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CONSTRUCTION OF A CLASS OF TRIVARIATE NONSEPARABLE COMPACTLY SUPPORTED WAVELETS WITH SPECIAL DILATION MATRIX

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ABSTRACT. We present a method for the construction of compactly supported $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ -wavelets under a mild condition. Wavelets inherit the symmetry of the corresponding scaling function and satisfies the vanishing moment condition originating in the symbols of the scaling function. As an application, an example is provided.

1. Introduction

Multivariate nonseparable wavelets have attracted the interest of many researchers. Although separable wavelet bases have a lot of advantages, they have a number of drawbacks. They are so special that they have very little design freedom. Furthermore, separability imposes an unnecessary product structure on the plane which is artificial for natural images. One way to avoid this is through the construction of nonseparable wavelets. Nonseparable wavelet basis offers the hope of a more isotropic analysis, see [1]- [6]. Numerical experiment with decomposition and reconstruction procedure using nonseparable wavelets reveals more

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feature in the high-frequency band than does by a separable wavelet, see [7]. Therefore, nonseparable wavelets were used in pattern recognition, texture analysis and edge detection. Note that the design of nonseparable orthogonal wavelets is still a challenging problem and only a few references have dealt with this subject, see [1-4] and [8-14]. In one dimension, it is well known that, there exists no compactly supported, symmetric or antisymmetric and orthogonal real-valued wavelet except for the Haar wavelet, see [3]. But it is not the case if orthogonality is replaced by Riesz basis. Chui and Wang present a method to construct a compactly supported Riesz basis and wavelets with symmetric or antisymmetric, see [15]. Their method depends on the determination of zeros of polynomials, which is not easy in higher dimension. Based on [5] and [8], it is a natural and interesting problem to construct a nonseparable compactly supported Riesz basis and wavelets with symmetry or antisymmetry in $L^2(\mathbf{R}^3)$.

The structure of this paper is as follows. In section 2, the basic concept is introduced. In section 3, we give the construction of trivariate nonseparable compactly supported orthogonal wavelets, and discuss the symmetry and vanishing moment of wavelets. Finally, an example is also given to demonstrate the general theory.

2. Basic Concept

Throughout this paper, let \mathbf{Z} and \mathbf{R} be the set of integers and real numbers, respectively. Furthermore, \mathbf{R}^3 will be the Euclidean threedimensional space and \mathbf{Z}^3 the set of 3-tuple integers. Also M is always referred to the matrix $M = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, M^{-T} denotes the transpose

of the matrix M^{-1} . The Fourier transform of f is defined by $\widehat{f}(\xi) =$ $\int_{\mathbf{R}^3} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\xi} d\mathbf{x} \text{ for } f \in L^2(\mathbf{R}^3). \text{ Let } L^2(\mathbf{R}^3) \text{ and } \ell^2(\mathbf{Z}^3) \text{ denote } \{f: \int_{\mathbf{R}^3} |f(\mathbf{x})|^2 d\mathbf{x} < \infty\} \text{ and } \{s_m: \sum_{m \in \mathbf{Z}^3} |s_m|^2 < \infty\} \text{ and respectively.}$

Definition 2.1. Let $\{V_k\}_{k \in \mathbb{Z}}$ be a sequence of the closed subspaces in $L^2(\mathbf{R}^3)$ satisfying

- (1) $V_k \subset V_{k+1}, k \in \mathbf{Z};$
- (2) $clos_{L^{2}(\mathbf{R}^{3})}(\bigcup_{k\in\mathbf{Z}}V_{k}) = L^{2}(\mathbf{R}^{3});$
- (3) $\bigcap_{k \in \mathbb{Z}} V_k = \{0\};$ (4) $f(\cdot) \in V_j$ if and only if $f(M \cdot) \in V_{j+1}, j \in \mathbb{Z};$

(5) There exists a function $\phi(\cdot) \in V_0$ such that $\{\phi(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis of V_0 .

Then $\{\{V_k\}_{k\in\mathbb{Z}}, \phi(\cdot)\}\$ is said to be a multiresolution analysis (MRA) related to M of $L^2(\mathbb{R}^3)$, $\phi(\cdot)$ is a corresponding scaling function. Since $V_0 \subset V_1$, ϕ satisfies some M-refinement equation

(2.1)
$$\phi(\cdot) = \sqrt{2} \sum_{n \in \mathbf{Z}^3} h_n \phi(M \cdot -n),$$

where $\{h_n\}_{n \in \mathbb{Z}^3}$ is called the mask. Implementing of the Fourier transform (2.1), we have

$$\widehat{\phi}(\cdot) = H_0(M^{-T} \cdot)\widehat{\phi}(M^{-T} \cdot).$$

and

(2.2)
$$H_0(\cdot) = \frac{\sqrt{2}}{2} \sum_{n \in \mathbf{Z}^3} h_n e^{-in \cdot}$$

which is said to be a symbol of ϕ .

Definition 2.2. A real-valued measurable function f defined on \mathbf{R}^3 is said to be symmetric (antisymmetric) about $\frac{\mathbf{x}_0}{2} \in \mathbf{R}^3$ if $f(\mathbf{x}) = f(\mathbf{x}_0 - \mathbf{x})(f(\mathbf{x}) = -f(\mathbf{x}_0 - \mathbf{x}))$.

Let W_j denote the orthogonal complement of V_j in V_{j+1} for $j \in \mathbb{Z}$. If we can find a ψ such that $\{\psi(\cdot - k) : k \in \mathbb{Z}^3\}$ is a Riesz basis for W_0 , then it is easy to check that $\{\psi_{j,k} : j \in \mathbb{Z} : k \in \mathbb{Z}^3\}$ is a Riesz basis for $L^2(\mathbb{R}^3)$, where $\psi_{j,k}(\cdot) = 2^{\frac{j}{2}}\psi(M^j \cdot -k)$ for any function ψ defined on \mathbb{R}^3 and $j \in \mathbb{Z}, k \in \mathbb{Z}^3$. In particular, when $\{\phi(\cdot - k) : k \in \mathbb{Z}^3\}$ is an orthonormal basis for V_0 , and

$$\psi(\cdot) = \sqrt{2} \sum_{n \in \mathbf{Z}^3} (-1)^{n_1 + n_2 + n_3} \overline{h_{(1,0,0)^T - n}} \phi(M \cdot - n).$$

It is easy to prove that $\{\psi(\cdot - k) : k \in \mathbf{Z}^3\}$ is an orthonormal basis for W_0 , and that $\{\psi_{j,k} : j \in \mathbf{Z} : k \in \mathbf{Z}^3\}$ is an orthonormal basis for $L^2(\mathbf{R}^3)$.

3. Main Results

In this section, under the assumptions that some trivariate polynomial has no zeros, we obtain a general approach to construct compactly supported M-wavelets, which inherits the symmetry of the corresponding scaling function and satisfies the vanishing moment condition originating in the symbols of the scaling function. Our main results can be stated as follows.

Theorem 3.1. Assume that ϕ is a scaling function of an MRA $\{V_j\}_{j \in \mathbb{Z}}$ satisfying (2.1), its symbol H_0 defined as in (2.2) is a Laurent polynomial, and W_0 is the orthogonal complement of V_0 in V_1 . Define

$$g_n = (-1)^{n_1 + n_2 + n_3} \langle \phi_{1,(1,0,0)^T - n}, \phi \rangle$$

for $n \in \mathbb{Z}^3$, and

$$\psi(\cdot) = \sqrt{2} \sum_{n \in \mathbf{Z}^3} g_n \phi(M \cdot -n).$$

Then

(1)
$$\psi(\cdot) \in W_0$$
;
(2) $\{\psi(\cdot - n) : n \in \mathbf{Z}^3\}$ is a Riesz basis for W_0 if and only if

$$\sum_{n \in \mathbf{Z}^3} \left(\sum_{l \in \mathbf{Z}^3} (-1)^{l_1 + l_2 + l_3} h_l g_{Mn + (1,0,0)^T - l}\right) e^{-in \cdot \xi}$$

has no zeros in $[-\pi,\pi]^3$.

Proof. First, we prove (1).

For $m \in \mathbf{Z}^3$, it is obvious that

$$\begin{split} \langle \psi(\cdot), \phi(\cdot - m) \rangle &= \\ \sum_{n \in \mathbf{Z}^3} (-1)^{n_1 + n_2 + n_3} \langle \phi_{1,(1,0,0)^T - n}(\cdot), \phi(\cdot) \rangle \langle \phi_{1,n-Mm}(\cdot), \phi(\cdot) \rangle. \end{split}$$

Let $n - Mm = (1, 0, 0)^T - \tilde{n}$. We have

$$\begin{split} \langle \psi(\cdot), \phi(\cdot - m) \rangle &= \\ & \sum_{\tilde{n} \in \mathbf{Z}^3} (-1)^{1+3m_1+m_2 - \tilde{n}_1 - \tilde{n}_2 - \tilde{n}_3} \\ & \langle \phi_{1,\tilde{n} - Mm}(\cdot), \phi(\cdot) \rangle \langle \phi_{1,(1,0,0)^T - \tilde{n}}(\cdot), \phi(\cdot) \rangle \\ &= -\sum_{\tilde{n} \in \mathbf{Z}^3} (-1)^{\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3} \\ & \langle \phi_{1,\tilde{n} - Mm}(\cdot), \phi(\cdot) \rangle \langle \phi_{1,(1,0,0)^T - \tilde{n}}(\cdot), \phi(\cdot) \rangle \\ &= - \langle \psi(\cdot), \phi(\cdot - m) \rangle. \end{split}$$

Hence, $\langle \psi(\cdot), \phi(\cdot - m) \rangle = 0$. So, $\psi \in W_0$.

Next, we divide the argument into three steps to prove (2). (i) $\{\psi(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for W_0 if and only if $\{\psi(\cdot - n), \phi(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for V_1 .

The necessity is obvious, we only need to prove sufficiency. Suppose $\{\psi(\cdot - n), \phi(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for V_1 . Let

$$\widetilde{W}_0 = \{ \sum_{n \in \mathbf{Z}^3} c_n \psi(\cdot - n) : c \in \ell^2(\mathbf{Z}^3) \}.$$

In the following, we prove that $W_0 = \widetilde{W}_0$. Since $\psi \in W_0$, and W_0 is invariant under integer shifts, then $\widetilde{W}_0 \subseteq W_0$. Since $W_0 \subset V_1$, for $f \in W_0$, we have

$$f(\cdot) = \sum_{n \in \mathbf{Z}^3} c_n \psi(\cdot - n) + \sum_{n \in \mathbf{Z}^3} d_n \phi(\cdot - n), c_n, d_n \in l^2(\mathbf{Z}^3).$$

Define $\widehat{\widetilde{\phi}}(\cdot) = \frac{\widehat{\phi}(\cdot)}{\sum\limits_{n \in \mathbf{Z}^3} |\widehat{\phi}(\cdot+2n\pi)|^2}$. It is easy to check that $\{\widetilde{\phi}(\cdot-n) : n \in \mathbf{Z}^3 : |\widehat{\phi}(\cdot+2n\pi)|^2$.

 \mathbf{Z}^3 } is the dual of $\{\phi(\cdot - n) : n \in \mathbf{Z}^3\}$. It follows that

$$0 = \langle f, \widetilde{\phi}(\cdot - m) \rangle = d_m, \text{ for } m \in \mathbb{Z}^3.$$

Thus

$$f(\cdot) = \sum_{n \in \mathbf{Z}^3} c_n \psi(\cdot - n) \in \widetilde{W}_0$$

and $W_0 \subset \widetilde{W}_0$. Combining above results, we obtain

$$W_0 = \widetilde{W}_0 = \{ \sum_{n \in \mathbf{Z}^3} c_n \psi(\cdot - n) : c \in \ell^2(\mathbf{Z}^3) \}.$$

Then $\{\psi(\cdot - n) : n \in \mathbf{Z}^3\}$ is a Riesz basis for W_0 . (ii) Let $\widetilde{\psi}(\cdot) = \sqrt{2} \sum_{n \in \mathbf{Z}^3} (-1)^{n_1 + n_2 + n_3} \overline{h_{(1,0,0)^T - n}} \phi(M \cdot - n)$. Then

 $\{\phi(\cdot - n), \widetilde{\psi}(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for V_1 .

Define $\phi_0(\cdot) = \phi(M \cdot), \ \phi_1(\cdot) = \phi(M \cdot -(1,0,0)^T)$. Since $\{\phi(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for $V_0, \{\phi(M \cdot -n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for V_1 . Hence, $\{\phi_0(\cdot - n), \phi_1(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for V_1 . It is easy to check that

$$\phi(\cdot) = \sqrt{2} \sum_{n \in \mathbf{Z}^3} h_{Mn} \phi_0(\cdot - n) + \sqrt{2} \sum_{n \in \mathbf{Z}^3} h_{Mn+(1,0,0)^T} \phi_1(\cdot - n),$$

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$$\widetilde{\psi}(\cdot) = \sqrt{2} \sum_{n \in \mathbf{Z}^3} \overline{h_{(1,0,0)^T - Mn}} \phi_0(\cdot - n) - \sqrt{2} \sum_{n \in \mathbf{Z}^3} \overline{h_{-Mn}} \phi_1(\cdot - n),$$

and

$$det \left(\begin{array}{cc} \sqrt{2} \sum_{n \in \mathbf{Z}^3} h_{Mn} e^{-in \cdot \xi} & \sqrt{2} \sum_{n \in \mathbf{Z}^3} h_{Mn+(1,0,0)^T} e^{-in \cdot \xi} \\ \sqrt{2} \sum_{n \in \mathbf{Z}^3} \overline{h_{(1,0,0)^T - Mn}} e^{-in \cdot \xi} & -\sqrt{2} \sum_{n \in \mathbf{Z}^3} \overline{h_{-Mn}} e^{-in \cdot \xi} \end{array} \right)$$
$$= -2[|H_0(M^{-T}\xi)|^2 + |H_0(M^{-T}\xi + (\pi, -\pi, -\pi)^T)|^2] \neq 0$$

for $\xi \in \mathbf{R}^3$, where the last inequality is due to the fact that $\{\phi(\cdot - n) : n \in \mathbf{Z}^3\}$ is a Riesz basis for V_0 . Therefore, $\{\phi(\cdot - n), \widetilde{\psi}(\cdot - n) : n \in \mathbf{Z}^3\}$ is a Riesz basis for V_1 .

(iii) $\{\phi(\cdot - n), \psi(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for V_1 if and only if $\sum (\sum (-1)^{l_1+l_2+l_3} h_{l_3} h_{l_3})$ $e^{-in\cdot\xi}$

$$\sum_{n \in \mathbf{Z}^3} \left(\sum_{l \in \mathbf{Z}^3} (-1)^{1+2+3} n_l g_{Mn+(1,0,0)^T-l} \right)^{d}$$

has no zeros in $[-\pi, \pi]^3$. Since $\{\phi(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for V_0 for $\xi \in \mathbb{R}^3$, we obtain that

$$\begin{split} &\sum_{n\in\mathbf{Z}^3} \left(\sum_{l\in\mathbf{Z}^3} h_{l+Mn}\overline{h_l}\right) e^{-in\cdot\xi} \\ &= |H_0(M^{-T}\xi)|^2 + |H_0(M^{-T}\xi + (\pi, -\pi, -\pi)^T)|^2 \neq 0. \\ &\text{Let} \\ &A(\xi) = \frac{\sum_{n\in\mathbf{Z}^3} (\sum_{l\in\mathbf{Z}^3} g_{l+Mn}\overline{h_l}) e^{-in\cdot\xi}}{\sum_{n\in\mathbf{Z}^3} (\sum_{l\in\mathbf{Z}^3} h_{l+Mn}\overline{h_l}) e^{-in\cdot\xi}}, \\ &B(\xi) = -\frac{\sum_{n\in\mathbf{Z}^3} (\sum_{l\in\mathbf{Z}^3} (-1)^{l_1+l_2+l_3}g_{(1,0,0)^T-l+Mn}h_l) e^{-in\cdot\xi}}{\sum_{n\in\mathbf{Z}^3} (\sum_{l\in\mathbf{Z}^3} h_{l+Mn}\overline{h_l}) e^{-in\cdot\xi}}, \\ &\widetilde{H_1}(\xi) = -e^{-i\xi_1}\overline{H_0(\xi + (\pi, -\pi, -\pi)^T)}, \ H_1(\xi) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbf{Z}^3} g_n e^{-in\cdot\xi}. \end{split}$$

It is easy to check that

$$H_1(M^{-T}\xi) = A(\xi)H_0(M^{-T}\xi) + B(\xi)\widetilde{H_1}(M^{-T}\xi).$$

Multiplying with $\widehat{\phi}(M^{-T}\xi)$, we have

$$\widehat{\psi}(\xi) = A(\xi)\widehat{\phi}(\xi) + B(\xi)\widehat{\widetilde{\psi}}(\xi),$$

where ψ is defined as in (ii). Using (ii), we deduce that $\{\phi(\cdot -n), \psi(\cdot -n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for V_1 if and only if $B(\xi) \neq 0$ for $\xi \in \mathbb{R}^3$, and equivalently,

$$\sum_{n \in \mathbf{Z}^3} \left(\sum_{l \in \mathbf{Z}^3} (-1)^{l_1 + l_2 + l_3} h_l g_{Mn+(1,0,0)^T - l}\right) e^{-in \cdot \xi} \neq 0$$

for $\xi \in [-\pi, \pi]^3$. Hence, (iii) holds.

Combining (iii) with (i), we deduce (2).

Remark 3.2. It is obvious that $\{g_n\}$ is finitely supported, and thus, ψ is compactly supported.

Theorem 3.3. Under the assumptions of Theorem 3.1, suppose ϕ is a real-valued, and $h_n \in \mathbf{R}$ for $n \in \mathbf{Z}^3$.

(1) If ϕ is symmetric about $x = (0, 0, 0)^T$, then ψ is symmetric about $x = (\frac{1}{2}, 0, \frac{1}{2})^T$;

(2) If ϕ is symmetric about $x = (\frac{1}{2}, 0, 0)^T$, then ψ is symmetric about $x = (\frac{1}{2}, 0, 0)^T$;

(3) If ϕ is symmetric about $x = (0, \frac{1}{2}, 0)^T$, then ψ is symmetric about $x = (\frac{1}{2}, 0, -\frac{1}{2})^T$;

(4) If ϕ is symmetric about $x = (0, 0, \frac{1}{2})^T$, then ψ is antisymmetric about $x = (1, -1, -\frac{1}{2})^T$;

(5) If ϕ is symmetric about $x = (\frac{1}{2}, \frac{1}{2}, 0)^T$, then ψ is symmetric about $x = (\frac{1}{2}, -1, -1)^T$;

(6) If ϕ is symmetric about $x = (\frac{1}{2}, 0, \frac{1}{2})^T$, then ψ is antisymmetric about $x = (1, -\frac{3}{2}, -1)^T$;

(7) If ϕ is symmetric about $x = (0, \frac{1}{2}, \frac{1}{2})^T$, then ψ is antisymmetric about $x = (1, -\frac{1}{2}, -\frac{1}{2})^T$;

(8) If ϕ is symmetric about $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$, then ψ is antisymmetric about $x = (1, -1, -1)^T$.

Proof. We only need to prove (1) under the condition that ϕ is symmetric about $x = (0, 0, 0)^T$, and the other parts can be proved analogously. Suppose ϕ is symmetric about $x = (0, 0, 0)^T$, then

$$\langle \phi_{1,n-(1,0,0)^T}, \phi \rangle = \langle \phi_{1,(1,0,0)^T-n}, \phi \rangle, \quad n \in \mathbf{Z}^3.$$

It follows that

$$g_n = g_{(2,0,0)^T - n}, \quad n \in \mathbf{Z}^3,$$

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which is equivalent to $H_1(\xi) = e^{-i2\xi_1}H_1(-\xi)$, where $H_1(\xi) = \frac{1}{\sqrt{2}}\sum_{n \in \mathbb{Z}^3} g_n e^{-in\cdot\xi}$. So we have

$$\widehat{\psi}(-\xi) = e^{i(\xi_1 + \xi_3)} \widehat{\psi}(\xi),$$

which implies that ψ is symmetric about $x = (\frac{1}{2}, 0, \frac{1}{2},)^T$.

Remark 3.4. A compactly supported *M*-refinable function ϕ must be a *M*-refinable function (i.e., 2-refinable), and satisfy $\hat{\phi}(\xi) \neq 0$ for a.e. $\xi \in \mathbf{R}^3$. If ϕ is real-valued and symmetric about $\frac{c}{2}$, then $c \in \mathbf{Z}^3$. For any ϕ compactly supported, *M*-refinable, real-valued and symmetric about some $c \in \mathbf{Z}^3$, one may take a reasonable integer shift so that the shifted version is symmetric about $x = (0, 0, 0)^T$, or $x = (\frac{1}{2}, 0, 0)^T$, or $x = (0, \frac{1}{2}, 0)^T$, or $x = (0, \frac{1}{2}, \frac{1}{2})^T$, or $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$, and preserve other properties. So the assumption of Theorem 3.3 on ϕ is reasonable.

Theorem 3.5. Under the assumptions of Theorem 3.1, suppose

$$H_0(\xi) = \left(1 - \theta_1 \sin^2 \frac{\xi_1}{2} - \theta_2 \sin^2 \frac{\xi_2}{2} - (1 - \theta_1 - \theta_2) \sin^2 \frac{\xi_3}{2}\right)^N \mathcal{L}(\xi)$$

for some positive integer N and some Laurent polynomial $\mathcal{L}(\xi), \ 0 \leq \theta_1, \theta_2, \theta_1 + \theta_2 \leq 1$. Then

$$\int_{\mathbf{R}^3} \mathbf{x}^{\alpha} \psi(\mathbf{x}) d\mathbf{x} = 0$$

for $|\alpha| \leq N-1$, where $|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3|$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{Z}^3, \alpha_i \geq 0, \ \mathbf{x}^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ for $\mathbf{x} \in \mathbf{R}^3$.

Proof. Since $\psi \perp V_0$, we have

$$\begin{split} 0 &= \sum_{n \in \mathbf{Z}^3} \hat{\psi}(\cdot + 2\pi n) \hat{\phi}(\cdot + 2\pi n) \\ &= \sum_{k=M^T l, l \in \mathbf{Z}^3} H_1(M^{-T}(\cdot + 2\pi k)) \overline{H_0(M^{-T}(\cdot + 2\pi k))} |\hat{\phi}(M^{-T}(\cdot + 2\pi k))|^2 \\ &+ \sum_{k=M^T l + (1,0,0)^T, l \in \mathbf{Z}^3} \{H_1(M^{-T}(\cdot + 2\pi k)) \overline{H_0(M^{-T}(\cdot + 2\pi k))} \\ &\quad \cdot |\hat{\phi}(M^{-T}(\cdot + 2\pi k))|^2 \} \\ &= \sum_{l \in \mathbf{Z}^3} H_1(M^{-T} \cdot + 2\pi l) \overline{H_0(M^{-T} \cdot + 2\pi l)} |\hat{\phi}(M^{-T} \cdot + 2\pi l)|^2 \\ &\quad + \sum_{l \in \mathbf{Z}^3} \left\{ H_1(M^{-T} \cdot + 2\pi l + (\pi, -\pi, -\pi)^T) \\ &\quad \overline{H_0(M^{-T} \cdot + 2\pi l + (\pi, -\pi, -\pi)^T)} \cdot |\hat{\phi}(M^{-T} \cdot + 2\pi l + (\pi, -\pi, -\pi)^T)|^2 \right\} \\ &= \sum_{l \in \mathbf{Z}^3} H_1(M^{-T} \cdot H_1(M^{-T} \cdot + (\pi, -\pi, -\pi)^T)) \frac{1}{2} \overline{H_0(M^{-T} \cdot + 2\pi l + (\pi, -\pi, -\pi)^T)} \\ &\quad \cdot |\hat{\phi}(M^{-T} \cdot + 2\pi l + (\pi, -\pi, -\pi)^T)|^2 \} \\ &= H_1(M^{-T} \cdot H_1(M^{-T} \cdot H_1(\pi, -\pi, -\pi)^T)) F(M^{-T} \cdot H_1(\pi, -\pi, -\pi)^T), \end{split}$$

where $H_1(\cdot) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbf{Z}^3} g_n e^{-in \cdot}$, $F(\cdot) = \overline{H_0(\cdot)} \sum_{l \in \mathbf{Z}^3} |\hat{\phi}(\cdot + 2\pi l)|^2$. It follows that

$$H_1(\cdot)F(\cdot) = -H_1(\cdot + (\pi, -\pi, -\pi)^T)F(\cdot + (\pi, -\pi, -\pi)^T)$$

and consequently,

(3.1)
$$\sum_{\substack{0 \le l \le \alpha}} {\alpha \choose l} D^l H_1(0) D^{\alpha - l} F(0) = -\sum_{\substack{0 \le l \le \alpha}} {\alpha \choose l} D^l H_1((\pi, -\pi, -\pi)^T) D^{\alpha - l} F((\pi, -\pi, -\pi)^T)$$

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for $|\alpha| \leq N-1$, where $0 \leq l \leq \alpha$ means that $0 \leq l_i \leq \alpha_i, i = 1, 2, 3, {\alpha \choose l} = {\alpha_1 \choose l_1} {\alpha_2 \choose l_2} {\alpha_3 \choose l_3}$, and $D^l f(x_1, x_2, x_3) = \frac{\partial^l f(x_1, x_2, x_3)}{\partial x_1^{l_1} \partial x_2^{l_2} \partial x_3^{l_3}}$ for $l = (l_1, l_2, l_3), \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{Z}^3$.

When $|\alpha| = 0$, $\alpha = (0, 0, 0)$. It follows from (3.1) that

$$H_1(0)F(0) = -H_1((\pi, -\pi, -\pi)^T)F((\pi, -\pi, -\pi)^T) = 0.$$

Since ϕ has stable integer shifts, which leads to $|H_0(\xi)|^2 + |H_0(\xi + (\pi, -\pi, -\pi)^T)|^2 \neq 0$, and $H_0((\pi, -\pi, -\pi)^T) = 0$, we have $F(0) \neq 0$. Hence

$$H_1(0) = 0$$

When $|\alpha| = 1$, $\alpha = (1, 0, 0)$, or $\alpha = (0, 1, 0)$, or $\alpha = (0, 0, 1)$. For $\alpha = (0, 0, 1)$, it follows (3.1) that $D^{(0,0,1)}H_1(0)F(0) = 0$, which implies

$$D^{(0,0,1)}H_1(0) = 0.$$

Analogously,

$$D^{(0,1,0)}H_1(0) = 0,$$

and

$$D^{(1,0,0)}H_1(0) = 0.$$

Assume that $D^{\alpha}H_1(0) = 0$ for $|\alpha| \leq s < N - 1$, then, for any α with $|\alpha| = s + 1 \leq N - 1$, using (3.1), we have

$$D^{\alpha}H_{1}(0)F(0) = -\sum_{0 \le l \le \alpha} {\binom{\alpha}{l}} D^{l}H_{1}((\pi, -\pi, -\pi)^{T})D^{\alpha-l}F((\pi, -\pi, -\pi)^{T}).$$

Therefore,

(3.2) $D^{\alpha}H_1(0) = 0, \quad |\alpha| \le N - 1.$

Since $\widehat{\psi}(\xi) = H_1(M^{-T}\xi)\widehat{\phi}(M^{-T}\xi)$, define

$$\eta = (\eta_1, \eta_2, \eta_3)^T = \left(\frac{\xi_1 - \xi_2 - \xi_3}{2}, \xi_2, \frac{-\xi_1 - \xi_2 + \xi_3}{2}\right)^T,$$
$$H_2(\xi) = H_1(M^{-T}\xi), G(\xi) = \widehat{\phi}(M^{-T}\xi).$$

Then $\widehat{\psi}(\xi) = H_1(\eta)G(\xi) = H_2(\xi)G(\xi)$. Hence, for $|\alpha| \leq N - 1$, we have

(3.3)
$$D^{\alpha}\widehat{\psi}(\xi) = \sum_{0 \le l \le \alpha} {\alpha \choose l} D^{l} H_{2}(\xi) D^{\alpha-l} G(\xi).$$

It is easy to check that, for any l with $|l| \leq N - 1$, $D^l H_2(\xi)$ can be represented as a linear combination of $D^s H_1(\eta)$ with |s| = |l|. Since $\eta = 0$ for $\xi = 0$, it follows from (3.2) and (3.3), we have

$$D^l H_2(0) = 0, \quad |l| \le N - 1.$$

This together with (3.3) yields that

$$D^{\alpha}\widehat{\psi}(0) = 0, \quad |\alpha| \le N - 1.$$

Therefore,

 $\int_{\mathbf{R}^3} \mathbf{x}^{\alpha} \psi(\mathbf{x}) d\mathbf{x} = 0, \quad |\alpha| \le N - 1.$

Analogously, we have

Theorem 3.6. Under the assumptions of Theorem 3.1, suppose

$$H_0(\xi) = \left(\frac{1 + \theta_1 e^{-i\xi_1} + \theta_2 e^{-i\xi_2} + (1 - \theta_1 - \theta_2)e^{-i\xi_3}}{2}\right)^N \mathcal{L}(\xi)$$

for some positive integer N and Laurent polynomial $\mathcal{L}(\xi), 0 \leq \theta_1, \theta_2, \theta_1 + \theta_2 \leq 1$. Then

$$\int_{\mathbf{R}^3} \mathbf{x}^{\alpha} \psi(\mathbf{x}) d\mathbf{x} = 0, \quad |\alpha| \le N - 1,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{Z}^3, \alpha \ge 0.$

Remark 3.7. For Theorem 3.5 and 3.6, the parameters θ_1 and θ_2 can be choose adaptively in [0,1] according to the actual needs.

Corollary 3.8. Under the assumptions of Theorem 3.1, suppose

$$H_0(\xi) = \left[1 - \frac{1}{3} \left(\sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} + \sin^2 \frac{\xi_3}{2}\right)\right]^N \mathcal{L}(\xi)$$

for some positive integer N and some Laurent polynomial $\mathcal{L}(\xi)$. Then

$$\int_{\mathbf{R}^3} \mathbf{x}^{\alpha} \psi(\mathbf{x}) d\mathbf{x} = 0$$

for $|\alpha| \leq N-1$, where $|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3|$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{Z}^3, \alpha_i \geq 0, \ \mathbf{x}^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ for $\mathbf{x} \in \mathbf{R}^3$.

Corollary 3.9. Under the assumptions of Theorem 3.1, for j = 1, 2, 3, suppose that

$$H_0(\xi) = (\frac{1 + e^{-i\xi_j}}{2})^N \mathcal{L}_j(\xi)$$

for some positive integer N and some Laurent polynomial $\mathcal{L}_{i}(\xi)$. Then

$$\int_{\mathbf{R}^3} x_j^{\alpha} \psi(\mathbf{x}) d\mathbf{x} = 0,$$

for $0 \le \alpha \le N - 1$.

Remark 3.10. The assumptions on H_0 in Corollary 3.8 and 3.9 is reasonable, which can be seen in [6].

Theorem 3.11. Suppose
$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ k_1 & 1 & 0 \\ l_1 & 2m_1 & 1 \end{pmatrix}$$
, $P_2 = \begin{pmatrix} 1 & 2m_2 & k_2 \\ 0 & 1 & l_2 \\ 0 & 0 & 1 \end{pmatrix}$,

 $P_3 = P(l,k)$, where $k_i + l_i$ are even numbers, $k_i, l_i, m_i \in \mathbb{Z}$, i=1, 2, P(l,k) is the elementary matrix obtained from the identity matrix by interchanging the lth and kth rows. Let

$$\widetilde{M} = (P_1 P_2 P_3)^{-1} M P_1 P_2 P_3.$$

Then

(1)
$$\widetilde{M}^3 = 2I;$$

(2) $\sum_{i=1}^{3} \widetilde{M}_{ij}$ is even numbers, $j = 1, 2, 3.$

Furthermore, trivariate nonseparable compactly supported wavelets with dilation matrix \widetilde{M} can be constructed by our method, and wavelets inherits the symmetry of the corresponding scaling function and satisfies the vanishing moment condition originating in the symbols of the scaling function.

Remark 3.12. Using matrix multiplication and matrix properties and the proof of Theorem 3.1, 3.3 and 3.5, it is easy to prove Theorem 3.11.

4. Numerical example

In this section, we give an example to demonstrate the general theory of section 3.

Example 4.1. Let

$$\phi(\cdot) = \begin{cases} (1 - |x_1 + x_2 - x_3|)(1 - |x_2 + x_3|)(1 - |x_2|), & |x_2| \le 1, \\ |x_2 + x_3| \le 1, \\ |x_1 - x_3| \le 1, \\ 0 & otherwise. \end{cases}$$

$$V_i = \overline{span\{\phi_{i\,k} : k \in \mathbf{Z}^3\}} \quad for \quad j \in \mathbf{Z}.$$

Then

(1) $\{V_j\}_{j \in \mathbb{Z}}$ is an MRA related to M with $\phi(\mathbf{x})$ being a corresponding scaling function.

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(2) Let W_0 be the orthogonal complement of V_0 in V_1 . Define

$$\psi(\cdot) = \sqrt{2} \sum_{n \in \mathbf{Z}^3} (-1)^{n_1 + n_2 + n_3} \langle \phi_{1,(1,0,0)^T - n}, \phi \rangle \phi(M \cdot - n).$$

Then $\{\psi(\cdot - n) : n \in \mathbb{Z}^3\}$ is a Riesz basis for W_0 , and ψ is symmetric about $x = (\frac{1}{2}, 0, \frac{1}{2})^T$, and

$$\int_{\mathbf{R}^3} \psi(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{R}^3} x_1 \psi(\mathbf{x}) d\mathbf{x} = 0.$$

Proof. (1) By computation, we obtain that

$$\begin{split} \widehat{\phi}(\xi) &= -\frac{e^{-i(4\xi_1 - \xi_2 + 2\xi_3)}(e^{i(\xi_1 + \xi_3)} - 1)^2(e^{i\xi_1} - 1)^2(e^{i(2\xi_1 - \xi_2 + \xi_3)} - 1)^2}{\xi_1^2(\xi_1 + \xi_3)^2(2\xi_1 - \xi_2 + \xi_3)^2},\\ \widehat{\phi}(M^{-T}\xi) &= -\frac{e^{-i(3\xi_1 - 4\xi_2 - \xi_3)}(e^{i\xi_1} - 1)^2(e^{i(\frac{1}{2}\xi_1 - \frac{1}{2}\xi_2 - \frac{1}{2}\xi_3)} - 1)^2}{\frac{1}{16}(\xi_1 - \xi_2 - \xi_3)^2(\xi_1 - \xi_2)^2},\\ \cdot \frac{(e^{i\frac{1}{2}(3\xi_1 - 5\xi_2 - \xi_3)} - 1)^2}{(3\xi_1 - 5\xi_2 - \xi_3)^2}. \end{split}$$

It is easy to check that $\widehat{\phi}(\xi) = H_0(M^{-T}\xi)\widehat{\phi}(M^{-T}\xi)$, where

$$(4.1) H_0(\omega) = \frac{1}{2} + \frac{1}{4}e^{i\xi_3} + \frac{1}{4}e^{-i\xi_3} = e^{i\xi_3}(\frac{1+e^{-i\xi_3}}{2})^2, \quad \xi \in \mathbf{R}^3.$$

So ϕ is *M*-refinable, and thus 2-refinable. It follow that

$$\bigcup_{j \in \mathbf{Z}} V_j = \bigcup_{j \in \mathbf{Z}} V_{3j} \quad \text{and} \quad \bigcap_{j \in \mathbf{Z}} V_j = \bigcap_{j \in \mathbf{Z}} V_{3j}.$$

Then applying [2] and [14], we have

$$\bigcap_{j \in \mathbf{Z}} V_j = \{0\} \text{ and } \overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{R}^3).$$

By simple computation, for $\xi \in \mathbf{R}^3$, we have

$$\begin{split} &\sum_{k \in \mathbf{Z}^3} |\widehat{\phi}(\xi + 2\pi k)|^2 \\ = &\frac{1}{8\pi^3} \left[\sum_{k \in \mathbf{Z}} \left(\frac{2}{\xi_3 + 2\pi k} \sin \frac{\xi_3 + 2\pi k}{2} \right)^4 \right] \\ &\cdot \left[\sum_{k \in \mathbf{Z}} \left(\frac{2}{\xi_1 + \xi_3 + 2\pi k} \sin \frac{\xi_1 + \xi_3 + 2\pi k}{2} \right)^4 \right] \\ &\cdot \left[\sum_{k \in \mathbf{Z}} \left(\frac{2}{\xi_1 - \xi_2 + 2\xi_3 + 2\pi k} \sin \frac{\xi_1 - \xi_2 + 2\xi_3 + 2\pi k}{2} \right)^4 \right] > 0. \end{split}$$

Since ϕ is compactly supported, $\sum_{k \in \mathbf{Z}^3} |\hat{\phi}(\xi + 2\pi k)|^2$ is continuous. Hence, $A \leq \sum_{k \in \mathbf{Z}^3} |\hat{\phi}(\xi + 2\pi k)|^2 \leq B$ for some $0 < A < B < \infty$, and thus $\{\phi(\cdot - n) : n \in \mathbf{Z}^3\}$ is a Riesz basis for V_0 . Note that ϕ is M-refinable. This together with (4.1) yield that $\{V_j\}_{j \in \mathbf{Z}^3}$ is an MRA related to M.

(2) For $n \in \mathbb{Z}^3$, let g_n be defined as in Theorem 3.1, we obtain that

$$\Big|\sum_{n\in\mathbf{Z}^3}\Big(\sum_{l\in\mathbf{Z}^3}(-1)^{l_1+l_2+l_3}h_lg_{Mn+(1,0,0)^T-l}\Big)e^{-in\xi}\Big|^2$$

which is nonzero in $[-\pi, \pi]^3$ by some estimation with the help of Matlab. Therefore, using Theorem 3.1, Theorem 3.3 and Corollary 3.9, $\{\psi(\cdot -n) : n \in \mathbf{Z}^3\}$ is a Riesz basis for W_0 , ψ is symmetric about $x = (\frac{1}{2}, 0, \frac{1}{2})^T$, and $\int_{\mathbf{R}^3} \psi(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{R}^3} x_3 \psi(\mathbf{x}) d\mathbf{x} = 0.$

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References

- Carl de Boor, R. A. Devore and A. Ron, On the construction of multivariate (pre) wavelets, *Constr. Approx.* 9 (1993), no. 2-3, 123–166.
- [2] E. Blogay and Y. Wang, Arbitrarily smooth orthogonal nonseparable wavelets in R², SIAM J. Math. Anal. **30** (1999), no. 3, 678–697.
- [3] I. Daubechies, Ten Lecture on Wavelets, CBMS-NSF Reg. Conf. Ser. in Appl. Math., 61, SIAM, Philadelphia, PA, 1992.
- [4] W. He and M. J. Lai, Examples of bivariate nonseparable compactly supported orthonormal continuous wavelets, IEEE Trans. Image Processing 9 (2000), 949– 953.
- [5] Y. Z. Li, On the construction of a class of bidimensional nonseparable compactly supported wavelets, *Proc. Amer. Math. Soc.* **133** (2005), no. 12, 3505–3513.
- [6] R. L. Long, High Dimensional Wavelet Analysis, World Book Publishing Corporation, Beijing, 1995.
- [7] B. Han, Symmetric multivariate orthogonal refinable function, Appl. Comput. Harmon. Anal. 17 (2004), no. 3, 277–292.
- [8] A. karoui, A note on the design of nonseparable orthonormal wavelet bases of $L^2(\mathbf{R}^3)$, Appl. Math. Letters 18 (2005), no. 3, 293–298.
- [9] R. Q. Jia and C. A. Micchelli, Using the Refinement Equation for the Construction of Pre-Wavelets, Acedemic Press, Boston, MA, 1991.
- [10] Q. Jiang, Multivariate matrix refinable functions with arbitrary matrix dilation, Trans. Amer. Math. Soc. 351 (1999), no. 6, 2407–2438.
- [11] A. karoui, A technique for the construction of compactly supported biorthogonal wavelets of $L^2(\mathbf{R}^n)$, $n \ge 2$, J Math. Anal. Appl. **249** (2000), no. 2, 367–392.
- [12] M. J. Lai, Construction of multivariate compactly supported orthonormal wavelets, Adv. Comput. Math. 25 (2006), no. 1-3, 41–56.
- [13] Y. D. Huang, Z. X. Cheng and J. W. Yang, Design of compactly supported trivatiate orthogonal wavelets, *Chaos. Solitons and Fractals* 29 (2006), 901–911.
- [14] A. Cohen and I. Daubechies, Nonseparable bidimensional wavelet bases, *Rev. Mat. Iberoamericana* 9 (1993), no. 1, 51–137.
- [15] C. K. Chui and J. Z. Wang, A cardinal spline approach to wavelets, Proc. Amer. Math. Soc. 113 (1991), no. 3, 785–793.

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