Bulletin of the Iranian Mathematical Society Vol. 38 *No. 1* (*2012*), *pp 123-130.*

ON TOPOLOGICAL TRANSITIVE MAPS ON OPERATOR ALGEBRAS

H. REZAEI

Communicated by Omid Ali Shehni Karamzadeh

Abstract. We consider the transitive linear maps on the operator algebra $\mathcal{B}(X)$ for a separable Banach space X. We show if a bounded linear map is norm transitive on $\mathcal{B}(X)$, then it must be hypercyclic with strong operator topology. Also we provide a SOT-transitive linear map without being hypercyclic in the strong operator topology.

1. **Introduction**

Let X be a Hausdorff locally convex space and $T : X \to X$ a continuous linear map. We say that T is transitive if for each pair of nonempty open subsets U; V of X there is $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. If the underlying space is a separable Baire space, transitivity is equivalent to hypercyclicity. A continuous linear map T acting on a separable Frechet space X is called hypercyclic if there exists a vector $f \in X$ such that its orbit; $\mathrm{orb}(T, f) = \{T^n f : n \geq 0\}$ is dense in X. Topological transitivity was introduced by G.D. Birkhoff [3] already in 1920 for the non-linear continuous function and it plays an important role in many definitions of chaos, in particular in [7]. For the space of all entire functions, Birkhoff [3] also proved that the translation operator $Tf(z) = f(z + 1)$ is hypercyclic, and MacLane [9] showed that the differentiation operator

MSC(2010): Primary: 47A16; Secondary: 47B48.
Keywords: Hypercyclic operator, transitive map, strong operator topology.
Received: 17 April 2010, Accepted: 9 September 2010.

$T(f) = f'$ is hypercyclic. When X is a separable Frechet space, Gethner and Shapiro [8] showed a useful sufficient condition for a continuous linear mapping $T : X \to X$ to be hypercyclic:

Hypercyclicity Criterion. Let T be a bounded operator on X and there exist a sequence (n_k), two dense subsets Y, Z of X and functions $S_k : Z \to X$ such that
(i) $T^{n_k} y \to 0$ for all $y \in Y$,
(ii) $S_k z \to 0$ and $T^{n_k} S_k z \to z$ for all $z \in Z$.
Then T is hypercyclic.

Good source of background information on transitive and hypercyclic operators includes [1]. While the existence of an element with a dense orbit implies that X must be separable, the transitivity of $T : X \to X$ does not require that the space X is separable. On the other hand, since Baire category theorem is essential in most of the fundamental results concerning hypercyclicity, the space X on which the operators are defined is usually assumed to be a Frechet space. However, several recent important results considered the existence of hypercyclic operators on locally convex spaces which are not metrizable and not Baire, see [4, 5], and [6].

If we look at the space of all bounded linear operators on X, i.e., $\mathcal{B}(X)$, the first problem we face is that it is usually non-separable and non-complete with the operator-norm topology. This forces us to consider weaker topologies on $\mathcal{B}(X)$. That was the reason why Chan [5], Chan and Taylor [6] and Bonet et. al [4] studied the hypercyclicity of left multiplication operators on the algebra $\mathcal{B}(X)$ endowed with the strong operator topology (SOT), i.e., the locally convex topology determined by the seminorms $x \to \|T(x)\|, T \in \mathcal{B}(X)$.

In the present paper we consider the transitivity of bounded linear map on $\mathcal{B}(X)$ with the norm and strong operator topology. In Corollary 2.3 below, we show each norm transitive linear map is hypercyclic with the SOT. In particular, the left multiplication operator L_T is norm transitive if and only if it is hypercyclic with the SOT. Finally, we provide an example of transitive linear map with the SOT without being hypercyclic with this topology.

2. **Main result**

In all that follows X will be a separable Banach space, and $\mathcal{B}(X)$ denotes the algebra of all bounded operator on X. We use $\mathcal{B}_N = \mathcal{B}_N(X)$ to denote the open N-ball of $\mathcal{B}(X)$, that is the set of all operators $T \in \mathcal{B}(X)$ with $\|T\| < N$ and $\bar{\mathcal{B}}_N = \bar{\mathcal{B}}_N(X)$ is used for the close N-ball. We agree that only those topological terms with the prefix "SOT" refer to the strong operator topology; otherwise, they refer to the operator norm topology.

Definition 2.1. *Let $L : \mathcal{B}(X) \to \mathcal{B}(X)$ be a bounded linear mapping and $D \subseteq \mathcal{B}(X)$. We say L is transitive (SOT-transitive) on D whenever for each pair (U, V) of nonempty open (SOT-open) sets in $\mathcal{B}(X)$ that $U \cap D \neq \emptyset$ and $V \cap D \neq \emptyset$, there exists some integer $r \geq 0$ such that*

$$L^r(U \cap D) \cap V \cap D \neq \emptyset.$$

When $D = \mathcal{B}(X)$, L is simply called transitive (SOT-transitive).

Theorem 2.2. *Suppose that the bounded linear mapping $L : \mathcal{B}(X) \to \mathcal{B}(X)$ is transitive on $\mathcal{B}_1$, then L is SOT-hypercyclic.*

Proof. Assume that U and V are two open sets in $\mathcal{B}(X)$. Choose an integer $m \geq 1$ in such a way that both $m^{-1}U \cap \mathcal{B}_1$ and $m^{-1}V \cap \mathcal{B}_1$ are nonempty. By transitivity of L on $\mathcal{B}_1$, we find that

$$L^r(m^{-1}U \cap \mathcal{B}_1) \cap (m^{-1}V \cap \mathcal{B}_1) \neq \emptyset$$

for some integer $r \geq 1$, consequently $L^r U \cap V \neq \emptyset$. It follows that L is transitive.
Let $E(X)$ be a countable SOT-dense subset of $\mathcal{B}(X)$. The bounded subsets of $\mathcal{B}(X)$ are SOT-metrizable and so are of first countable in the SOT. For $T \in E(X)$ and positive integer $i \geq 1$, let

$$\{V_{i,j}(T) \cap \bar{\mathcal{B}}_i(X)\}_{j \geq 1}$$

be a countable basis of SOT-neighborhoods of T in $\bar{\mathcal{B}}_i$; where $V_{i,j}(T)$ is a SOT-open set in $\mathcal{B}(X)$ containing T. Consider the index set

$$I_{i,j}(T) = \{r \in \mathbb{N} : L^{-r}V_{i,j}(T) \cap \mathcal{B}_1 \neq \emptyset\} \quad (T \in E(X), i, j \in \mathbb{N})$$

which is nonempty by the transitivity of L. Then for $r \in I_{i,j}(T)$, each of the sets

$$G(T, i, j, r) = \bigcup_{n \in \mathbb{N}} L^{-n}(L^{-r}V_{i,j}(T) \cap \mathcal{B}_1)$$

are open in $\mathcal{B}(X)$ and dense in $\bar{\mathcal{B}}_1$. To prove the density, let $T \in E(X)$, $i, j \in \mathbb{N}$, $r \in I_{i,j}(T)$ and let $U \cap \bar{\mathcal{B}}_1$ be a nonempty open set in $\bar{\mathcal{B}}_1$. Then

$$U \cap \mathcal{B}_1 \neq \emptyset, \quad L^{-r}V_{i,j}(T) \cap \mathcal{B}_1 \neq \emptyset \text{ (because } r \in I_{i,j}(T))$$

and $L^{-r}V_{i,j}(T)$ is SOT-open set in $B(X)$. By transitivity of L on $\mathcal{B}_1$,

$$L^{-n}(L^{-r}V_{i,j}(T) \cap \mathcal{B}_1) \cap U \cap \mathcal{B}_1 \neq \emptyset$$

for some $n \geq 1$. This implies that $G(T, i, j, r)$ is norm dense in $\bar{\mathcal{B}}_1$. Using the Baire Category Theorem for $\bar{\mathcal{B}}_1$, we see the set

$$H_{\text{sot}}(L) = \bigcap_{T \in E} \bigcap_{i,j \in \mathbb{N}} \bigcap_{r \in I_{i,j}(T)} G(T, i, j, r)$$

is also dense in $\bar{\mathcal{B}}_1$. Now, we claim that every element of $H_{\text{sot}}(L)$ is SOT-hypercyclic for L. For this let $S \in H_{\text{sot}}(L)$ and let $\mathbf{V}$ be an arbitrary SOT-open set in $\mathcal{B}(X)$. Then there is a SOT-neighborhood $V(T)$ of some $T \in E(X)$ contained in $\mathbf{V}$. By transitivity of L, $L^r\mathcal{B}_1 \cap V(T) \neq \emptyset$ for some integer $r \geq 1$. Choose a positive integer $i > \|L\|^r$ in such a way that the set $L^r\mathcal{B}_1 \cap V(T) \cap \mathcal{B}_i$ and so $L^{-r}(V(T) \cap \mathcal{B}_i) \cap \mathcal{B}_1$ are nonempty. The first countability of $\bar{\mathcal{B}}_i$ in SOT implies that for some $j \in \mathbb{N}$,

$$L^{-r}(V_{i,j}(T) \cap \bar{\mathcal{B}}_i) \cap \mathcal{B}_1 \neq \emptyset.$$

This yields that $L^{-r}V_{i,j}(T) \cap \mathcal{B}_1$ is also nonempty and hence $r \in I_{i,j}(T)$. Since $S \in G(T, i, j, r)$, there is some $n \in \mathbb{N}$, such that

$$L^{n+r}S \in V_{i,j}(T) \text{ and } L^nS \in \mathcal{B}_1.$$

But $\|L^{r+n}S\| \leq \|L\|^r < i$ and $L^{r+n}S \in V_{i,j}(T) \cap \bar{\mathcal{B}}_i$ which is a subset of $V(T) \cap \bar{\mathcal{B}}_i$. Thus $L^{r+n}S \in \mathbf{V}$. Therefore S is a SOT-hypercyclic vector for L and L is SOT-hypercyclic. □

Corollary 2.3. *If the bounded linear mapping L on $\mathcal{B}(X)$ is transitive then it is SOT-hypercyclic.*

It is shown by Bonet et. al in [4] that left multiplication operators $L_T : B(X) \to B(X)$ by $L_T(S) = TS$ with the SOT are hypercyclic provided that T satisfies the Hypercyclicity Criterion. The following theorem characterizes SOT-transitive left multiplication operators on $B(X)$.

Theorem 2.4. *For $T \in \mathcal{B}(X)$, the bounded linear mapping L_T is SOT-transitive on $\mathcal{B}(X)$ if and only if $T \oplus T$ is transitive on $X \oplus X$.*

Proof. Assume that $T \oplus T$ is transitive on $X \oplus X$. Consider two non-empty SOT-open subsets

$$U_i = \{S \in \mathcal{B}(X) : \|(S - S_1)(f_{i,j})\| < \varepsilon_i, \text{ for } 1 \leq j \leq m\} \quad (i = 1, 2).$$

Let $E = \{f_{i,j} : i = 1, 2 \text{ and } j = 1, 2, \ldots, m\}$. Pick a linearly independent subset $\{f_1, f_2, \ldots, f_k\}$ of E such that $E \subseteq \text{span}\{f_j : 1 \leq j \leq k\}$. For any $i = 1, 2$ let $f_{i,j} = \lambda_{i,1} f_1 + \cdots + \lambda_{i,k} f_k$ for some scalars $\lambda_{i,1}, \ldots, \lambda_{i,k}$ that are not all equal to zero. If

$$\delta_i = \min\{\varepsilon_i, \frac{\varepsilon_i}{n \max_j\{|\lambda_{i,j}|\}}\} \quad (i = 1, 2)$$

and

$$V_i = \{S \in \mathcal{B}(X) : \|(S - S_i)(f_j)\| < \delta_i \text{ for } 1 \leq j \leq k\}$$

for $i = 1, 2$, then $V_i \subseteq U_i$. By an application of Hahn-Banach Theorem there are bounded linear functionals $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that $\varphi_j(f_i) = 0$ for $i \neq j$, $\varphi_j(f_j) = 1$ and $\|f_j\| = d^{-1}(f_j, \text{ span } \{f_i : i \neq j\})$. Since $T \oplus T$ is transitive, it is hypercyclic and by [2, Theorem 2.3], T satisfies the Hypercyclicity Criterion on X. Hence, $\bigoplus_{j=1}^{k} T$ also satisfies the criterion which implies the hypercyclicity and transitivity of $\bigoplus_{j=1}^{k} T$ on $\bigoplus_{j=1}^{k} X$. Thus there exist a finite subset $\{z_j : j = 1, 2, \ldots, n\}$ of X and positive integer n satisfying

$$\|z_j - S_1(f_j)\| < \delta_1 \text{ and } \|T^n z_j - S_2(f_j)\| < \delta_2.$$

Let $S(v) = \sum_{j=1}^{k} \varphi_j(v) z_j$, then $S(f_j) = z_j$ and $L_T^n S(f_j) = T^n(z_j)$. It follows that $S \in V_1$ and $L_T^n S \in V_2$ and hence $L_T^n(U_1) \cap U_2 \neq \emptyset$. For the converse, let L_T be SOT-transitive on $\mathcal{B}(X)$. Fix two vectors $a, b \in X$ and define the linear mapping $\Lambda : \mathcal{B}(X) \to X \oplus X$

$$\Lambda(S) = Sa \oplus Sb \quad (S \in \mathcal{B}(X))$$

Hence, Λ is bounded and $\Lambda L_T = (T \oplus T)\Lambda$. Now for two non-empty open subsets $U = U_1 \otimes U_2$ and $V = V_1 \otimes V_2$ of $X \oplus X$, we have

$$\begin{aligned} \Lambda^{-1}(U \cap (T \oplus T)^{-n}(V)) &= \Lambda^{-1}(U) \cap ((T \oplus T)^n \Lambda)^{-1}(V) \\ &= \Lambda^{-1}(U) \cap (\Lambda L_T^n)^{-1}(V) \\ &= \Lambda^{-1}(U) \cap L_T^{-n}(\Lambda^{-1} V). \end{aligned}$$

Since L_T is SOT-transitive on $\mathcal{B}(X)$, by the last relation, we deduce that $T \oplus T$ is transitive on $X \oplus X$. $\square$

Proposition 2.5. *For any $T \in \mathcal{B}(X)$ the continuous linear mapping L_T is transitive if and only if it is SOT-hypercyclic.*

Proof. By Theorem 2.2, we need to prove only one implication of proposition. Let L_T be SOT-hypercyclic on $\mathcal{B}(X)$. Fix a vector $f \in X^*$ and define $\Lambda : X \oplus X \to \mathcal{B}(X)$ by

$$\Lambda(x \oplus y) = (x + y) \otimes f, \quad (x, y \in X)$$

where $(x + y) \otimes f$ is defined by $((x + y) \otimes f)(z) = f(z)(x + y)$. Then Λ is a bounded linear map, $\Lambda(T \oplus T) = L_T\Lambda$ and for two open sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{B}(X)$,

$$\begin{aligned} \Lambda^{-1}(\mathcal{U} \cap L_T^{-n}(\mathcal{V})) &= \Lambda^{-1}(\mathcal{U}) \cap (L_T^n\Lambda)^{-1}(\mathcal{V}) \\ &= \Lambda^{-1}(\mathcal{U}) \cap (\Lambda(T \oplus T)^n)^{-1}(\mathcal{V}) \\ &= \Lambda^{-1}(\mathcal{U}) \cap (T \oplus T)^{-n}(\Lambda^{-1}\mathcal{V}). \end{aligned}$$

Since L_T is SOT-hypercyclic, hence is SOT-transitive and by the last theorem $T \oplus T$ is transitive on $X \oplus X$. Now from the above relations, we find that L_T is norm transitive. □

The following example provides a SOT-transitive linear mapping on $\mathcal{B}(X)$ which is not SOT-hypercyclic.

For any Hilbert space H and vectors x, y in H, define the rank one operator $x \otimes y(h) =< h, y > x$. It is easy to check for $T \in \mathcal{B}(H)$, $T(x \otimes y) = Tx \otimes y$ and $(x \otimes y)T^* = x \otimes Ty$.

Example 2.6. *Let $\{e_n : n \in \mathbb{Z}\}$ be an orthonormal basis of $H = \ell^2(\mathbb{Z})$ and let $S \in \ell^2(\mathbb{Z})$ be the bilateral forward shift, which is defined by $Se_n = e_{n+1}$ for all integers n. The Hilbert space adjoint S^* is the bilateral backward shift, which satisfies $S^*(e_n) = e_{n-1}$ for all integers n. Define a linear mapping $L : \mathcal{B}(H) \to \mathcal{B}(H)$ by $L(T) = STS^*$ for all T in $\mathcal{B}(H)$. Since $\|L(T)\| \leq \|S\|\|T\|\|S^*\| \leq \|T\|$, the linear mapping L is bounded and $\|L\| \leq 1$. In addition, we define a linear mapping $A : \mathcal{B}(H) \to \mathcal{B}(H)$ by $A(T) = S^*TS$, giving the identity $LA = I$. Furthermore, let $\{x_k : k \in \mathbb{Z}\}$ be a dense subset of H and $D(H)$ be the sets of all operators T in $\mathcal{B}(H)$ such that $T = \sum_{k=-n}^{n} x_k \otimes e_k$. The set $D(H)$ is a countable SOT-dense subset of $\mathcal{B}(H)$. Observe that, if T is an operator in $D(H)$,*

$$L^nT = \sum_{k=-m}^{m} S^n x_k \otimes S^n e_k = \sum_{k=-m}^{m} S^n x_k \otimes e_{k+n}$$

$$A^nT = \sum_{k=-m}^{m} S^{*n}x_k \otimes S^{*n}e_k = \sum_{k=-m}^{m} S^{*n}x_k \otimes e_{k-n}$$

and $L^nA^n(T) = T$. *This proves that,*

$$L^nT \to 0 \text{ and } A^nT \to 0 \text{ in SOT.}$$

Now let $\mathcal{U}$ *and* $\mathcal{V}$ *be two SOT-open set in* $\mathcal{B}(H)$. *Choose* $T_1 \in \mathcal{U} \cap D(H)$, $T_2 \in \mathcal{V} \cap D(H)$ *and let* $T_n = T_1 + A^n(T_2)$. *Then* $T_n \to T_1$ *and* $L^n(T_n) \to T_2$ *in SOT. It gives that* $T_n \in \mathcal{U}$ *and* $L^nT_n \in \mathcal{V}$ *for n large enough, that is,*

$$L^n\mathcal{U} \cap \mathcal{V} \neq \emptyset$$

for some integer $n \geq 1$. *Therefore L is SOT-transitive. On the other hand,* $\|L\| \leq 1$ *and so for each* $T \in \mathcal{B}(H)$, $orb(L, T)$ *is a norm bounded subset of* $\mathcal{B}(H)$ *which can not be SOT-dense in* $\mathcal{B}(H)$. *Therefore L is not SOT-hypercyclic.*

Acknowledgments

The author is very grateful to the referee for many helpful suggestions and interesting comments about the paper.

References

[1] F. Bayart and E. Matheron, Dynamics of Linear Operators, Cambridge University Press, Cambridge, 2009.

[2] J. Bes and A. Peris. Hereditarily hypercyclic operators, *J. Funct. Anal.* **167** (1999), 94–112.

[3] G. D. Birkhoff, Demonstration d'un theoreme elementaire sur les fonctions entieres, *C.R. Acad. Sci. Paris* **189** (1929), 473–475.

[4] J. Bonet, F. Martínez-Giménez and A. Peris, Universal and chaotic multipliers on spaces of operators, *J. Math. Anal. Appl.* **297**, (2004) 599–611.

[5] K. C. Chan, Hypercyclicity of the operator algebra for a separable Hilbert space, *J.Operator Theory.* **42** (1999), 231–244.

[6] K. C. Chan and R. D. Taylor, Hypercyclic subspaces of a Banach space, *Integral Equations Operator Theory* **41** (2001), 381–388.

[7] R.L. Devaney, An Introduction to Chaotic Dynamical Systems, Addison-Wesley, Reedwood City, 1989.

[8] R. M. Gethner and J. H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, *Proc. Amer. Math. Soc.* **100** (1987), 281–288.

[9] G.R. MacLane, Sequences of derivatives and normal families, *J. Analyse Math.* **2** (1952), 72–87.

Hamid Rezaei
Department of Mathematics, University of Yasouj, P.O. Box 75918-74831,
Yasouj, Iran
Email: rezaei@mail.yu.ac.ir