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APPLICATION OF FUNDAMENTAL RELATIONS ON n-ARY POLYGROUPS

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ABSTRACT. The class of n-ary polygroups is a certain subclass of n-ary hypergroups, a generalization of Dörnte n-ary groups and a generalization of polygroups. The β^*-relation and the γ^*-relation are the smallest equivalence relations on an n-ary polygroup P such that P/β^* and P/γ^* are an n-ary group and a commutative n-ary group, respectively. We use the β^*-relation and the γ^*-relation on a given n-ary polygroup and obtain some new results and some fundamental theorems in this respect. In particular, we prove that the relation γ is transitive on an n-ary polygroup.

1. **Introduction**

The concept of a hypergroup which is a generalization of the concept of a group, was first introduced by Marty at the 8^{th} International Congress of Scandinavian Mathematicians [20]. Applications of hypergroups have mainly appeared in special subclasses. For example, polygroups which form a certain subclass of hypergroups are used to study color algebra [1, 2].

The fundamental relation β^* which is the transitive closure of the relation β was introduced on hypergroups by Koskas [17] and was studied

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mainly by Corsini [3] and Vougiouklis [21]. The commutative fundamental equivalence relation γ^* which is the transitive closure of the relation γ, was studied on hypergroups by Freni [14, 15], also see [9]. Applications of fundamental relations β^* and γ^* on hypergroups and polygroups were used by Corsini and Leoreanu [4, 5], Davvaz [6, 7, 8] and Vougiouklis [21, 22].

On the other hand, the first paper about the concept of an n-ary group has been published about 80 years ago by Dörnte in [13], which is a natural generalization of the notion of group. Recently, the notion of n-ary hypergroups is defined and considered by Davvaz and Vougiouklis in [10], as a generalization of hypergroups in the sense of Marty and a generalization of Dörnte n-ary groups. Davvaz and Vougiouklis [10] introduced the relation β on an n-ary semihypergroup H such that β^* is the smallest equivalence relation and the quotient $(H/\beta^*, f/\beta^*)$ is a fundamental n-ary semigroup, see also [11, 18]. Leoreanu-Fotea and Davvaz [19] proved that the relation β is transitive. Davvaz et. al. [12] defined the relation γ on an n-ary semihypergroup and studied the relaton γ^* as the smallest equivalence relation such that the quotient $(H/\gamma^*, f/\gamma^*)$ is a commutative n-ary semigroup. Ghadiri and Waphare [16] defined the notation of n-ary polygroups, as a subclass of n-ary hypergroups and as a generalization of polygroups.

In this paper, we consider the fundamental relation β^* and the commutative fundamental relation γ^* on an n-ary polygroup, in a similar way as in the case of n-ary hypergroups, and we obtain some new results in this respect. In particular, we prove that the relation γ is transitive on an n-ary polygroup.

2. **Basic Definitions and Results**

Let H be a non-empty set and f a mapping $f : H \times H \longrightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H. Then f is called a *binary* (*algebraic*) *hyperoperation* on H. As it is well-known a binary hyperoperation f on H is *associative*, if $f(f(x, y), z) = f(x, f(y, z))$, for all $x, y, z \in H$. A binary hypergroupoid with the associative hyperoperation is called a *semihypergroup*. A hypergroupoid (H, f) satisfying the *reproducibility axiom*: $f(a, H) = f(H, a) = H$ for all $a \in H$, is called a *quasihypergroup*. A quasihypergroup which is a semihypergroup is called a *hypergroup*. Moreover, according to [1], a *polygroup* is a multivalued system $(P, \cdot, e, ^{-1})$ where $e \in P$, $^{-1} : P \to P$, $\cdot : P \times P \to \mathcal{P}^*(P)$ and the following axioms hold for all $x, y, z \in P$

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(ii) $e \cdot x = x \cdot e = x$,

(iii) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

Every commutative polygroup is called a *canonical hypergroup*.

In general, a mapping $f : H \times \ldots \times H \longrightarrow \mathcal{P}^*(H)$, where H appears n times, is called an *n-ary hyperoperation*. An algebraic system (H, f), where f is an n-ary hyperoperation defined on H, is called an *n-ary hypergroupoid*. Since we identify the set $\{x\}$ with the element x, any n-ary groupoid is an n-ary hypergroupoid.

We shall use the following abbreviated notation:
The sequence $x_i, x_{i+1}, \cdots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty symbol. In this convention

$$f(x_1, \cdots, x_i, y_{i+1}, \cdots, y_j, z_{j+1}, \ldots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. In the case when $y_{i+1} = \ldots = y_j = y$ the last expression will be write in the form $f(x_1^i, \overset{(j-i)}{y}, z_{j+1}^n)$. For non-empty subsets $A_1, \ldots, A_n$ of H we define

$$f(A_1^n) = f(A_1, \ldots, A_n) = \cup \{ f(x_1^n) \mid x_i \in A_i,\ i = 1, \ldots n \}.$$

An n-ary hyperoperation f is called *associative*, if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

holds for every $1 \leq i < j \leq n$ and all $x_1, x_2, \ldots, x_{2n-1} \in H$. An n-ary hypergroupoid with the associative n-ary hyperoperation is called an *n-ary semihypergroup*. An n-ary hypergroupoid (H, f) in which the relation

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n) \qquad (*)$$

has a solution $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an *n-ary quasihypergroup*, In addition, when (H, f) is an n-ary semihypergroup, (H, f) is called an *n-ary hypergroup*. An n-ary hypergroupoid (H, f) is *commutative* if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H^n$ we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. If $a_1^n \in H^n$ we denote $a_{\sigma(1)}^{\sigma(n)}$ as the $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$.

An element $e \in H$ is called *neutral element* if $x \in f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e})$, for every $1 \leq i \leq n$ and for every $x \in H$. An element $e \in H$ is called *scalar neutral element* if $x = f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e})$, for every $1 \leq i \leq n$ and for every $x \in H$. If $m = k(n-1)+1$, then the m-ary hyperoperation h given by

$$h(x_1^{k(n-1)+1}) = \underbrace{f(f(\cdots(f(f}_{k}(x_1^n), x_{n+1}^{2n-1}), \cdots), x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

will be denoted by $f_{(k)}$. If $k = 0$ then $m = 1$ and we denote $f_{(0)}(z_1^m) = z_1$. According to [16], an n-ary polygroup is a multivalued system $\mathbb{P} = (P, f, e, ^{-1})$, where $e \in P$, $^{-1}$ is a unitary operation on P, f is an n-ary hyperoperation on P and the following axioms hold for all $1 \leq i, j \leq n$ and $x, x_1^{2n-1} \in P$:

(i) $f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$, i.e., f is associative,

(ii) element e is a scalar neutral of P, i.e., $x = f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e})$,

(iii) $x \in f(x_1^n)$ implies $x_i \in f(x_{i-1}^{-1}, \ldots, x_1^{-1}, x, x_n^{-1}, \ldots, x_{i+1}^{-1})$.

An n-ary subpolygroup N of an n-ary polygroup P is *normal* in P if for every $a \in P$, $f(a^{-1}, N, a, \overset{(n-3)}{e}) \subseteq N$. Let $\mathbb{A} = (A, f, e_1, ^{-1})$ and $\mathbb{B} = (B, g, e_2, ^{-1})$ be two n-ary polygroups. A *homomorphism* from A into B is a mapping $\phi : A \longrightarrow B$ such that $\phi(f(a_1^n)) = g(\phi(a_1), \ldots, \phi(a_n))$ holds for all $a_1, \ldots, a_n \in A$, and $\phi(e_1) = e_2$.

3. Application of Fundamental Relation β^* and Commutative Fundamental Relation γ^* on n-ary Polygroups

Davvaz and Vougiouklis in [10] defined the relation β on an n-ary semihypergroup (H, f) as follows:

β_0 is the diagonal relation, i.e., $\beta_0 = \{(x, x) | x \in H\}$, and, for every integer $k > 0$, β_k is the relation defined as follows:

$$x \ \beta_k \ y \quad \Leftrightarrow \quad \exists z_1^m \in H : \{x, y\} \subseteq f_{(k)}(z_1^n), \text{ where } m = k(n-1)+1.$$

Now, set

$$\beta = \bigcup_{k \geq 0} \beta_k,$$

then $x \ \beta \ y$ if and only if $x \ \beta_k \ y$ for some $k \geq 0$.

If β^* is the smallest strongly compatible equivalence relation on an n-ary semihypergroup (H, f) such that the quotient $(H/\beta^*, f/\beta^*)$ is an n-ary group, then β^* is transitive closure of the relation β (for a proof see[10]). The n-ary operation f/β^* is as follows:

$$f/\beta^*(\beta^*(a_1), \ldots, \beta^*(a_n)) = \beta^*(a),$$

for all $a \in f(\beta^*(a_1), \ldots, \beta^*(a_n)) = \beta^*(a)$. Also, Leoreanu and Davvaz [19] showed that the relation β is transitive on an n-ary hypergroup. The relation β^* is called the *fundamental relation* and $(H/\beta^*, f/\beta^*)$ is called the *fundamental n-ary group*.

When (H, f) is an n-ary semihypergroup, Davvaz, Dudek and Mirvakili [12] studied the relation $\gamma = \bigcup_{k\geq 0} \gamma_k$, where γ_0 is the diagonal relation and for every integer $k \geq 1$, γ_k is the relation defined as follows: there exist $z_1^m \in H^m$ and $\sigma \in \mathbb{S}_m$ such that $x\gamma_k y$ if and only if $x \in f_{(k)}(z_1^m)$ and $y \in f_{(k)}(z_{\sigma(1)}^{\sigma(m)})$, where $m = k(n-1)+1$.

Let (H, f) be an n-ary semihypergroup. We define γ^* as the smallest equivalence relation such that the quotient $(H/\gamma^*, f/\gamma^*)$ is a commutative n-ary semigroup, where H/γ^* is the set of all equivalence classes. The equivalence relation γ^* is called *commutative fundamental relation* and $(H/\gamma^*, f/\gamma^*)$ is called *commutative fundamental n-ary semigroup*.

The relation γ (respectively, γ^*) was introduced on hypergroups (2-ary hypergroups) by Freni [14, 15].

Theorem 3.1. [12] *Let (H, f) be an n-ary hypergroup. Then we have:*

(1) *The fundamental relation γ^* is the transitive closure of the relation γ.*
(2) *Relation γ is a strongly compatible relation on (H, f).*
(3) *If (H, f) is commutative then $\beta = \gamma$.*

Let $\mathbb{P} = (P, f, e, ^{-1})$ be an n-ary polygroup, $\phi : P \to P/\beta^*$ and $\varphi : P \to P/\gamma^*$ canonical projections. Then w_P and $D(P)$ are the kernels of ϕ and φ, respectively. In fact, w_P is a neutral element of P/β^* and $D(P)$ is a neutral element of P/γ^*. We have $w_P \subseteq D(P)$, since $\beta^* \subseteq \gamma^*$. Also, it is not difficult to see that

$$w_P = \beta^*(e) \quad \text{and } \beta^*(x^{-1}) = \beta^*(x)^{-1} \text{ for all } x \in P,$$
$$D(P) = \gamma^*(e) \quad \text{and } \gamma^*(x^{-1}) = \gamma^*(x)^{-1} \text{ for all } x \in P.$$

Theorem 3.2. [12] *If (H, f) is an n-ary hypergroup with a neutral (identity) element such that $H/\gamma*$ is i-cancellative then γ is transitive.*

So we have:

Corollary 3.3. *If (P, f) is an n-ary polygroup, then γ is an equivalence relation on P and $\gamma = \gamma^*$.*

Theorem 3.4. *Let P be an n-ary polygroup and $a_1^m, b_1^m \in P$ such that $a_j \ \gamma^* \ b_j$ for all $j = 1, 2, \ldots, m$, where $m = k(n-1)+1$. Then for all $x \in f_{(k)}(a_1^{\delta_1}, \ldots, a_m^{\delta_m})$ and $y \in f_{(k)}(b_1^{\delta_1}, \ldots, b_m^{\delta_m})$ where $\delta_i \in \{1, -1\}(i = 1, 2, \ldots, m)$, we have $x \ \gamma^* \ y$. Also, this theorem is true for β^*-relation.*

Proof. Suppose that $a_j \ \gamma^* \ b_j$ for all $j = 1, 2, \ldots, m$, then there exist $k_j \in \mathbb{N} \cup \{0\}$ and $z_{j1}^{jn_j} \in P$ where $n_j = k_j(n-1)+1$, and there exists permutation $\sigma_j \in \mathbb{S}_{n_j}$ such that $a_j \in f_{(k_j)}(z_{j1}, \ldots, z_{jn_j})$ and $b_j \in f_{(k_j)}(z_{j\sigma_j(1)}, \ldots, z_{j\sigma_j(n_j)})$. Therefore,

$$f_{(k)}(a_1^m) \subseteq f_{(k)}(f_{(k_1)}(z_{11}, \ldots, z_{1n_1}), \ldots, f_{(k_m)}(z_{m1}, \ldots, z_{mn_m})) \quad \text{and}$$

$$f_{(k)}(b_1^m) \subseteq f_{(k)}(f_{(k_1)}(z_{1\sigma_1(1)}, \ldots, z_{1\sigma_1(n_1)}), \ldots, f_{(k_m)}(z_{m\sigma_m(1)}, \ldots, z_{m\sigma_m(n_m)})),$$

and so we conclude that

$$x \in f_{(k)}(a_1^m) \subseteq f_{(k+k_1+\ldots+k_m)}(z_{11}, \ldots, z_{1n_1}, \ldots, z_{m1}, \ldots, z_{mn_m}) \quad \text{and}$$

$$y \in f_{(k)}(b_1^m) \subseteq f_{(k+k_1+\ldots+k_m)}(z_{1\sigma_1(1)}, \ldots, z_{1\sigma_1(n_1)}, \ldots, z_{m\sigma_m(1)}, \ldots, z_{m\sigma_m(n_m)}).$$

Thus, we obtain $x \ \gamma^* \ y$. Since $a_j \ \gamma^* \ b_j$ implies $a_j^{-1} \ \gamma^* \ b_j^{-1}$, so by the similar way for all $x \in f_{(k)}(a_1^{\delta_1}, \ldots, a_m^{\delta_m})$ and $y \in f_{(k)}(b_1^{\delta_1}, \ldots, b_m^{\delta_m})$ where $\delta_i \in \{1, -1\}(i = 1, 2, \ldots, m)$, we obtain $x \ \gamma^* \ y$. □

By the above theorem and definition of γ^*-relation, we obtain:

Corollary 3.5. *Let P be an n-ary polygroup and $a_1^m, b_1^m \in P$ such that $a_j \ \gamma^* \ b_j$ for all $j = 1, 2, \ldots, m$, where $m = k(n-1)+1$. Then for every $\tau \in \mathbb{S}_m$ and every $x \in f_{(k)}(a_1^{\delta_1}, \ldots, a_m^{\delta_m})$ and $y \in f_{(k)}(b_{\tau(1)}^{\delta_{\tau(1)}}, \ldots, b_{\tau(m)}^{\delta_{\tau(m)}})$ where $\delta_i \in \{1, -1\}(i = 1, 2, \ldots, m)$, we have $x \ \gamma^* \ y$.*

Theorem 3.6. *Let P be an n-ary polygroup. If there exist $A, A' \subseteq \gamma^*(z)$ and $B, B' \subseteq \gamma^*(z^{-1})$ for some $z \in P$ such that $f(\overset{(i-1)}{A}, x, \overset{(n-i)}{A}) \cap B \neq \emptyset$ and $f(\overset{(i-1)}{A'}, y, \overset{(n-i)}{A'}) \cap B' \neq \emptyset$, then $x \ \gamma^* \ y$.*

Proof. Suppose that there exist $A, A' \subseteq \gamma^*(z)$ and $B, B' \subseteq \gamma^*(z^{-1})$ for some $z \in P$ such that $f(\overset{(i-1)}{A}, x, \overset{(n-i)}{A}) \cap B \neq \emptyset$ and $f(\overset{(i-1)}{A'}, y, \overset{(n-i)}{A'}) \cap B' \neq$

$\emptyset$. Then we have

$$f/\gamma^*(\overset{(i-1)}{\gamma^*(A)}, \gamma^*(x), \overset{(n-i)}{\gamma^*(A)}) \cap \gamma^*(B) \neq \emptyset,$$

$$f/\gamma^*(\overset{(i-1)}{\gamma^*(A')}, \gamma^*(y), \overset{(n-i)}{\gamma^*(A')}) \cap \gamma^*(B') \neq \emptyset.$$

Therefore, we conclude that $f/\gamma^*(\overset{(i-1)}{\gamma^*(z)}, \gamma^*(x), \overset{(n-i)}{\gamma^*(z)}) = \gamma^*(z^{-1})$ and $f/\gamma^*(\overset{(i-1)}{\gamma^*(z)}, \gamma^*(y), \overset{(n-i)}{\gamma^*(z)}) = \gamma^*(z^{-1})$. Since P/γ^* is an n-ary group, $\gamma^*(x) = \gamma^*(y)$. □

Theorem 3.7. *Let (P, f) be an n-ary polygroup.*

(1) *If $x_1^n \in D(P)$ then for every $a \in P$, there exists $A \subseteq \gamma^*(a)$ such that for every $i \in \{1, 2, \ldots, n\}$, we have $f(x_1^{i-1}, A, x_{i+1}^n) \cap A \neq \emptyset$.*

(2) *Let $a, x_1^n \in P$ such that $x_1\gamma^* \ldots \gamma^* x_n$. If there exist $A \subseteq \gamma^*(a)$ and $i \in \{1, \ldots, n\}$ such that $f(x_1^{i-1}, A, x_{i+1}^n) \cap A \neq \emptyset$ and $D(P) = \gamma^*(e)$ is a unique neutral element of P/γ^* then $x_1^{i-1}, x_{i+1}^n \in D(P)$.*

Proof. (1) Suppose that $x_1^n \in D(P)$ and set $A = \gamma^*(a)$ for an arbitrary $a \in P$. So for every $1 \leq i \leq n$, we have:

$$\begin{aligned}&\varphi(f(x_1^{i-1}, a, x_{i+1}^n))\\ &= f/\gamma^*(\gamma^*(x_1), \ldots, \gamma^*(x_{i-1}), \gamma^*(a), \gamma^*(x_{i+1}), \ldots, \gamma^*(x_n))\\ &= f/\gamma^*(\overset{(i-1)}{D(P)}, \gamma^*(a), \overset{(n-i)}{D(P)}) = \gamma^*(a).\end{aligned}$$

Thus, $f(x_1^{i-1}, \gamma^*(a), x_{i+1}^n) \cap \gamma^*(a) \neq \emptyset$, so $f(x_1^{i-1}, A, x_{i+1}^n) \cap A \neq \emptyset$.

(2) If $f(x_1^{i-1}, A, x_{i+1}^n) \cap A \neq \emptyset$, then

$$f/\gamma^*(\gamma^*(x_1), \ldots, \gamma^*(x_{i-1}), \gamma^*(a), \gamma^*(x_{i+1}), \ldots, \gamma^*(x_n)) = \gamma^*(a).$$

Since $D(P)$ is the unique identity element of P/γ^* and $\gamma^*(x_1) = \ldots = \gamma^*(x_n)$ then $\gamma^*(x_i) = D(P)$ and $x_i \in D(p)$, when $i \in \{1, \ldots, i-1, i+1, \ldots, n\}$. □

Let P be an n-ary polygroup and $a \in P$, we define a^l, where $l \in \mathbb{N} \cup \{0\}$, as follows

$$\begin{cases} a^l = e & \text{if } l = 0, \\ a^l = f_{(k)}(\overset{(l)}{a}, \overset{(m-l)}{e}) & \text{if } (k-1)(n-1)+1 < l \leq m = k(n-1)+1. \end{cases}$$

Then, by the above notation we have:

Theorem 3.8. *Let P be a n-ary polygroup. For every $a \in P$, and $r, r' \in \mathbb{N} \cup \{0\}$ such that $r' \leq r$, if $a^r \cap a^{r'} \neq \emptyset$ then $a^{r-r'} \subseteq D(P)$.*

Proof. Let $(k-1)(n-1)+1 < r \leq m = k(n-1)+1$ and $(k'-1)(n-1)+1 < r' \leq m' = k'(n-1)+1$. Now, we have $(k-k'-2)(n-1)+1 < r-r' = r'' < m-m'+1 = (k-k')(n-1)+1 = m''$ or $r'' = r-r' = (k-k'-2)(n-1)+1 = m''$. First, suppose that $(k-k'-2)(n-1)+1 < r-r' = r'' < m-m'+1 = (k-k')(n-1)+1 = m''$, then $a^r \cap a^{r'} \neq \emptyset$ implies $f_{(k)}(\overset{(r)}{a}, \overset{(m-r)}{e}) \cap f_{(k')}(\overset{(r')}{a}, \overset{(m'-r')}{e}) \neq \emptyset$. So $\varphi(f_{(k)}(\overset{(r)}{a}, \overset{(m-r)}{e})) = \varphi(f_{(k')}(\overset{(r')}{a}, \overset{(m'-r')}{e}))$ which implies that

$$(f/\gamma^*)_{(k)}(\overset{(r)}{\gamma^*(a)}, \overset{(m-r)}{\gamma^*(e)}) = (f/\gamma^*)_{(k')}(\overset{(r')}{\gamma^*(a)}, \overset{(m'-r')}{\gamma^*(e)}),$$

hence

$$\begin{aligned}&(f/\gamma^*)_{(k')}(\overset{(r')}{\gamma^*(a)}, (f/\gamma^*)_{(k-k')}(\overset{(r-r')}{\gamma^*(a)}, \overset{(m-m'+1-(r-r'))}{\gamma^*(e)}), \overset{(m'-r'-1)}{\gamma^*(e)})\\ &= (f/\gamma^*)_{(k')}(\overset{(r')}{\gamma^*(a)}, \overset{(m'-r')}{\gamma^*(e)}).\end{aligned}$$

Therefore,

$$\begin{aligned}&(f/\gamma^*)_{(k')}(\overset{(r')}{\gamma^*(a)}, (f/\gamma^*)_{(k'')}(\overset{(r'')}{\gamma^*(a)}, \overset{(m''-r'')}{\gamma^*(e)}), \overset{(m'-r'-1)}{\gamma^*(e)})\\ &= (f/\gamma^*)_{(k')}(\overset{(r')}{\gamma^*(a)}, \overset{(m'-r')}{\gamma^*(e)}).\end{aligned}$$

Since P/γ^* is an n-ary group, we have $(f/\gamma^*)_{(k'')}(\overset{(r'')}{\gamma^*(a)}, \overset{(m''-r'')}{\gamma^*(e)}) = \gamma^*(e)$ and so $\gamma^*(f_{(k'')}(\overset{(r'')}{a}, \overset{(m''-r'')}{e})) = \gamma^*(e)$. Therefore, $f_{(k'')}(\overset{(r'')}{a}, \overset{(m''-r'')}{e}) \subseteq D(P)$, thus $a^{r-r'} = a^{r''} \subseteq D(P)$.

If $r'' = r-r' = (k-k'-2)(n-1)+1 = m''$, then by a similar way we obtain $a^{r-r'} \subseteq D(P)$. □

Remark 3.9. *If we use β^*, w_P and ϕ instead of γ^*, $D(P)$ and φ respectively, then Theorems 3.6, 3.7 and 3.8, are still valid.*

Let A be a non-empty subset of P. The intersection of β-parts of P which contains A is called β-closure of A in P. It will be denoted by $C_\beta(A)$. Also, we define γ-closure of A in P (i.e., $C_\gamma(A)$) by a similar way.

Similar to Theorem 63 in [3], we have:

Theorem 3.10. *Let B be a non-empty subset of an n-ary polygroup P. Then*

(1) $C_\beta(B) = \bigcup_{b\in B} C_\beta(b)$,

(2) $C_\gamma(B) = \bigcup_{b\in B} C_\gamma(b)$.

Theorem 3.11. *Let P be an n-ary polygroup. If A is a non-empty subset of P. Then for every $i \in \{1, 2, \ldots, n-1\}$ we have*

1) $f(\overset{(i-1)}{w_P}, A, \overset{(n-i)}{w_P}) = \phi^{-1}(\phi(A))$,

2) $f(\overset{(i-1)}{D(P)}, A, \overset{(n-i)}{D(P)}) = \varphi^{-1}(\varphi(A))$.

Proof. We prove (2), the proof of (1) is similar. For every $x \in f(\overset{(i-1)}{D(P)}, A, \overset{(n-i)}{D(P)})$, there exist $d_2, \ldots, d_n \in D(P)$ and $a \in A$ such that $x \in f(d_2^{i-1}, a, d_{i+1}^n)$, so $\varphi(x) = f/\gamma^*(\overset{(i-1)}{e_{P/\gamma^*}}, \varphi(a), \overset{(n-i)}{e_{P/\gamma^*}}) = \varphi(a)$, therefore $x \in \varphi^{-1}(\varphi(x)) = \varphi^{-1}(\varphi(a)) \subseteq \varphi^{-1}(\varphi(A))$.

For the converse, take $x \in \varphi^{-1}(\varphi(A))$, so an element $b \in A$ exists such that $\varphi(x) = \varphi(b)$. Since P is an n-ary polygroup, thus $a \in P$ exists such that $x \in f(a, \overset{(i-2)}{e}, b, \overset{(n-i)}{e})$, so $\varphi(b) = \varphi(x) = f/\gamma^*(\varphi(a), \overset{(i-2)}{\varphi(e)}, \varphi(b), \overset{(n-i)}{\varphi(e)}) = f/\gamma^*(\varphi(a), \overset{(i-2)}{e_{P/\gamma^*}}, \varphi(b), \overset{(n-i)}{e_{P/\gamma^*}})$. But $f/\gamma^*(\overset{(i-1)}{e_{P/\gamma^*}}, \varphi(b), \overset{(n-i)}{e_{P/\gamma^*}}) = \varphi(b)$, thus P is 1-cancellative and so $\varphi(a) = e_{P/\gamma^*}$ and $a \in \varphi^{-1}(e_{P/\gamma^*}) = D(P)$. Hence $x \in f(a, \overset{(i-2)}{e}, b, \overset{(n-i)}{e}) \subseteq f(\overset{(i-1)}{D(P)}, A, \overset{(n-i)}{D(P)})$. Therefore we obtain $f(\overset{(i-1)}{D(P)}, A, \overset{(n-i)}{D(P)}) = \varphi^{-1}(\varphi(A))$. □

Theorem 3.12. *If A is a non-empty subset of an n-ary polygroup P, then for every $i \in \{1, 2, \ldots, n\}$,*

(1) $f(\overset{(i-1)}{w_P}, A, \overset{(n-i)}{w_P}) = C_\beta(A)$,

(2) $f(\overset{(i-1)}{D(P)}, A, \overset{(n-i)}{D(P)}) = C_\gamma(A)$,

where $C_\beta(A)$ and $C_\gamma(A)$ are β-closure and γ-closure of A in P, respectively.

Proof. We prove (2), the proof of (1) is similar. If $x \in \varphi^{-1}(\varphi(A))$, then $a \in A$ there exists such that $\varphi(x) = \varphi(a)$ and so $\gamma^*(x) = \gamma^*(a)$. Therefore, $x \in \gamma^*(a) \subseteq C_\gamma(a)$. Also, if $x \in C_\gamma(a)$ for some $a \in A$, then we have $x\ \gamma^*\ a$ and so $\varphi(x) = \varphi(a)$. Thus, we obtain $x \in \varphi^{-1}(\varphi(A))$ and so:

$$\varphi^{-1}(\varphi(A)) = \{x \in P \mid \exists a \in A : x \in C_\gamma(a)\} = \bigcup_{b \in B} C_\gamma(b).$$

By Theorems 3.10 and 3.11, we obtain $f(\overset{(i-1)}{D(P)}, A, \overset{(n-i)}{D(P)}) = C_\gamma(A)$. □

Corollary 3.13. *If A is a non-empty subset of an n-ary polygroup P, then for every $1 \le i, j \le n$ we have:*

(1) $f(\overset{(i-1)}{w_P}, A, \overset{(n-i)}{w_P}) = f(\overset{(j-1)}{w_P}, A, \overset{(n-j)}{w_P})$,
(2) $f(\overset{(i-1)}{D(P)}, A, \overset{(n-i)}{D(P)}) = f(\overset{(j-1)}{D(P)}, A, \overset{(n-j)}{D(P)})$.

Corollary 3.14. *Let P be an n-ary polygroup and $A \in \wp(P)^*$. If A is a γ-part then for evey $i \in \{1, 2, \ldots, n\}$ we have $f(\overset{(i-1)}{D(P)}, A, \overset{(n-i)}{D(P)}) = A$. Conversely, if for some $i \in \{1, 2, \ldots, n\}$ we have $f(\overset{(i-1)}{D(P)}, A, \overset{(n-i)}{D(P)}) = A$, then A is a γ-part of P. Also, this corollary is true for the β^*-relation.*

Proof. By Theorem 3.12, the proof is straightforward. □

Theorem 3.15. *If P is an n-ary polygroup, then*

(1) *w_P is a β-part of P,*
(2) *$D(P)$ is a γ-part of P.*

Proof. (1) See the proof of (2) and set $\sigma = id$.

(2) Let $m = k(n-1)+1$, $z_1^m \in P$. We have $f_{(k)}(z_1^m) \bigcap D(P) \neq \emptyset$. Thus, there exists $x \in f_{(k)}(z_1^m) \bigcap D(P)$ and so we obtain $\varphi(x) = \varphi(D(P)) = e_{P/\gamma^*}$ and $\varphi(x) = \varphi(f_{(k)}(z_1^m)) = \gamma^*(f_{(k)}(z_1^m))$. Now, for every $\sigma \in \mathbb{S}_m$ and for every $y \in f_{(k)}(z_{\sigma(1)}^{\sigma(m)})$ we have $x\ \gamma^* y$, because $x \in f_{(k)}(z_1^m)$. Therefore, $e_{P/\gamma^*} = \gamma^*(f_{(k)}(z_1^m)) = \gamma^*(x) = \gamma^*(y) = \gamma^*(f_{(k)}(z_{\sigma(1)}^{\sigma(m)})$. Thus, $f_{(k)}(z_{\sigma(1)}^{\sigma(m)}) \subseteq \varphi^{-1}(e_{P/\gamma^*}) = D(P)$, this shows that $D(P)$ is a γ-part of P. □

Theorem 3.16. *Let P be an n-ary polygroup. If A_i is a γ-part of P for some $i \in \{1, 2, \ldots, n\}$, then for every $\sigma \in \mathbb{S}_n$ and for every $A_j \subseteq P$, $i \neq j \in \{1, 2, \ldots, n\}$, the image $f(A_{\sigma(1)}^{\sigma(n)})$ is a γ-part of P. Also, this theorem is true for the β^*-relation.*

Proof. Set $B = f(A_{\sigma(1)}^{\sigma(n)})$ and suppose that $\sigma(k) = i$. We prove that $f(B, \overset{(n-1)}{D(P)}) = B$ and then by Corollary 3.14, B is a γ-part of P. Since A_i is a γ-part of P, by Corollary 3.14, we have $f(\overset{(j-1)}{D(P)}, A_i, \overset{(n-j)}{D(P)}) = A_i$, for every $j \in \{1, 2, \ldots, n\}$. Now, by Corollary 3.13, we obtain:

$$
\begin{aligned}
& f(B, \overset{(n-1)}{D(P)}) = f(f(A_{\sigma(1)}^{\sigma(n)}), \overset{(n-1)}{D(P)}) = f(A_{\sigma(1)}^{\sigma(n-1)}, f(A_{\sigma(n)}, \overset{(n-1)}{D(P)})) \\
& = f(A_{\sigma(1)}^{\sigma(n-1)}, f(\overset{(n-1)}{D(P)}, A_{\sigma(n)})) = f(A_{\sigma(1)}^{\sigma(n-2)}, f(A_{\sigma(n-1)}, \overset{(n-1)}{D(P)}), A_{\sigma(n)}) \\
& \ldots \\
& = f(A_{\sigma(1)}^{\sigma(k-1)}, f(A_{\sigma(k)}, \overset{(n-1)}{D(P)}), A_{\sigma(k+1)}^{\sigma(n)}) \\
& = f(A_{\sigma(1)}^{\sigma(k-1)}, f(A_i, \overset{(n-1)}{D(P)}), A_{\sigma(k+1)}^{\sigma(n)}) \\
& = f(A_{\sigma(1)}^{\sigma(k-1)}, A_i, A_{\sigma(k+1)}^{\sigma(n)}) = f(A_{\sigma(1)}^{\sigma(n)}) = B.
\end{aligned}
$$

Therefore, $f(B, \overset{(n-1)}{D(P)}) = B$ and the proof is completed. □

Let P be an n-ary polygroup, and $\prod(P)$ be the set of m-ary hyperproducts of elements of P. In fact:

$$\prod(P) = \{f_{(k)}(z_1^m) \mid m = k(n-1)+1, k \in \mathbb{N} \cup \{0\}, z_1^m \in P\}.$$

Also, consider $\prod(P)$ with an n-ary hyperoperation F defined as follows:

$$F(A_1, \ldots, A_n) = \{C \in \prod(P) \mid C \subseteq f(A_1, \ldots, A_n)\},$$

for all $A_1^n \in \prod(P)$. We consider the following condition:
(*) $X \in F(A_1^n)$, if for every $a_i \in A_i (i = 1, ..., n)$, there exists $x \in X$ such that $x \in f(a_1^n)$.
Then we have the following theorem:

Theorem 3.17. (Construction) *If P is an n-ary polygroup which satisfies the (*), then $(\prod(P), F)$ is an n-ary polygroup.*

Proof. (1) First, we show that n-ary hyperoperation F on $\prod(P)$ is associative. Let $A_1^{2n-1} \in \prod(P)$. Then for every $1 \leq i, j \leq n$ we have:

$$
\begin{aligned}
& F(A_1^{i-1}, F(A_i^{n+i-1}), A_{n+i}^{2n-1}) \\
& = \{F(A_1^{i-1}, C, A_{n+i}^{2n-1}) \mid C \subseteq f(A_i^{n+i-1})\} \\
& = \{D \mid D \subseteq f(A_1^{i-1}, f(A_i^{n+i-1}), A_{n+i}^{2n-1})\} \\
& = \{D \mid D \subseteq f(A_1^{j-1}, f(A_j^{n+j-1}), A_{n+j}^{2n-1})\} \\
& = \{F(A_1^{j-1}, C, A_{n+j}^{2n-1}) \mid C \subseteq f(A_j^{n+j-1})\} \\
& = F(A_1^{j-1}, F(A_j^{n+j-1}), A_{n+j}^{2n-1}).
\end{aligned}
$$

(2) Let $E = \{e\}$. Then for all $A \in \prod(P)$ and for every $i \in \{1, 2, \ldots, n\}$, it is easy to see that $f(\overset{(i-1)}{E}, A, \overset{(n-i)}{E})) = A$, and $E^{-1} = \{e\}^{-1} = \{e^{-1}\} = \{e\} = E$.

(3) We define the unitary operation $^{-I}$ as follows

$$^{-I} : \prod(P) \longrightarrow \prod(P)$$

$$(f_{(k)}(x_1, \ldots, x_m))^{-I} = f_{(k)}(x_m^{-1}, \ldots, x_1^{-1}),$$

where $m = k(n-1) + 1$ and $x_1^m \in P$. Now, let $A_1 = f_{(k_1)}(a_{11}^{1m_1})$, $A_2 = f_{(k_2)}(a_{21}^{2m_2})$, $\ldots$, $A_n = f_{(k_n)}(a_{n1}^{nm_n})$ and $A_{n+1} = f_{(k_{n+1})}(a_{(n+1)1}^{(n+1)m_{n+1}})$ be elements of $\prod(P)$ such that $A_{n+1} \in F(A_1, \ldots, A_n)$. Let $a_i \in A_i$, $1 \leq i \leq n$ be arbitrary. Then, there exists $a_{n+1} \in A_{n+1}$ such that $a_{n+1} \in f(a_1^n)$. Since P is an n-ary polygroup, thus,

$$a_i \in f(a_{i-1}^{-1}, \ldots, a_1^{-1}, a_{n+1}, a_n^{-1}, \ldots, a_{i+1}^{-1}).$$

But for every $1 \leq j \leq n$ we have $a_j^{-1} \in f_{(k_j)}(a_{jm_j}^{-1}, \ldots, a_{j1}^{-1}) = A_j^{-1}$, and so we obtain

$$
\begin{aligned}
a_i \ & \in f(a_{i-1}^{-1}, \ldots, a_1^{-1}, a_{n+1}, a_n^{-1}, \ldots, a_{i+1}^{-1}) \\
& \subseteq f(A_{i-1}^{-1}, \ldots, A_1^{-1}, A_{n+1}, A_n^{-1}, \ldots, A_{i+1}^{-1}).
\end{aligned}
$$

Since for every $a_i \in A_i$ the above equation is true,

$$A_i \subseteq f(A_{i-1}^{-1}, \ldots, A_1^{-1}, A_{n+1}, A_n^{-1}, \ldots, A_{i+1}^{-1}),$$

or $A_i \in F(A_{i-1}^{-1}, \ldots, A_1^{-1}, A_{n+1}, A_n^{-1}, \ldots, A_{i+1}^{-1})$. Thus, by (1), (2), (3) and definition of n-ary polygroups, we conclude that $(\prod(P), F)$ is an n-ary polygroup. $\square$

Corollary 3.18. *If A is an n-ary subpolygroup of P and A belongs to $\prod(P)$, then A is contained in w_P and $D(P)$.*

Proof. Since A is an n-ary subpolygroup thus $e \in A$. But $A \in \prod(P)$ and so for every $x \in A$ we have $\beta^*(x) = \beta^*(A) = \beta^*(e)$. This means that $x \in w_P$ or $A \subseteq w_P$. Since $w_P \subseteq D(P)$, $A \subseteq D(P)$. $\square$

The following example shows that not all n-ary subpolygroups of an n-ary polygroup are in $\prod(P)$.

Example 3.19. *Let $P = \{e, a, b, c\}$ and let (P, f) be a commutative ternary hypergroupoid with two scalar neutral elements $e \in P$ and $a \in P$. Assume that and 3-ary hyperoperation f defined as follows:*

$$\begin{array}{lll} f(e,a,b) = b, & f(e,a,c) = c, & f(e,b,b) = \{e,a,c\}, \\ f(e,b,c) = \{b,c\}, & f(e,c,c) = \{e,a,b\}, & f(a,b,b) = \{e,a,c\}, \\ f(a,b,c) = \{b,c\}, & f(a,c,c) = \{e,a,b\}, & f(b,b,b) = \{b,c\}, \\ f(b,b,c) = \{e,a,b\}, & f(b,c,c) = P, & f(c,c,c) = \{b,c\}. \end{array}$$

Then the 3-ary hyperoperation f is associative. We define unitary operation $x^{-1} = x$ for every $x \in P$. Then $\mathbb{P} = (P, f, e, ^{-1})$ is a ternary polygroup. Furthere, $A = \{e, a\}$ is a 3-ary subpolygroup of P, but $A \notin \prod(P)$. We also have $w_P = D(P) = f(b, c, c) = P \in \prod(P)$.

If P is an n-ary polygroup, we denote the set of n-ary hyperproducts A of elements of P by, $\prod_{C_\beta}(P)$ ($\prod_{C_\gamma}(P)$) such that $C_\beta(A) = A$ ($C_\gamma(A) = A$).

Theorem 3.20. *Let P be an n-ary polygroup and let x_1^m, where $m = k(n-1)+1$, are elements of P such that $f_{(k)}(x_1^m) \in \prod_{C_\gamma}(P)$. Then there exists $y_1^m \in P$ such that $f_{(2k+1)}(x_1^m, y_m^1, \overset{(n-2)}{e}) = D(P)$. This theorem is true for β^*-relation, too.*

Proof. Suppose that $a_j \in D(P)$, where $1 \le j \le m$. Then there exists $y_j \in P$ such that $a_j \in f(x_j, y_j, \overset{(n-2)}{e})$. Since $D(P)$ is a γ-part, then $f(x_j, y_j, \overset{(n-2)}{e}) \subseteq D(P)$, and so $f(x_j, y_j, \overset{(n-3)}{e}, D(P)) = D(P)$. By Corollaries 3.13 and 3.14, we obtain

$$\begin{aligned} f(f_{(k)}(x_1^m), y_m, \overset{(n-2)}{e}) &= f(f(f_{(k)}(x_1^m), \overset{(n-1)}{D(P)}), y_m, \overset{(n-2)}{e}) \\ &= f(f_{(k)}(x_1^m), f(\overset{(n-1)}{D(P)}, y_m), \overset{(n-2)}{e}) \\ &= f(f_{(k)}(x_1^m), f(y_m, \overset{(n-1)}{D(P)}), \overset{(n-2)}{e}) \\ &= f_{(k+1)}(x_1^{m-1}, f(x_m, y_m, \overset{(n-3)}{e}, D(P)), \overset{(n-2)}{D(P)}, e) \\ &= f_{(k+1)}(x_1^{m-1}, \overset{(n-1)}{D(P)}, e), \end{aligned}$$

and so

$$\begin{aligned} & f(f_{(k)}(x_1^m), y_m, y_{m-1}, \overset{(n-3)}{e}) \\ &= f(f_{(k)}(x_1^m), y_m, f(\overset{(n-2)}{e}, y_{m-1}, e), \overset{(n-3)}{e}) \\ &= f(f(f_{(k)}(x_1^m), y_m, \overset{(n-2)}{e}), y_{m-1}, \overset{(n-2)}{e}) \\ &= f(f_{(k+1)}(x_1^{m-1}, \overset{(n-1)}{D(P)}, e), y_{m-1}, \overset{(n-2)}{e}) \\ &= f_{(k+1)}(x_1^{m-1}, \overset{(n-1)}{D(P)}, f(e, y_{m-1}, \overset{(n-2)}{e})) \\ &= f_{(k+1)}(x_1^{m-1}, \overset{(n-1)}{D(P)}, f(y_{m-1}, \overset{(n-1)}{e})) \\ &= f_{(k+1)}(x_1^{m-1}, f(\overset{(n-1)}{D(P)}, y_{m-1}), \overset{(n-1)}{e}) \\ &= f_{(k+1)}(x_1^{m-1}, f(y_{m-1}, \overset{(n-1)}{D(P)}), \overset{(n-1)}{e}) \\ &= f_{(k+1)}(x_1^{m-2}, f(x_{m-1}, y_{m-1}, \overset{(n-3)}{e}, D(P)), \overset{(n-2)}{D(P)}, e, e) \\ &= f_{(k+1)}(x_1^{m-2}, \overset{(n-1)}{D(P)}, \overset{(2)}{e}). \end{aligned}$$

If we continue in the same way, then we obtain

$$f_{(k)}(f_{(k)}(x_1^m),\ y_m^2) = f_{(k+1)}(x_1,\ \overset{(n-1)}{D(P)},\ \overset{(m-1)}{e}),$$

and since e is a scalar neutral element of P, then $f_{(k)}(f_{(k)}(x_1^m), y_m^2) = f(x_1, \overset{(n-1)}{D(P)})$. Finally

$$f_{(2k+1)}(x_1^m, y_m^2, y_1, \overset{(n-2)}{e}) = f(f_{(2k)}(x_1^m, y_m^2), y_1, \overset{(n-2)}{e})$$
$$= f(f(x_1, \overset{(n-1)}{D(P)}), y_1, \overset{(n-2)}{e}) = f(f(x_1, y_1, \overset{(n-3)}{e}, D(P)), e, \overset{(n-2)}{D(P)})$$
$$= f(D(P), e, \overset{(n-2)}{D(P)}) = D(P).$$

Therefore, $f_{(2k+1)}(x_1^m, y_m^1, \overset{(n-2)}{e}) = D(P)$. □

Corollary 3.21. *Let P be an n-ary polygroup. Then*

(1) *If $\prod_{C_\beta}(P) \neq \emptyset$ then $w_P \in \prod_{C_\beta}(P)$ and w_P is m-ary hyperproduct.*

(2) *If $\prod_{C_\gamma}(P) \neq \emptyset$ then $D(P) \in \prod_{C_\gamma}(P)$ and $D(P)$ is m-ary hyperproduct.*

Theorem 3.22. *Let P be an n-ary polygroup. Then*

(1) *If $P\backslash w_P$ is an m-ary hyperproduct, then w_P is m-ary hyperproduct and $w_P \in \prod_{C_\beta}(P)$.*

(2) *If $P\backslash D(P)$ is an m-ary hyperproduct, then $D(P)$ is m-ary hyperproduct and $D(P) \in \prod_{C_\gamma}(P)$.*

Proof. (1) Since w_P is a β-part, then $P\backslash w_P$ is also β-part.
Now, $P\backslash w_P \in \prod_{C_\beta}(P)$ and by Corollary 3.21, the proof is completed.
The proof of (2) is similar. □

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