Bulletin of the Iranian Mathematical Society Vol. 38 *No. 1* (*2012*), *pp 85-99.*

COMPACT COMPOSITION OPERATORS ON CERTAIN ANALYTIC LIPSCHITZ SPACES

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Communicated by Mohammad Sal Moslehian

Abstract. We investigate compact composition operators on ceratin Lipschitz spaces of analytic functions on the closed unit disc of the plane. Our approach also leads to some results about composition operators on Zygmund type spaces.

1. Introduction

Let A and B be Banach spaces of analytic functions on the plane set X. For a selfmap ϕ of X and a complex-valued mapping ψ on X, the *weighted composition operator* $\psi C_\phi : A \to B$ is the operator given by $(\psi C_\phi f)(z) = \psi(z) f(\phi(z))$ for all $z \in X$ and $f \in A$. In the special case of $\psi = 1$ we get the *composition operator* $(C_\phi f)(z) = f(\phi(z))$. There has been growing interest in the study of (weighted) composition operators between Banach spaces of analytic functions. Boundedness and compactness of composition operators on Bloch spaces (see Section 2 for the definition) were first studied by Roan [13] and later by Madigan [9, 10] and Matheson [10]. Moreover, Ohno, Stroethoff and Zhao studied weighted composition operators between Bloch type spaces in [12]. The

MSC(2010): Primary: 47B38; Secondary: 47B33, 46J15, 46E15.
Keywords: Compact operators, Bloch type spaces, Zygmund type spaces, analytic Lipschitz spaces, differentiable Lipschitz spaces.
Received: 19 June 2010, Accepted: 28 August 2010.
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compactness of composition operators on certain Banach algebras of analytic and differentiable Lipschitz functions was investigated in [1].

For a bounded plane set X and $0 < \alpha \leq 1$, the *Lipschitz* algebra of order α, $Lip(X, \alpha)$, is the algebra of all complex-valued functions f on X for which

$$p_{\alpha,X}(f) = \sup\left\{\frac{|f(z) - f(w)|}{|z - w|^{\alpha}} : z, w \in X \text{ and } z \neq w\right\} < \infty.$$

These Lipschitz algebras were first studied by Sherbert [14, 15]. The algebra $Lip(X, \alpha)$ is a Banach function algebra when equipped with the norm

$$\|f\|_{Lip(X,\alpha)} = \|f\|_X + p_{\alpha,X}(f) \qquad (f \in Lip(X, \alpha)),$$

where $\|f\|_X = \sup\limits_{z \in X} |f(z)|$.

Let X be a compact plane set with nonempty interior and $A(X)$ the Banach function algebra of all continuous complex-valued functions on X which are analytic on $\mathrm{int}X$. For $0 < \alpha \leq 1$, define

$$Lip_A(X, \alpha) = Lip(X, \alpha) \cap A(X).$$

Then the analytic Lipschitz algebra $(Lip_A(X, \alpha), \|\cdot\|_{Lip(X,\alpha)})$ is a Banach function algebra on X.

A complex-valued function f on a perfect plane set X is called *differentiable* if at each point $z_0 \in X$, the limit

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \in X}} \frac{f(z) - f(z_0)}{z - z_0},$$

exists. Let X be a perfect bounded plane set, $n \in \mathbb{N}$, and $0 < \alpha \leq 1$. The algebra of all complex-valued functions f on X whose derivatives up to order n exist and $f^{(k)} \in Lip(X, \alpha)$ for each k $(0 \leq k \leq n)$, is denoted by $Lip^n(X, \alpha)$. These differentiable Lipschitz algebras were first studied in [7, 11]. The algebra $Lip^n(X, \alpha)$ $(n \in \mathbb{N}, 0 < \alpha \leq 1)$ with the norm

$$\|f\|_{n,\alpha} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_{Lip(X,\alpha)}}{k!} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X + p_{\alpha,X}(f^{(k)})}{k!},$$

is a normed function algebra on X which is not necessarily complete. However, for the closed unit disc $\overline{\mathbb{D}}$, the algebra $Lip^n(\overline{\mathbb{D}}, \alpha)$ is a Banach function algebra on $\overline{\mathbb{D}}$.

Let ϕ be a selfmap of $\overline{\mathbb{D}}$. In [1], it was proved that $\phi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$ is a sufficient condition for the compactness of the composition operator C_ϕ on $Lip_A(\overline{\mathbb{D}}, \alpha)$ and on $Lip^n(\overline{\mathbb{D}}, \alpha)$ when $0 < \alpha \leq 1$. It was also proved

that this condition is necessity when $\alpha = 1$ and it was conjectured that the same result is true in the case $0 < \alpha < 1$. Later in [2], these results were extended to more general compact plane sets X. In Section 2, using a different approach from the one given in [1], we show that the condition $\phi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$ is a necessary condition for the compactness of the composition operator C_ϕ on $Lip_A(\overline{\mathbb{D}}, \alpha)$ in the case $0 < \alpha < 1$. Indeed, we modify the problem of compactness of a composition operator C_ϕ on $Lip_A(\overline{\mathbb{D}}, \alpha)$ to an equivalent problem, i.e. compactness of a composition operator C_φ on a Bloch type space for suitable choice of $\varphi : \mathbb{D} \to \mathbb{D}$. Thus, we consider the analytic Lipschitz algebra $Lip_A(\overline{\mathbb{D}}, \alpha)$ as a Banach function *space* on $\overline{\mathbb{D}}$. In Section 3, we show that the condition $\phi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$ is also a necessary condition for the compactness of the composition operator C_ϕ on $Lip^n(\overline{\mathbb{D}}, \alpha)$ in the case $0 < \alpha < 1$. Indeed, we invoke to this problem by applying Julia-Caratheodory Theorem to an equivalent problem, i.e. compactness of a weighted composition operator on a Bloch type space. Our approach also yields some new results about composition operators on Zygmund type spaces (see Section 3 for the definition). We also consider the differentiable Lipschitz algebra $Lip^n(\overline{\mathbb{D}}, \alpha)$ as a Banach function *space* on $\overline{\mathbb{D}}$.

2. The analytic Lipschitz space $Lip_A(\overline{\mathbb{D}}, \alpha)$

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disc $\mathbb{D}$. For $0 < \alpha < \infty$, we denote by $\mathcal{B}^\alpha$ the *Bloch type space* of all functions $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\sup_{z \in \mathbb{D}} (1 - |z|)^\alpha \left| f'(z) \right| < \infty.$$

The space $\mathcal{B}^\alpha$ is a Banach space when equipped with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha \left| f'(z) \right| \qquad (f \in \mathcal{B}^\alpha).$$

In the case $\alpha = 1$ we have the classical *Bloch space* $\mathcal{B} = \mathcal{B}^1$ (see [16]).

By [16, Theorem 7.9], for each $0 < \alpha < 1$ the space $\mathcal{B}^{1-\alpha}$ can be identified with the analytic Lipschitz space $H\Lambda_\alpha(\mathbb{D}) := \mathcal{H}(\mathbb{D}) \cap Lip(\mathbb{D}, \alpha)$, which is a closed subspace of $(Lip(\mathbb{D}, \alpha), \|\cdot\|_{Lip(\mathbb{D},\alpha)})$. Since the norm topologies on $\mathcal{B}^{1-\alpha}$ and $H\Lambda_\alpha(\mathbb{D})$ are stronger than compact-open topology, the Closed Graph Theorem implies that the norms $\|\cdot\|_{\mathcal{B}^{1-\alpha}}$ and

$\|\cdot\|_{Lip(\mathbb{D},\alpha)}$ are equivalent on $H\Lambda_\alpha(\mathbb{D}) = \mathcal{B}^{1-\alpha}$, that is

$$C_1\|f\|_{\mathcal{B}^{1-\alpha}} \leq \|f\|_{Lip(\mathbb{D},\alpha)} \leq C_2\|f\|_{\mathcal{B}^{1-\alpha}} \quad (f \in H\Lambda_\alpha(\mathbb{D}) = \mathcal{B}^{1-\alpha}), \tag{2.1}$$

for some constants $C_1, C_2 > 0$. Using this, we show that the spaces $Lip_A(\overline{\mathbb{D}}, \alpha)$ and $\mathcal{B}^{1-\alpha}$ are isomorphic. Note first that every $f \in Lip(\mathbb{D}, \alpha)$ has a unique continuous extension F to $\overline{\mathbb{D}}$. To see this, consider any sequence (z_n) in $\mathbb{D}$ converging to $z_0 \in \partial\mathbb{D}$. Since $f \in Lip(\mathbb{D}, \alpha)$ the sequence $(f(z_n))$ is a Cauchy and hence is a convergent sequence. Define $F(z_0) := \lim_{n\to\infty} f(z_n)$, then F is well-defined and it is the unique continuous extension of f.

Proposition 2.1. *Let $0 < \alpha < 1$. Then $F \in Lip_A(\overline{\mathbb{D}}, \alpha)$ if and only if $f = F|_{\mathbb{D}} \in \mathcal{B}^{1-\alpha}$, or equivalently, f belongs to $\mathcal{B}^{1-\alpha}$ if and only if F, the continuous extension of f to $\overline{\mathbb{D}}$ belongs to $Lip_A(\overline{\mathbb{D}}, \alpha)$. Moreover,*

$$C_1\|f\|_{\mathcal{B}^{1-\alpha}} \leq \|F\|_{Lip(\overline{\mathbb{D}},\alpha)} \leq C_2\|f\|_{\mathcal{B}^{1-\alpha}} \quad (f \in \mathcal{B}^{1-\alpha}),$$

where C_1 and C_2 are the constants described in (2.1).

Proof. If $F \in Lip_A(\overline{\mathbb{D}}, \alpha)$ then clearly $f = F|_{\mathbb{D}} \in H\Lambda_\alpha(\mathbb{D}) = \mathcal{B}^{1-\alpha}$ and $\|f\|_{Lip(\mathbb{D},\alpha)} \leq \|F\|_{Lip(\overline{\mathbb{D}},\alpha)}$. Hence, by (2.1) we have $C_1\|f\|_{\mathcal{B}^{1-\alpha}} \leq \|F\|_{Lip(\overline{\mathbb{D}},\alpha)}$, which implies that the restriction operator

$$R : Lip_A(\overline{\mathbb{D}}, \alpha) \to \mathcal{B}^{1-\alpha} \quad R(F) = F|_{\mathbb{D}}\ , \tag{2.2}$$

is well-defined and bounded with $\|R\| \leq \frac{1}{C_1}$.

Now, let $f \in \mathcal{B}^{1-\alpha}$ and let F be the continuous extension of f to $\overline{\mathbb{D}}$. For each $z, w \in \overline{\mathbb{D}}$, let (z_n) and (w_n) be sequences in $\mathbb{D}$ with $z_n \to z$ and $w_n \to w$ as $n \to \infty$. Then

$$\begin{aligned} |F(z) - F(w)| &= \lim_{n\to\infty} |f(z_n) - f(w_n)| \\ &\leq p_{\alpha,\mathbb{D}}(f) \lim_{n\to\infty} |z_n - w_n|^\alpha \\ &= p_{\alpha,\mathbb{D}}(f)\, |z - w|^\alpha\,. \end{aligned}$$

Consequently, $F \in Lip(\overline{\mathbb{D}}, \alpha)$ and $p_{\alpha,\overline{\mathbb{D}}}(F) \leq p_{\alpha,\mathbb{D}}(f)$. On the other hand, since $F|_{\mathbb{D}} = f \in \mathcal{H}(\mathbb{D})$, we have $F \in A(\overline{\mathbb{D}})$. Hence $F \in Lip_A(\overline{\mathbb{D}}, \alpha)$ and also $\|F\|_{\overline{\mathbb{D}}} = \|F\|_{\mathbb{D}} = \|f\|_{\mathbb{D}}$. Therefore, $\|F\|_{Lip(\overline{\mathbb{D}},\alpha)} \leq \|f\|_{Lip(\mathbb{D},\alpha)}$ and by (2.1) we get $\|F\|_{Lip(\overline{\mathbb{D}},\alpha)} \leq C_2\|f\|_{\mathcal{B}^{1-\alpha}}$. This completes the proof and also shows that the extension operator

$$E : \mathcal{B}^{1-\alpha} \to Lip_A(\overline{\mathbb{D}}, \alpha) \quad E(f) = F, \tag{2.3}$$

is well-defined and bounded with $\|E\| \leq C_2$. $\square$

Remark 2.2. *It is worth mentioning that in the proof of Proposition 2.1, in order to show that $E(f) \in Lip(\overline{\mathbb{D}}, \alpha)$ one could also use the fact that $E(f) \in Lip(\partial\mathbb{D}, \alpha)$ (by* [16, Theorem 7.9] *and* [5, Theorem 5.1]*) and hence, by* [6, Lemma 4]*, $E(f) \in Lip(\overline{\mathbb{D}}, \alpha)$.*

Let $C_\phi : Lip_A(\overline{\mathbb{D}}, \alpha) \to Lip_A(\overline{\mathbb{D}}, \alpha)$ be a composition operator induced by the non-constant selfmap $\phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Consider the selfmap $\varphi = R(\phi) = \phi|_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$. Then φ induces a composition operator $C_\varphi : \mathcal{B}^{1-\alpha} \to \mathcal{B}^{1-\alpha}$. To see this, let $f \in \mathcal{B}^{1-\alpha}$. Then $E(f) \in Lip_A(\overline{\mathbb{D}}, \alpha)$ and hence $E(f) \circ \phi \in Lip_A(\overline{\mathbb{D}}, \alpha)$. Consequently, $f \circ \varphi = R\left(E(f) \circ \phi\right) \in \mathcal{B}^{1-\alpha}$. Conversely, if φ induces the composition operator $C_\varphi : \mathcal{B}^{1-\alpha} \to \mathcal{B}^{1-\alpha}$ then $\phi = E(\varphi) : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ induces the composition operator $C_\phi : Lip_A(\overline{\mathbb{D}}, \alpha) \to Lip_A(\overline{\mathbb{D}}, \alpha)$. This follows from the fact that if $F \in Lip_A(\overline{\mathbb{D}}, \alpha)$, then $R(F) \circ \varphi \in \mathcal{B}^{1-\alpha}$ and hence, by the uniqueness of the continuous extension we have $F \circ \phi = E\left(R(F) \circ \varphi\right) \in Lip_A(\overline{\mathbb{D}}, \alpha)$.

Theorem 2.3. *Let $0 < \alpha < 1$ and let $C_\phi : Lip_A(\overline{\mathbb{D}}, \alpha) \to Lip_A(\overline{\mathbb{D}}, \alpha)$ be a composition operator induced by the non-constant selfmap $\phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Let φ denote the restriction of ϕ to $\mathbb{D}$. Then $C_\phi : Lip_A(\overline{\mathbb{D}}, \alpha) \to Lip_A(\overline{\mathbb{D}}, \alpha)$ is compact if and only if $C_\varphi : \mathcal{B}^{1-\alpha} \to \mathcal{B}^{1-\alpha}$ is compact.*

Proof. Let R and E denote the restriction and the extension operators described in (2.2) and (2.3), and note that by the discussion right before this theorem, we have

$$C_\phi = E \circ C_\varphi \circ R \quad \text{and} \quad C_\varphi = R \circ C_\phi \circ E. \tag{2.4}$$

By the same argument as in the proof of Proposition 2.1, the operators $R : Lip_A(\overline{\mathbb{D}}, \alpha) \to \mathcal{B}^{1-\alpha}$ and $E : \mathcal{B}^{1-\alpha} \to Lip_A(\overline{\mathbb{D}}, \alpha)$ are bounded. Therefore, by (2.4), $C_\varphi : \mathcal{B}^{1-\alpha} \to \mathcal{B}^{1-\alpha}$ is compact if and only if $C_\phi : Lip_A(\overline{\mathbb{D}}, \alpha) \to Lip_A(\overline{\mathbb{D}}, \alpha)$ is compact. $\square$

For the rest of this paper, we need the following important theorem proved by MacCluer and Zhao in [8] for weighted composition operators on the Bloch type spaces.

Theorem 2.4. [8, Theorem 5] *Let $0 < \alpha < 1$, $\zeta \in \partial\mathbb{D}$ and $u, \varphi \in H(\mathbb{D})$, where φ is a selfmap of $\mathbb{D}$. If $uC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is compact, then $u(\zeta) = 0$ whenever $\lim_{r\to 1^-} \varphi(r\zeta)$ exists and has modulus 1.*

For $0 < \alpha \leq 1$, consider the composition operator $C_\phi : Lip_A(\overline{\mathbb{D}}, \alpha) \to Lip_A(\overline{\mathbb{D}}, \alpha)$ induced by the non-constant selfmap $\phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. It was proved in [1, Theorem 3.3] that C_ϕ is compact provided that $\phi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$. It was also shown that for $\alpha = 1$, this condition is necessary. Here we will prove that the same condition is necessary for $0 < \alpha < 1$.

Theorem 2.5. *Let $0 < \alpha \leq 1$ and let $C_\phi : Lip_A(\overline{\mathbb{D}}, \alpha) \to Lip_A(\overline{\mathbb{D}}, \alpha)$ be a composition operator induced by the non-constant selfmap $\phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Then C_ϕ is compact if and only if $\phi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$.*

Proof. We only need to show that $\phi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$ whenever C_ϕ is compact and $0 < \alpha < 1$. By Theorem 2.3, if $C_\phi : Lip_A(\overline{\mathbb{D}}, \alpha) \to Lip_A(\overline{\mathbb{D}}, \alpha)$ is compact then C_φ is a compact operator on $\mathcal{B}^{1-\alpha}$ $(0 < \alpha < 1)$. Now, by contrary let $\phi(\zeta) = \eta \in \partial\mathbb{D}$ for some $\zeta \in \partial\mathbb{D}$. Then $|\lim_{r\to 1^-} \varphi(r\zeta)| = |\eta| = 1$. Therefore, Theorem 2.4 leads to a contradiction and completes the proof of the theorem. □

3. The differentiable Lipschitz space $Lip^n(\overline{\mathbb{D}}, \alpha)$

The Zygmund space $\mathcal{Z}$ is the class of all functions $f \in \mathcal{H}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ with

$$\sup_{\substack{e^{i\theta}\in\partial\mathbb{D} \\ h>0}} \frac{\left|f\left(e^{i(\theta+h)}\right) + f\left(e^{i(\theta-h)}\right) - 2f\left(e^{i\theta}\right)\right|}{h} < \infty.$$

By [5, Theorem 5.3], an analytic function f belongs to $\mathcal{Z}$ if and only if $f' \in \mathcal{B}$, or equivalently $\sup_{z\in\mathbb{D}} (1 - |z|) |f''(z)| < \infty$. For $0 < \alpha < \infty$ we denote by $\mathcal{Z}^\alpha$ the *Zygmund type space* of those functions $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}} (1 - |z|)^\alpha \left|f''(z)\right| < \infty.$$

The space $\mathcal{Z}^\alpha$ is a Banach space with the norm

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + \left|f'(0)\right| + \sup_{z\in\mathbb{D}} (1 - |z|)^\alpha \left|f''(z)\right| \quad (f \in \mathcal{Z}^\alpha).$$

Boundedness of composition operators on $\mathcal{Z}$ was first studied by Choe, Koo and Smith in [4].

Now, in general, for each $n \in \mathbb{N}$ and $0 < \alpha < \infty$ we define the space $\mathcal{Z}_n^\alpha$ of those functions $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}} (1-|z|)^\alpha \left| f^{(n+1)}(z) \right| < \infty.$$

The space $\mathcal{Z}_n^\alpha$ is a Banach space when equipped with the norm

$$\|f\|_{\mathcal{Z}_n^\alpha} = |f(0)| + \left|f'(0)\right| + \cdots + \left|f^{(n)}(0)\right| + \sup_{z\in\mathbb{D}} (1-|z|)^\alpha \left| f^{(n+1)}(z) \right|.$$

Note that for a differentiable function F on $\overline{\mathbb{D}}$, we have $R(F') = R(F)'$, where R is the restriction operator $R(F) = F\,|_{\mathbb{D}}$. We also have $F \in Lip^n(\overline{\mathbb{D}}, \alpha)$ if and only if $F^{(n)} \in Lip_A(\overline{\mathbb{D}}, \alpha)$. Hence, for $n \in \mathbb{N}$ and $0 < \alpha < 1$, if $F \in Lip^n(\overline{\mathbb{D}}, \alpha)$ then by Proposition 2.1, $R(F)^{(n)} = R(F^{(n)}) \in \mathcal{B}^{1-\alpha}$. This shows that the restriction operator

$$R : Lip^n(\overline{\mathbb{D}}, \alpha) \to \mathcal{Z}_n^{1-\alpha} \quad R(F) = F\,|_{\mathbb{D}}\ , \tag{3.1}$$

is well-defined. This operator is also bounded with $\|R\| \le n! \max\{1, \frac{1}{C_1}\}$. Note also that if $C_\phi : Lip^n(\overline{\mathbb{D}}, \alpha) \to Lip^n(\overline{\mathbb{D}}, \alpha)$ $(0 < \alpha < 1)$ is a composition operator induced by the non-constant selfmap $\phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, then $\varphi = R(\phi) = \phi\,|_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ induces the composition operator $C_\varphi : \mathcal{Z}_n^{1-\alpha} \to \mathcal{Z}_n^{1-\alpha}$. To see this, let $f \in \mathcal{Z}_n^{1-\alpha}$ and note that $f^{(n)} \in \mathcal{B}^{1-\alpha}$. Hence by Proposition 2.1, $F_n := E(f^{(n)}) \in Lip_A(\overline{\mathbb{D}}, \alpha)$. Define

$$F_{n-1}(z) := \int_0^z F_n(\zeta)d\zeta + f^{(n-1)}(0) = \int_0^z E(f^{(n)})(\zeta)d\zeta + f^{(n-1)}(0),$$

for $z \in \overline{\mathbb{D}}$. It follows that $F'_{n-1} = F_n = E(f^{(n)}) \in Lip_A(\overline{\mathbb{D}}, \alpha)$ and hence $F_{n-1} \in Lip^1(\overline{\mathbb{D}}, \alpha)$. On the other hand, $R(F_{n-1}) = f^{(n-1)}$ which implies that $f^{(n-1)} \in H\Lambda_\alpha(\mathbb{D})$. Also $F_{n-1} = E(f^{(n-1)})$, because $f^{(n-1)}$ has a unique continuous extension to $\overline{\mathbb{D}}$. Setting

$$F(z) := \int_0^z F_1(\zeta)d\zeta + f(0) = \int_0^z E(f')(\zeta)d\zeta + f(0) \quad (z \in \overline{\mathbb{D}}),$$

it yields $F \in Lip^n(\overline{\mathbb{D}}, \alpha)$, $F = E(f)$ and $F^{(n)} = F_n = E(f^{(n)})$. Moreover, $R(F^{(k)}) = R(F_k) = f^{(k)}$ and $E(f^{(k)}) = F_k = F^{(k)} = E(f)^{(k)}$ for each $0 \le k \le n$. Since ϕ induces the composition operator C_ϕ on $Lip^n(\overline{\mathbb{D}}, \alpha)$, we have $F \circ \phi \in Lip^n(\overline{\mathbb{D}}, \alpha)$ or equivalently $(F \circ \phi)^{(n)} \in Lip_A(\overline{\mathbb{D}}, \alpha)$.

Therefore, Proposition 2.1 implies that

$$(f \circ \varphi)^{(n)} = (R(F \circ \phi))^{(n)} = R\left((F \circ \phi)^{(n)}\right) \in \mathcal{B}^{1-\alpha},$$

meaning that $f \circ \varphi \in \mathcal{Z}_n^{1-\alpha}$, so φ induces the composition operator $C_\varphi : \mathcal{Z}_n^{1-\alpha} \to \mathcal{Z}_n^{1-\alpha}$.

Conversely, if $\varphi : \mathbb{D} \to \mathbb{D}$ induces a composition operator $C_\varphi : \mathcal{Z}_n^{1-\alpha} \to \mathcal{Z}_n^{1-\alpha}$, then $\phi = E(\varphi) : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ induces the composition operator $C_\phi : Lip^n(\overline{\mathbb{D}}, \alpha) \to Lip^n(\overline{\mathbb{D}}, \alpha)$. To see this, let $F \in Lip^n(\overline{\mathbb{D}}, \alpha)$. As mentioned, $F \circ \phi \in Lip^n(\overline{\mathbb{D}}, \alpha)$ if and only if $(F \circ \phi)^{(n)} \in Lip_A(\overline{\mathbb{D}}, \alpha)$. According to Proposition 2.1, this is equivalent to

$$(R(F) \circ \varphi)^{(n)} = (R(F \circ \phi))^{(n)} = R\left((F \circ \phi)^{(n)}\right) \in \mathcal{B}^{1-\alpha}.$$

On the other hand, since φ induces the composition operator $C_\varphi : \mathcal{Z}_n^{1-\alpha} \to \mathcal{Z}_n^{1-\alpha}$ and $R(F) \in \mathcal{Z}_n^{1-\alpha}$, we have $R(F) \circ \varphi \in \mathcal{Z}_n^{1-\alpha}$, or equivalently, $(R(F) \circ \varphi)^{(n)} \in \mathcal{B}^{1-\alpha}$. Hence, $\phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ induces the composition operator $C_\phi : Lip^n(\overline{\mathbb{D}}, \alpha) \to Lip^n(\overline{\mathbb{D}}, \alpha)$. □

Lemma 3.1. *Let $n \in \mathbb{N}$ and $0 < \alpha < 1$. Then there exists a constant $C > 0$ such that*

$$\|g^{(\ell)}\|_{\mathbb{D}} \leq C\|g\|_{\mathcal{Z}_n^\alpha},$$

for all $g \in \mathcal{Z}_n^\alpha$ and $0 \leq \ell \leq n$.

Proof. Let $g \in \mathcal{Z}_n^\alpha$. Using the Fundamental Theorem of Calculus we have

$$g^{(\ell)}(z) = \int_0^z g^{(\ell+1)}(\zeta)d\zeta + g^{(\ell)}(0) \quad (z \in \mathbb{D}),$$

for each $0 \leq \ell \leq n-1$, which implies that

$$\|g^{(\ell)}\|_{\mathbb{D}} \leq \|g^{(\ell+1)}\|_{\mathbb{D}} + |g^{(\ell)}(0)|. \tag{3.2}$$

Hence, by applying $(n-\ell)$-times (3.2), we have

$$\|g^{(\ell)}\|_{\mathbb{D}} \leq |g^{(\ell)}(0)| + |g^{(\ell+1)}(0)| + \cdots + |g^{(n-1)}(0)| + \|g^{(n)}\|_{\mathbb{D}}. \tag{3.3}$$

On the other hand, $g^{(n)} \in \mathcal{B}^\alpha$. So, by Proposition 2.1 we get

$$\|g^{(n)}\|_{\mathbb{D}} \leq \|E(g^{(n)})\|_{\overline{\mathbb{D}}} \leq \|E(g^{(n)})\|_{Lip(\overline{\mathbb{D}},1-\alpha)} \leq C_2\|g^{(n)}\|_{\mathcal{B}^\alpha}. \tag{3.4}$$

Now, applying (3.3) and (3.4), it follows that

$$\|g^{(\ell)}\|_{\mathbb{D}} \leq \max\{1, C_2\}\|g\|_{\mathcal{Z}_n^\alpha},$$

for all $0 \leq \ell \leq n$. $\square$

Theorem 3.2. *Let $0 < \alpha < 1$, $n \in \mathbb{N}$ and let $C_\phi : Lip^n(\overline{\mathbb{D}}, \alpha) \to Lip^n(\overline{\mathbb{D}}, \alpha)$ be a composition operator induced by the selfmap $\phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Let φ denote the restriction of ϕ to $\mathbb{D}$. Then $C_\phi : Lip^n(\overline{\mathbb{D}}, \alpha) \to Lip^n(\overline{\mathbb{D}}, \alpha)$ is compact if and only if $C_\varphi : \mathcal{Z}_n^{1-\alpha} \to \mathcal{Z}_n^{1-\alpha}$ is compact.*

Proof. Let R denote the restriction operator $R : Lip^n(\overline{\mathbb{D}}, \alpha) \to \mathcal{Z}_n^{1-\alpha}$ described in (3.1). Consider the operator $T : \mathcal{Z}_n^{1-\alpha} \to Lip^n(\overline{\mathbb{D}}, \alpha)$ given by

$$T(f)(z) = \int_0^z E(f')(\zeta)d\zeta + f(0) \quad (f \in \mathcal{Z}_n^{1-\alpha}, z \in \overline{\mathbb{D}}). \tag{3.5}$$

Note that by the discussion before Lemma 3.1, the operator T is well-defined and $R \circ T = id$. We now show that T is bounded. Let $(f_m) \subseteq \mathcal{Z}_n^{1-\alpha}$ with $f_m \to f$ in $\mathcal{Z}_n^{1-\alpha}$ and $T(f_m) \to g$ in $Lip^n(\overline{\mathbb{D}}, \alpha)$. By Lemma 3.1, we have

$$\|T(f_m) - T(f)\|_{\mathbb{D}} \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

On the other hand, $\|T(f_m) - g\|_{\overline{\mathbb{D}}} \leq \|T(f_m) - g\|_{n,\alpha} \to 0$ as $n \to \infty$. This along with (3.6) implies that $T(f) = g$ on $\mathbb{D}$ and hence, $T(f) = g$ on $\overline{\mathbb{D}}$. Thus, by the Closed Graph Theorem, the operator T is bounded.

Considering the bounded operators $R : Lip^n(\overline{\mathbb{D}}, \alpha) \to \mathcal{Z}_n^{1-\alpha}$ and $T : \mathcal{Z}_n^{1-\alpha} \to Lip^n(\overline{\mathbb{D}}, \alpha)$, we have $C_\varphi = R \circ C_\phi \circ T$ and $C_\phi = T \circ C_\varphi \circ R$. Therefore the compactness of $C_\phi : Lip^n(\overline{\mathbb{D}}, \alpha) \to Lip^n(\overline{\mathbb{D}}, \alpha)$ and $C_\varphi : \mathcal{Z}_n^{1-\alpha} \to \mathcal{Z}_n^{1-\alpha}$ are equivalent. $\square$

In order to state the main results of this section, we need a few preliminary lemmas. In what follows, set $Lip^0(\overline{\mathbb{D}}, \alpha) = Lip_A(\overline{\mathbb{D}}, \alpha)$ and denote by $\mathcal{Z}_0^\alpha$ the Bloch type space $\mathcal{B}^\alpha$.

Lemma 3.3. *Let $0 < \alpha < 1$ and let k, n be two nonnegative integers. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be in $\mathcal{Z}_{n+2}^\alpha$. Then $P \equiv P_{k,n} : \mathcal{Z}_n^\alpha \to \mathcal{Z}_n^\alpha$ given by*

$$P(f)(z) = \varphi'(z)^k \varphi''(z) \int_0^{\varphi(z)} f(\zeta)d\zeta \quad (f \in \mathcal{Z}_n^\alpha, z \in \mathbb{D}),$$

is a compact operator.

Proof. First we show that P is well-defined. To see this, let $f \in \mathcal{Z}_n^\alpha$. Then by Lemma 3.1 there exists a constant $C > 0$ such that

$$\|f^{(j)}\|_{\mathbb{D}} \leq C\|f\|_{\mathcal{Z}_n^\alpha} \quad (0 \leq j \leq n), \tag{3.7}$$

and

$$\|\varphi^{(i)}\|_{\mathbb{D}} \leq C\|\varphi\|_{\mathcal{Z}_{n+2}^\alpha} \quad (0 \leq i \leq n+2). \tag{3.8}$$

On the other hand, considering the bounded extension operator T given in (3.5), one can see that the extension $T(\varphi)$ of φ to $\overline{\mathbb{D}}$ belongs to $Lip^{n+2}(\overline{\mathbb{D}}, 1-\alpha)$, because $\varphi \in \mathcal{Z}_{n+2}^\alpha$. Hence, by the discussion before Lemma 3.1, $T(\varphi^{(i)}) = T(\varphi)^{(i)} \in Lip_A(\overline{\mathbb{D}}, 1-\alpha)$ for each $0 \leq i \leq n+2$. Therefore, by Proposition 2.1, $\varphi^{(i)} \in \mathcal{B}^\alpha$ or equivalently,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\varphi^{(i+1)}(z)| < \infty \quad (0 \leq i \leq n+2). \tag{3.9}$$

Now, consider

$$\begin{aligned}\|P(f)\|_{\mathcal{Z}_n^\alpha} &= \sum_{i=0}^{n}\left|P(f)^{(i)}(0)\right| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\left|P(f)^{(n+1)}(z)\right| \\ &= \sum_{i=0}^{n}\left|\left(\varphi'(z)^k\varphi''(z)\int_0^{\varphi(z)} f(\zeta)d\zeta\right)^{(i)}_{z=0}\right| \\ &\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\left|\left(\varphi'(z)^k\varphi''(z)\int_0^{\varphi(z)} f(\zeta)d\zeta\right)^{(n+1)}\right|,\end{aligned} \tag{3.10}$$

and note that the terms in (3.10) are all dominated by the terms of the type (3.7), (3.8) and (3.9). This implies that (3.10) is bounded. Thus, $P(f) \in \mathcal{Z}_n^\alpha$ and P is well-defined.

To see the compactness of P, let (f_m) be a bounded sequence in $\mathcal{Z}_n^\alpha$. It follows from boundedness of the extension operator T, given in (3.5), that $(T(f_m))$ is a bounded sequence in $Lip^n(\overline{\mathbb{D}}, 1-\alpha)$. Consequently, for each $0 \leq j \leq n$, the sequence $(T(f_m)^{(j)})$ is equicontinuous and hence, up to subsequence, there exists a continuous function F on $\overline{\mathbb{D}}$ such that

$$\|T(f_m^{(j)}) - F^{(j)}\|_{\overline{\mathbb{D}}} = \|T(f_m)^{(j)} - F^{(j)}\|_{\overline{\mathbb{D}}} \to 0 \quad \text{as } m \to \infty, \tag{3.11}$$

for each $0 \leq j \leq n$. Let $f = R(F)$ be the restriction of F to $\mathbb{D}$, then by applying $R(T(f_m)) = f_m$ to (3.11), we have $\|f_m^{(j)} - f^{(j)}\|_{\mathbb{D}} \to 0$ as $m \to \infty$, thereby giving that

$$\|f_p^{(j)} - f_q^{(j)}\|_{\mathbb{D}} \to 0 \quad \text{as } p, q \to \infty, \tag{3.12}$$

for each $0 \le j \le n$. Now, to prove the convergence of $(P(f_m))$, we show that it is a Cauchy sequence in $\mathcal{Z}_n^\alpha$. Considering

$$\|P(f_p) - P(f_q)\|_{\mathcal{Z}_n^\alpha} =$$
$$\sum_{i=0}^{n} \left| \left(\varphi'(z)^k \varphi''(z) \int_0^{\varphi(z)} (f_p(\zeta) - f_q(\zeta)) d\zeta \right)^{(i)}_{z=0} \right|$$
$$+ \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \left| \left(\varphi'(z)^k \varphi''(z) \int_0^{\varphi(z)} (f_p(\zeta) - f_q(\zeta)) d\zeta \right)^{(n+1)} \right|, \tag{3.13}$$

one can see that the terms in (3.13) are all dominated by the terms of the type (3.8) and (3.9), and they all contain a term of the type (3.12). Therefore, (3.13) along with (3.12) implies that $\|P(f_p) - P(f_q)\|_{\mathcal{Z}_n^\alpha} \to 0$ as $p, q \to \infty$. □

Lemma 3.4. *Let $0 < \alpha < 1$ and let k, n be two integers with $n \ge 0$ and $k \ge 1$. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be in $\mathcal{Z}_{n+2}^\alpha$. Then $(\varphi')^k C_\varphi : \mathcal{Z}_{n+1}^\alpha \to \mathcal{Z}_{n+1}^\alpha$ is compact if and only if $(\varphi')^{k+1} C_\varphi : \mathcal{Z}_n^\alpha \to \mathcal{Z}_n^\alpha$ is compact.*

Proof. First, we recall that the differentiation and integration operators given by

$$D : \mathcal{Z}_{n+1}^\alpha \to \mathcal{Z}_n^\alpha \quad D(f) = f',$$

and

$$S : \mathcal{Z}_n^\alpha \to \mathcal{Z}_{n+1}^\alpha \quad S(f)(z) = \int_0^z f(\zeta) d\zeta,$$

are bounded, indeed, $\|D\| \le 1$ and $\|S\| \le 1$. Now, consider the following diagram,

$$\begin{tikzcd} \mathcal{Z}_{n+1}^\alpha \arrow[r, "(\varphi')^k C_\varphi"] & \mathcal{Z}_{n+1}^\alpha \arrow[d, "D"] \\ \mathcal{Z}_n^\alpha \arrow[u, "S"] \arrow[r, "Q"] & \mathcal{Z}_n^\alpha, \end{tikzcd}$$

where the operator $Q : \mathcal{Z}_n^\alpha \to \mathcal{Z}_n^\alpha$ is given by

$$Q = D \circ (\varphi')^k C_\varphi \circ S. \tag{3.14}$$

Therefore,

$$S \circ Q \circ D = (\varphi')^k C_\varphi + P_0, \tag{3.15}$$

where $P_0 : \mathcal{Z}^\alpha_{n+1} \to \mathcal{Z}^\alpha_{n+1}$ is the compact operator given by $P_0(f) = -f(0)(\varphi')^k - f(\varphi(0))\varphi'(0)^k + f(0)\varphi'(0)^k$. Note that by (3.14) and (3.15) and boundedness of the operators S and D, one can conclude that the operator $(\varphi')^k C_\varphi : \mathcal{Z}^\alpha_{n+1} \to \mathcal{Z}^\alpha_{n+1}$ is compact if and only if the operator $Q : \mathcal{Z}^\alpha_n \to \mathcal{Z}^\alpha_n$ is compact. Now, for each $f \in \mathcal{Z}^\alpha_n$ and $z \in \mathbb{D}$ we have

$$\begin{aligned} Q(f)(z) =& D\left(\varphi'(z)^k \left(\int_0^{\varphi(z)} f(\zeta)d\zeta\right)\right) \\ =& k\varphi'(z)^{k-1}\varphi''(z)\left(\int_0^{\varphi(z)} f(\zeta)d\zeta\right) + \varphi'(z)^{k+1} f(\varphi(z)) \\ =& kP_{k-1,n}(f)(z) + (\varphi')^{k+1} C_\varphi(f)(z), \end{aligned}$$

where $P_{k-1,n} : \mathcal{Z}^\alpha_n \to \mathcal{Z}^\alpha_n$ is the compact operator given in Lemma 3.3. Consequently, $Q : \mathcal{Z}^\alpha_n \to \mathcal{Z}^\alpha_n$ is compact if and only if $(\varphi')^{k+1} C_\varphi : \mathcal{Z}^\alpha_n \to \mathcal{Z}^\alpha_n$ is compact, which is the desired result. □

We remark that the result of Lemma 3.4 also holds in the case $k = 0$ without assuming $\varphi \in \mathcal{Z}^\alpha_{n+2}$. Indeed, if $k = 0$, then $Q = \varphi' C_\varphi$ and hence $C_\varphi : \mathcal{Z}^\alpha_{n+1} \to \mathcal{Z}^\alpha_{n+1}$ is compact if and only if $\varphi' C_\varphi : \mathcal{Z}^\alpha_n \to \mathcal{Z}^\alpha_n$. In fact, the following result holds.

Corollary 3.5. *Let $0 < \alpha < 1$, n a nonnegative integer, and φ an analytic selfmap of $\mathbb{D}$. Then $C_\varphi : \mathcal{Z}^\alpha_{n+1} \to \mathcal{Z}^\alpha_{n+1}$ is compact if and only if $\varphi' C_\varphi : \mathcal{Z}^\alpha_n \to \mathcal{Z}^\alpha_n$ is compact.*

We are now ready to state our main results in this section.

Theorem 3.6. *Let $0 < \alpha < 1$, $n \in \mathbb{N}$ and let $C_\varphi : \mathcal{Z}^\alpha_n \to \mathcal{Z}^\alpha_n$ be a composition operator induced by $\varphi : \mathbb{D} \to \mathbb{D}$. Then C_φ is compact if and only if the weighted composition operator $(\varphi')^n C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is compact.*

Proof. The case $n = 1$ is done by Corollary 3.5. Let $n \geq 2$ and consider the composition operator $C_\varphi : \mathcal{Z}^\alpha_n \to \mathcal{Z}^\alpha_n$. Then by the same argument as in the proof of Lemma 3.4, one has

$$\begin{aligned} &\varphi' C_\varphi = D \circ C_\varphi \circ S, \\ &S \circ \varphi' C_\varphi \circ D = C_\varphi + P_0, \end{aligned}$$

where $P_0 : \mathcal{Z}_n^\alpha \to \mathcal{Z}_n^\alpha$ is the compact operator given by $P_0(f) = -f(\varphi(0))$. Hence, $C_\varphi : \mathcal{Z}_n^\alpha \to \mathcal{Z}_n^\alpha$ is compact if and only if $\varphi' C_\varphi : \mathcal{Z}_{n-1}^\alpha \to \mathcal{Z}_{n-1}^\alpha$ is compact. Now, applying $(n-1)$-times Lemma 3.4 implies that, this is equivalent to the compactness of $(\varphi')^n C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ which completes the proof of the theorem. □

Applying [12, Theorem 3.1] to Theorem 3.6, one can get the following characterization for the compactness of the composition operators on Zygmund type spaces.

Corollary 3.7. *Let $0 < \alpha < 1$, $n \in \mathbb{N}$ and let $C_\varphi : \mathcal{Z}_n^\alpha \to \mathcal{Z}_n^\alpha$ be a composition operator induced by the selfmap φ of $\mathbb{D}$. Then C_φ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1^-} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\alpha |\varphi'(z)|^{n+1} = 0.$$

Next, we apply Theorem 3.2 and Theorem 3.6 to improve the result of [1, Theorem 4.3 and Remark 4.4] to the case $0 < \alpha \leq 1$. We first recall the concept of angular derivative and Julia-Caratheodory Theorem.

Let f be a complex-valued function on $\mathbb{D}$ and $w \in \partial\mathbb{D}$. We say that the *angular* (or *non-tangential*) *limit* of f at w is L, denoted by $\angle \lim_{z\to w} f(z) = L$, if $f(z) \to L$ as $z \to w$ through any triangle in $\mathbb{D}$ that has one of its vertices at w. An analytic selfmap $g : \mathbb{D} \to \mathbb{D}$ has an *angular derivative* at a point $w \in \partial\mathbb{D}$ if for some $\eta \in \partial\mathbb{D}$

$$\angle g'(w) = \angle \lim_{z\to w} \frac{\eta - g(z)}{w - z},$$

exits (finitely).

By Julia-Caratheodory Theorem, if the angular derivative of a non-constant analytic selfmap $g : \mathbb{D} \to \mathbb{D}$ exists at some point $w \in \partial\mathbb{D}$, then $\angle g'(w) \neq 0$ [3, Chapter I of Part Six].

Let a selfmap $g : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be continuously differentiable. If $g(w) = \eta \in \partial\mathbb{D}$ for some $w \in \partial\mathbb{D}$, then clearly the angular derivative of g at w exists and $\angle g'(w) = g'(w)$. Therefore, by Julia-Caratheodory Theorem, if g is non-constant then $g'(w) \neq 0$.

Theorem 3.8. *Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$ and let $C_\phi : Lip^n(\overline{\mathbb{D}}, \alpha) \to Lip^n(\overline{\mathbb{D}}, \alpha)$ be a composition operator induced by the non-constant selfmap ϕ of $\overline{\mathbb{D}}$. Then C_ϕ is compact if and only if $\phi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$.*

Proof. We only need to show $\phi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$ whenever C_ϕ is compact and $0 < \alpha < 1$. By Theorem 3.2, if $C_\phi : Lip^n(\overline{\mathbb{D}}, \alpha) \to Lip^n(\overline{\mathbb{D}}, \alpha)$ is compact, then the composition operator $C_\varphi : \mathcal{Z}_n^{1-\alpha} \to \mathcal{Z}_n^{1-\alpha}$ is compact, where $\varphi = R(\phi) = \varphi|_{\mathbb{D}}$. Hence, by Theorem 3.6, the weighted composition operator $(\varphi')^n C_\varphi : \mathcal{B}^{1-\alpha} \to \mathcal{B}^{1-\alpha}$ is compact. Now, by contrary let $\phi(\zeta) = \eta \in \partial\mathbb{D}$ for some $\zeta \in \partial\mathbb{D}$. Then $|\lim_{r\to 1^-} \varphi(r\zeta)| = |\eta| = 1$ and hence by Theorem 2.4, we get $(\phi'(\zeta))^n = 0$. On the other hand, by the discussion before this Theorem, Julia-Caratheodory Theorem implies that $\phi'(\zeta) \neq 0$ which leads to a contradiction and completes the proof of the theorem. □

Remark 3.9. *Choe, Koo and Smith in* [4, Theorem 2.3] *proved the results of Theorem 2.5 and Theorem 3.8 for the spaces* $Lip(\partial\mathbb{D}, \alpha) \cap A(\overline{\mathbb{D}})$ *and* $Lip^n(\partial\mathbb{D}, \alpha) \cap A(\overline{\mathbb{D}})$ *(*$n \in \mathbb{N}$*,* $0 < \alpha \leq 1$*). On the other hand, by* [6, Lemma 4] *for a continuous function* f *on* $\overline{\mathbb{D}}$ *which is analytic on* $\mathbb{D}$*,* $f \in Lip(\partial\mathbb{D}, \alpha)$ *if and only if* $f \in Lip(\overline{\mathbb{D}}, \alpha)$*. Hence,* [4, Theorem 2.3] *along with* [6, Lemma 4] *provides another proof to Theorem 2.5 and Theorem 3.8. Our approach has the advantage to lead us to some new results stated in Theorem 3.6 and Corollary 3.7 besides giving a new proof to* [4, Theorem 2.3].

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