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ON CO-NOETHERIAN DIMENSION OF RINGS

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ABSTRACT. We define and study co-Noetherian dimension of rings for which the injective envelope of simple modules have finite Krull-dimension. This is a Morita invariant dimension that measures how far the ring is from being co-Noetherian. The co-Noetherian dimension of certain rings, including commutative rings, are determined. It is shown that the class $\mathcal{W}_n$ of rings with co-Noetherian dimension $\leq n$ is closed under homomorphic images and finite normalizing extensions, and that for each n there exist rings with co-Noetherian dimension n. The possible relations between Krull and co-Noetherian dimensions are investigated, and examples are provided to show that these dimensions are independent of each other.

1. **Introduction and Preliminaries**

Throughout rings have nonzero identity elements and modules are unitary. Unless otherwise stated, we shall consider right hand properties, e.g., right modules, right Noetherian, etc. The category of all right R-modules is denoted by Mod-R. Dualizing the notion of finitely generated (f.g.), Vamos [12] defined *finitely embedded* (*f.e.*) module M_R by the condition $\mathrm{E}(M) = \oplus_{i=1}^{n}\mathrm{E}(S_i)$ where each S_i is simple. In [8], Jans called a ring R *co-Noetherian* if factors of f.e. modules are f.e., and proved that R is co-Noetherian if and only if all f.e. modules are Artinian. More

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recently the term finitely embedded is replaced by *finitely cogenerated*, and the books [2, 13] contain detailed accounts of this notion. We shall define in §2, the co-Noetherian dimension of a ring R in terms of the Krull dimension of finitely cogenerated R-modules. A useful characterization of co-Noetherian rings states that R is co-Noetherian if and only if for every M_R, the intersection of submodules N with M/N Artinian, is zero; see [8] and [7]. Our first result, Theorem 2.1 (stated below) is a generalization of this fact. Hirano in [7, Theorem 2.2], proved that if R is co-Noetherian and S is a finite normalizing extension of R, then S is also co-Noetherian. This result is generalized in Theorem 2.2 for rings of higher co-Noetherian dimension. We use formal triangular matrix rings to prove in Theorem 2.4 that for a given positive integer n there exists a ring R with co-Noetherian dimension n. Co-Noetherian dimension of certain rings is investigated in Proposition 2.6 and, for a commutative ring R, it is shown in Theorem 2.5 that the co-Noetherian dimension (when it exists) is equal to $\mathrm{Sup}_{P\in X}\{\ \mathrm{K.dim}(R/P)\}$ where X is a suitable subset of $\mathrm{Spec}(R)$. In the final part of the paper the existence of rings with infinite co-Noetherian dimension is shown and, by way of examples, some possible relations which may exist between co-Noetherian dimension and Krull dimension of rings are illustrated. We now state some terminology that will be used in the paper, and the reader is referred to [10] and [11] which deal in depth with Krull dimension. However, note the following:

Standing hypothesis. Throughout the paper we shall only deal with modules that have finite Krull dimension, and the symbol " K.dim " will stand for Krull dimension. When we write K.dim(M), it is tacitly assumed that M dose have Krull dimension which is n for some non-negative integer n.
For each non-negative integer n, and $M \in$Mod-R, let $\tau_n(M_R) = \cap N$, where the intersection runs through the set of R-submodules N of M with $\mathrm{K.dim}(M/N) \leq n$. Let $\mathcal{W}_n$ denote the class of rings R with the property that *all finitely cogenerated R-modules have Krull dimension at most n*. Thus $\mathcal{W}_0$ is the class of co-Noetherian rings.
For M_R let $\tau(M_R) = \cap N$, where the intersection runs through the set of R-submodules N such that M/N has Krull dimension. Moreover, $\mathcal{W}$ will denote the class of rings R such that $\mathrm{E}(S_R)$ has Krull dimension for all simple modules $S \in$ Mod-R. If there is no confusion, we simply write

$\tau_n(M)$ (respectively $\tau(M)$) instead of $\tau_n(M_R)$ (respectively $\tau(M_R)$).

Next, we present some facts on formal triangular matrix rings for later use.

Let A and B be rings, ${}_BM_A$ a bimodule and $R = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ the formal triangular matrix ring. We apply the equivalence of category Mod-R with the category of triples $(X,Y)_f$, and use methods of [4]. Thus if V is a right R-module it is often identified with the triple $(Ve_1, Ve_2)_f$ where $(Ve_1)_A$, $(Ve_2)_B$ are obtained from the idempotents $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Here, f is an A-linear map $: Ve_2 \otimes_B M \to Ve_1$, given by the rule $f(ve_2 \otimes m) = ve_2 \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} = ve_2 \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} e_1$. Hence, it is easily verified (or use the description of maximal right ideals of R in [6]) that if V_R is simple then either $V = (X,0)_0$ or $V = (0,Y)_0$ where X_A and Y_B are simple. We are interested in triples that can represent an injective envelope of the simple module V_R. In fact, if $V = (0,Y)_0$ then $\mathrm{E}(V) = (0, \mathrm{E}(Y))_0$, and if $V = (X,0)_0$, then $\mathrm{E}(V) = [\mathrm{E}(X), \mathrm{Hom}_A(M, \mathrm{E}(X))]_\delta$ where $\delta : \mathrm{Hom}_A(M, \mathrm{E}(X)) \otimes_B M \to \mathrm{E}(X)$ with $\delta(\eta \otimes m)) = \eta(m)$ for all $\eta \in \mathrm{Hom}_A(M, \mathrm{E}(X))$, $m \in M$; see [4] or [6] for details.

Given an arbitrary R-module $L = (C,D)_g$, according to [5], $\mathrm{K.dim}(L_R) = \mathrm{Max}\ \{\mathrm{K.dim}(C_A), \mathrm{K.dim}(D_B)\}$. In particular, we have $\mathrm{K.dim}(R_R) = \mathrm{Max}\ \{\mathrm{K.dim}(A \oplus M)_A, \mathrm{K.dim}(B_B)\}$, and for the simple R-modules V of the form $(0,Y)_0$ we have $\mathrm{K.dim}(\mathrm{E}(V))_R = \mathrm{K.dim}(\mathrm{E}(Y))_B$ while for the simple R-modules V of the form $(X,0)_0$, $\mathrm{K.dim}(\mathrm{E}(V))_R = \mathrm{Max}\ \{\mathrm{K.dim}(\mathrm{E}(X))_A, \mathrm{K.dim}[\mathrm{Hom}_A(M,\mathrm{E}(X)]_B\}$.

2. **Co-Noetherian dimension**

In this section we use the notation introduced in §1. Let $R \in \mathcal{W}$. If there exists an integer $n \geq 1$ such that $R \in \mathcal{W}_n \backslash \mathcal{W}_{n-1}$, we say that R has *co-Noetherian dimension* n, and write *co-N.dim*$(R) = n$. If on the other hand, $R \notin \mathcal{W}_m$ for all m, we say that R has infinite co-Noetherian dimension, and write co-N.dim$(R) = \infty$. Finally, we set co-N.dim$(R) = 0$

whenever R is co-Noetherian. We begin with the following result that characterizes when a ring R belongs to $\mathcal{W}_n$ or to $\mathcal{W}$.

Theorem 2.1. *If $n \geq 0$ and R is a ring, then $R \in \mathcal{W}_n$ (respectively $R \in \mathcal{W}$) if and only if for all $M \in$ Mod-R, $\tau_n(M) = 0$ (respectively $\tau(M) = 0$).*

Proof. First suppose that $\tau_n(M) = 0$ for all $M \in$ Mod-R. Let M be an f.e. R-module. Because $\tau_n(M) = 0$ and M is f.e., there exist submodules $N_i (i = 1, \cdots, t)$ such that $\cap_{i=1}^t N_i = 0$ and the Krull dimension of each M/N_i is at most n. It follows that M embedding in $\bigoplus_{i=1}^t (M/N_i)$ has Krull dimension at most n, as desired. For the converse, let $\{S_i \mid i \in I\}$ be a complete set of non-isomorphic simple R-modules. The module $C = \oplus_{i \in I} \mathrm{E}(S_i)$ is a co-generator for Mod-R, hence if $0 \neq M_R$ is given, there are a set Λ and an embedding $f : M \to \prod_{\alpha \in \Lambda} C_\alpha$ where $C_\alpha = C$ for all $\alpha \in \Lambda$. For $i \in I$, let $\pi_{(\alpha,i)}$ be the projection of $\prod_{\alpha \in \Lambda} C_\alpha$ into the summand $\mathrm{E}(S_i)$ of C_α. By assumption that the Krull dimension of each $\mathrm{E}(S_i)$ is at most n, we see that $\mathrm{K.dim}(M/\ker(\pi_{(\alpha,i)} f)) \leq n$. Clearly, $\tau_n(M) \subseteq \cap_{\alpha,i} \ker(\pi_{(\alpha,i)} f) = 0$. A slight modification of the above argument will prove the other statement. □

Theorem 2.2. *All classes $\mathcal{W}_n$, as well as $\mathcal{W}$, are closed under Morita equivalence, homomorphic images, and finite normalizing extensions.*

Proof. That $\mathcal{W}$ and each $\mathcal{W}_n$ is closed under Morita equivalence follows from the standard techniques of Morita theory since "simple", "essential monomorphism", " injective" and "Krull dimension" are all Morita invariant properties. Next we prove that $\mathcal{W}_n$ $(n \geq 1)$ is closed under homomorphic images. Suppose that $R \in \mathcal{W}_n$, I is a proper ideal of R, S is a simple R/I-module and $E := \mathrm{E}(S_R)$. It is well known that the set $E' := \{e \in E \mid eI = 0\}$ is the injective envelope of $S_{R/I}$. Hence $\mathrm{K.dim}(E'_{R/I}) = \mathrm{K.dim}(E'_R) \leq \mathrm{K.dim}(E_R) \leq n$. This shows that $R/I \in \mathcal{W}_n$. We now prove that if $R \in \mathcal{W}_n$ and T is a finite normalizing extension of R, then $T \in \mathcal{W}_n$. In view of Theorem 2.1, we need to show that $\tau_n(M_T) = 0$ whenever M is a nonzero T-module. By assumption $T = \sum_{i=1}^t a_i R$ with a_i normalizing R. If N is an R-submodule of M, then according to [10]; p345 $Na_i^{-1} = \{m \in M \mid ma_i \in N\}$ is an R-submodule of M and the group monomorphism $M/Na_i^{-1} \to M/N$ given by $m + Na_i^{-1} \mapsto ma_i + N$ induces a lattice embedding of the R-submodules of M/Na_i^{-1} into the R-submodules of M/N. If

$b(N) = \cap_{i=1}^{t} Na_i^{-1}$, then $b(N)$ is the largest T-submodule of M contained in N and there is an R-monomorphism $M/b(N) \hookrightarrow \oplus_i(M/Na_i^{-1})$. Consequently, if $\mathrm{K.dim}(M/N)_R \leq n$ we deduce that $\mathrm{K.dim}[M/b(N)]_R \leq n$, hence $\mathrm{K.dim}[M/b(N)]_T \leq n$. Now by hypothesis and Theorem 1, $\tau_n(M_R) = 0$, thus it follows that $\tau_n(M_T) = 0$. The proof of the implication "$R \in \mathcal{W} \Rightarrow T \in \mathcal{W}$ " is now evident. □

Immediate consequences are given in the following.

Corollary 2.3. *(i) Let A and B be Morita equivalent rings under the category equivalence α : Mod-A → Mod-B. If $M \in$ Mod-A and $n \geq 1$, then $\alpha(\tau_n(M)) \simeq \tau_n(\alpha(M))$ and $\alpha(\tau(M)) \simeq \tau(\alpha(M))$. Furthermore co-N.dim($A$) = co-N.dim($B$).*
(ii) If $R \in \mathcal{W}$, I is a proper ideal of R and T is a finite normalizing extension of R, then co-N.dim(R) $\geq$ Max {co-N.dim(R/I), co-N.dim(T)}.

Theorem 2.4. *For each integer $n \geq 1$, there exists a ring with co-Noetherian dimension n*

Proof. Given a positive integer n, let A be any commutative Noetherian local ring with $\mathrm{K.dim}(A) = n$. If P is the unique maximal ideal of A, let $M = \mathrm{E}(A/P)$. It is well-known that M_A is Artinian. Put $B = \mathrm{End}(M_A)$, which by a classical result of Matlis [9] is isomorphic to $\hat{A}$, the P-adic completion of A, and consequently by [3, Proposition 10.16], B is also a local ring. Now consider the formal triangular matrix ring $R = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$. Up to isomorphisms, the only simple modules over R are $V_1 = (A/P, 0)_0$ and $V_2 = (0, B/\mathrm{J}(B))_0$. Thus we have the following descriptions: $\mathrm{E}(V_1) = (\mathrm{E}(A/P), \mathrm{Hom}_A(M, \mathrm{E}(A/P))_\delta$, and $\mathrm{E}(V_2) = (0, \mathrm{E}(B/\mathrm{J}(B))_0$. It follows that $\mathrm{K.dim}(R_R) = n = \mathrm{Sup}\{ \mathrm{K.dim}(\mathrm{E}(V_R)) \mid V_R$ is simple$\}$. Thus we deduce that co-N.dim$(R) = n$. □

Our next result determines the co-Noetherian dimension of commutative rings when they exist. For any ring R, let
$\mathrm{Ass}_E(R) = \{ P \mid P \in \mathrm{Ass}(N)$ where N is a sub-factor of $\mathrm{E}(S)$ for some simple $S_R\}$.

Theorem 2.5. *Co-N.dim(R) = Sup{ K.dim(R/P) where $P \in Ass_E(R)$} for every commutative ring R in $\mathcal{W}$.*

Proof. Let co-N.dim$(R) = n$. Note that if $P \in$Ass(N) for some N_R, then there exists a cyclic submodule C of N such that $P = \mathrm{ann}_R(C)$. It follows that R/P embeds in N. Thus K.dim$(N) \geq$ K.dim(R/P) (if they exist). Now since $R \in \mathcal{W}$, Sup{K.dim (E(S)) $\mid S_R$ is simple} $= n$. Hence $n \geq$ Sup{K.dim$(R/P) \mid P \in \mathrm{Ass}_E(R)$}. On the other hand, if S is a simple R-module, then by a known result, K.dim(E(S)) = Sup{K.dim$(C) \mid C$ is a critical R-module contained in a sub-factor of E(S)}; see [11, 4.19] or Corollary 3.2.12. Since for every critical R-module C, K.dim$(xR) =$ K.dim(C) for all $0 \neq x \in C$, we can conclude that K.dim(E(S)) = Sup{ K.dim$(C) \mid C$ is a critical cyclic R-module contained in a sub-factor of E(S)}. The result is now clear by the well known fact that critical cyclic R-modules have the form R/P for some $P \in$ Spec(R); see for example [1, Lemma 3.5]. □

We already know that any commutative Noetherian ring is necessarily co-Noe-therian. Thus there are plenty of commutative rings R with co-N.dim$(R) <$ K.dim(R). Our next result provides a class of rings that are not necessarily commutative, yet for them the strict inequality holds.

Proposition 2.6. (*i*) *Let R be a semiprime right Goldie ring. If R has non-zero Krull dimension, then either $R \notin \mathcal{W}$ or co-N.dim$(R) <$ K.dim(R).*
(*ii*) *Let R be a right semi-Artinian ring. Then either $R \notin \mathcal{W}$ or co-N.dim$(R) = 0$.*

Proof. Let $R \in \mathcal{W}$ and S be a simple R-module.
(i) Let K.dim$(R) = n \geq 1$. By hypothesis, K.dim(E(S)) and, hence K.dim(E$(S)/S$) exist. Let $x \in$ E$(S)/S$. Thus $xR \simeq R/A$ for some essential right ideal A of R. Because R is semiprime right Goldie, A contains a regular element c of R. Thus we have K.dim$(R/A) \leq$ K.dim$(R/cR) < n$ by [10, Lemma 6.3.9]. It follows that K.dim(E$(S)/S$) $\leq n-1$ by [10, Lemma 6.2.17]. Consequently, one obtains
co-N.dim(R) = Sup{ K.dim(E(S_R)) $\mid S_R$ is simple} $\leq n-1 < n$.
(ii) Let B = the sum of all Artinian submodules of E(S). Then since E(S) has Krull dimension so dose B. It follows that B is an Artinian R-module. Now if E$(S)/B$ is non-zero, it contains a simple submodule L/B by our assumption on R. But then L is an Artinian submodule of E(S) and so $L \subseteq B$, a contradiction. Therefore, E$(S) = B$ is an Artinian R-module, proving that co-N.dim$(R) = 0$. □

Corollary 2.7. *Let R be a ring with Krull-dimension and P be a prime ideal of R. If co-N.dim(R/P) = K.dim(R/P), then P is a maximal ideal of R.*

Proof. By [10, Proposition 6.3.5], R/P is a prime right Goldie ring. Hence, K.dim(R/P) must be zero by Proposition 2.6(i). It follows that R/P is a right Artinian ring and hence P is a maximal ideal of R. $\square$

We note that the converse to the above Corollary is not true in general. For instance, if $R = K[x, D]$ is the Cozzens's example where K is a universal field with derivation D, then it is well known that R is a simple principal right (and left) ideal domain with a unique (up to isomorphism) simple injective R-module and K.dim$(R) = 1 \neq$ co-N.dim(R).

3. Examples and concluding remarks

This section is devoted to explore possible relations between Krull dimension and co-Noetherian dimension of rings. If R is a ring, provided that for R both Krull dimension and co-Noetherian dimension exist, we have co-N.dim$(R) \leq$ K.dim(R). The proof of Theorem 2.4 shows that equality can occur.

Example 3.1. If p is a fixed prime number, the ring $\begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_{p^\infty} & \mathbb{Z} \end{bmatrix}$ is not of the type used in the proof of Theorem 2.4, yet for it the Krull dimension coincides with the co-Noetherian dimension and this common value is 1. This example also shows that the semiprime condition in Proposition 2.6(i) cannot be deleted.

The following examples show that in general co-Noetherian dimension and Krull dimension of a ring are independent of each other.

Example 3.2. There is an Artinian ring with a simple module whose injective envelope has no Krull dimension: Let $B \subseteq A$ denote Cohn's division rings such that A_B has finite dimension while the left B-module A has infinite dimension. Then the ring $R = \begin{bmatrix} A & 0 \\ A & B \end{bmatrix}$ is right Artinian and the injective envelope of the simple R-module $(A,0)_0$ has no Krull dimension. Thus $R \notin W$, so co-N.dim(R) is not defined.

Example 3.3. It is well-known that the first Weyl algebra $A = \mathrm{A}_1(k)$ over a field of characteristic zero is a simple Noetherian domain of Krull dimension 1, and that it is not co-Noetherian, as shown in [7, Example 2.1]. Thus by Proposition 2.6(i), for A even the co-Noetherian dimension cannot exist. This fact in particular shows that in general, co-Noetherian dimension is not well behaved under Ore ring extension. Now set $R = \begin{bmatrix} A & 0 \\ M & k \end{bmatrix}$ where M is an infinite direct sum of any non-zero A-module. It is clear that R does not have Krull-dimension. Moreover, as A is a homomorphic image of R, by Corollary 2.3, we deduce that R does not have co-Noetherian dimension.

Example 3.4. We construct a co-Noetherian ring that has no Krull dimension. Let $A = \mathbb{C}[x]$, and consider an infinite sequence of pairwise non-isomorphic simple A-modules $\{S_i\}_{i\geq 1}$. Put $M = \oplus_i S_i$ and consider the ring $R = \begin{bmatrix} A & 0 \\ M & \mathbb{C} \end{bmatrix}$. Suppose $V = (S',0)_0$ is a simple R-module. Then $\mathrm{E}(V) = (\mathrm{E}(S'),\mathrm{Hom}_A(M,\mathrm{E}(S')))_\delta$. We have $T := \mathrm{Hom}_A(M,\ \mathrm{E}(S')) \simeq \prod_{i\geq 1} \mathrm{Hom}_A(S_i,\mathrm{E}(S'))$. If $S_i \not\simeq S'$ for all i, then $T = 0$. Suppose $S_j \simeq S'$ for some j. In this case, we have:

$$T \simeq \mathrm{Hom}_A(S_j, \mathrm{E}(S_j)) \hookrightarrow \mathrm{Hom}_A(S_j, S_j) = \mathbb{C}$$

Thus in any case T is a finite dimensional $\mathbb{C}$-space. Since $\mathrm{E}(S')$ is an Artinian A-module, we deduce that $\mathrm{E}(V)$ is an Artinian R-module. All other simple R-modules are of the form $(0,\mathbb{C})_0$ which is Artinian and injective. Therefore R is co-Noetherian.

Remark 3.5. (i) It is possible to modify the definition of co-Noetherian dimension by restricting to those simple modules whose injective envelopes do have Krull-dimensions (not necessarily finite). Then for any ring this new dimension exists, it is less than or equal to the global Krull-dimension of the ring in the sense of Albu and Smith [11], and results similar to Theorems 2.1 and 2.2 hold. But when this new dimension is zero (as it is the case for the first Weyl algebra over a field with zero characteristic) the ring is not necessarily co-Noetherian. Hence it cannot measure distance from being co-Noetherian. On the other hand, our construction of rings with co-Noetherian dimension n cannot be carried over to the case of infinite ordinal numbers. These considerations indicate the reason for our Standing Hypothesis. But allowing modules with arbitrary Krull dimensions the following problem is naturally imposed:

Open Problem. Given an arbitrary ordinal number γ, does there exist a ring R with a simple module S such that $\mathrm{E}(S)$ has Krull-dimension γ?

(ii) By using " Noetherian dimension ", it is possible to define " co-Artinian dimension " of a ring and prove results analogous to Theorems 2.1 and 2.2 However, one must first prove the existence of rings with arbitrary co-Artinian dimensions.

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