_1)) = 0 \) for \( r = (-2, r_2) \) with \( r_2 \in \mathbb{Z} \).

**Case 2:** \( r_1 = 0 \). Suppose \( \varphi(D(e_2, e_1)) = D(w, e_1 + r) \), we choose an element \( D(u', r) \in g_r \) with
\[
u' = (-\frac{w_1}{r_2}, -\frac{w_1 + r_2w_2}{r_2^2}).
\]

Then we get
\[
(\varphi - adD(u', r))(D(e_2, e_1)) = 0.
\]
Now we assume \( \varphi(D(e_2, -e_1)) = D(\bar{w}, -e_1 + r) \) with \( \bar{w} = (\bar{w}_1, \bar{w}_2) \).
Similarly as before (see (4.1)),

\[
[[D(e_2, e_1)D(e_2, -e_1)]D(e_2, e_1)] = 0
\]

implies that we also have

\[
\left\{
\begin{array}{l}
-r_2^2 \tilde{w}_1 = 0 \\
-r_2^3 \tilde{w}_2 + 2r_2 \tilde{w}_1 = 0.
\end{array}
\right.
\tag{4.2}
\]

Thus \( \varphi(D(e_2, -e_1)) = 0 \).

If \( r_2 \neq -2 \), \( \varphi(D(e_1, e_2)) = 0 \) due to the fact that

\[
\varphi([[D(e_2, e_1)D(e_1, e_2)]D(e_2, -e_1)]) = \varphi(-D(e_1, e_2)).
\]

If \( r_2 = -2 \), i.e. \( r = (0, -2) \), we also have

\[
\varphi([[D(e_2, e_1)D(e_1, e_2)]D(e_1, -e_2)]) = \varphi(D(e_2, e_1)) = 0.
\]

Similar discussion on \( r_2 = 2 \) or \( r_2 \neq 2 \), gives \( \varphi(D(e_1, -e_2)) = 0 \).

Therefore \( \varphi(D(e_i, e_j)) = 0 \), \( 1 \leq i \neq j \leq 2 \). By Lemma 3.2 we also have \( \varphi(\mathfrak{g}) = 0 \) for all nonzero \( r \in \mathbb{Z}^2 \). This completes the proof. \( \square \)

**Theorem 4.2.** \( \text{Der}^{(3)} \mathfrak{g} \cong \mathfrak{z} \).

**Proof.** We first define the action of \( D(u, 0) \in \mathfrak{h} \) on \( \mathfrak{g} \) as follows:

\[
D(u, 0).D(v, s) = [D(u, 0), D(v, s)] = (u, s)D(v, s), \ \forall s \in \mathbb{Z}^2 \setminus \{0\},
\]

which is a 0-homogeneous outer derivation of \( \mathfrak{g} \).

We note that \( \varphi(D(e_i, \pm e_j)) = c_{ij}^\pm D(e_i, \pm e_j) \) and

\[
[[D(e_i, e_j)D(e_j, e_i)]D(e_j, -e_i)]) = D(e_i, e_j), \ \text{if} \ \ 1 \leq i \neq j \leq 2.
\]

By applying \( \varphi \) to both sides it follows that \( c_{ij}^+ = -c_{ij}^- \).

We then define

\[
h = c_{21}^+ D(e_1, 0) + c_{12}^+ D(e_2, 0).
\]

Then we can check \( (\varphi - h)(\mathfrak{g}) = 0 \) when \( r = 0 \).
Therefore by Section 3 and Lemma 4.1 we have

\[
\text{Der}^{(3)} \mathfrak{g} = \bigoplus_{r \in \mathbb{Z}_2} \text{Der}^{(3)} \mathfrak{g}_r \\
\cong \bigoplus_{r \in \mathbb{Z}^2 \setminus \{0\}} ad \mathfrak{g}_r \oplus \mathfrak{h} \\
= ad \mathfrak{g} \oplus \mathfrak{h} \\
= ad \mathfrak{L} \\
= \mathfrak{L}.
\]

\[\square\]

4.2. **The Structure of Der^{(3)} \mathfrak{L}.** In this section we will prove $\text{Der}^{(3)} \mathfrak{L} \cong \mathfrak{L}$ by introducing the following lemma.

**Lemma 4.3.** Suppose $\varphi \in \text{Der}^{(3)} \mathfrak{L}$. If $\varphi(\mathfrak{g}) = 0$, then $\varphi(\mathfrak{h}) = 0$.

**Proof.** We only consider $D(e_1, 0)$.

Assume that $\varphi(D(e_1, 0)) = D(\tilde{u}, r)$, where $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$.

**Case 1:** $r \neq 0$.

If $r_1 \neq 0$, applying $\varphi$ on both sides of

\[
[[D(e_1, e_2), D(e_1, 0)], D(e_1, e_2)] = 0,
\]

we have

\[
2r_1 \tilde{u}_2 - r_1^2 \tilde{u}_1 = 0 \\
r_1^2 \tilde{u}_2 = 0,
\]

which follows that $\tilde{u} = 0$ and $\varphi(D(e_1, 0)) = 0$.

If $r_2 \neq 0$, it follows from

\[
[[D(e_2, e_1), D(e_1, 0)], D(e_2, -e_1)] = 0.
\]

**Case 2:** $r = 0$.

Note that

\[
[[D(e_1, e_2), D(e_1, 0)], D(e_2, e_1)] = 0.
\]

According to our assumption, the operation of $\varphi$ on the both sides is

\[
[[D(e_1, e_2), \tilde{u}_1 D(e_1, 0) + \tilde{u}_2 D(e_2, 0)], D(e_j, e_i)] = 0.
\]

By a simple calculation we can get $\tilde{u}_2 = 0$.

The same action on

\[
[[D(e_2, e_1), D(e_1, 0)], D(e_1, 0)] = D(e_2, e_1)
\]
tells us $\tilde{u}_1 = 0$. This completes the proof. \[\square\]
We therefore have:

**Theorem 4.4.** $\text{Der}^{(3)} \mathcal{L} \cong \mathcal{L}$.

**Proof.** Let $\phi$ be a triple derivation of $\mathcal{L}$. By Theorem 3.1 we have

$$\phi = \sum_{r \in \mathbb{Z}^2} \phi_r.$$

Repeat the process of Section 4.1, we deduce that there exists an element $x_r \in \mathcal{L}_r$ such that $(\phi_r - adx_r)(\mathfrak{g}) = 0$, for all $r \in \mathbb{Z}^2$. By Lemma 4.3, we obtain $(\phi_r - adx_r)(\mathcal{L}) = 0$. So

$$\text{Der}^{(3)} \mathcal{L} = \bigoplus_{r \in \mathbb{Z}^2} \text{Der}^{(3)} \mathcal{L}_r = \bigoplus_{r \in \mathbb{Z}^2} ad \mathcal{L}_r = ad \mathcal{L} \cong \mathcal{L}.$$ 

\[\square\]

**Corollary 4.5.** Suppose $\mathfrak{g}$ and $\mathcal{L}$ are defined as above, Then

$$\text{Der} \mathfrak{g} = \text{Der}^{(3)} \mathfrak{g} = \text{Der} \mathcal{L} = \text{Der}^{(3)} \mathcal{L} \cong \mathcal{L}.$$

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