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# A VARIATIONAL APPROACH TO THE PROBLEM OF OSCILLATIONS OF AN ELASTIC HALF CYLINDER

M. HASANSOY

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Abstract. This paper is devoted to the spectral theory (more precisely, to the variational theory of the spectrum) of guided waves in an elastic half cylinder. We use variational methods to investigate several aspects of propagating waves, including localization (see Figure 1), existence criteria and the formulas to find them. We approach the problem using two complementary methods: The variational methods for non-overdamped operator pencils to describe eigenvalues in definite spectral zones, and Ljusternik-Schnirelman critical point theory to investigate eigenvalues in the mixed spectral zone where the classical variational theory of operator pencils is not applicable.

## 1. **Introduction and preliminary facts**

Oscillations in an elastic continuum (see [10]) shaped like a semicylinder $Q = R^+_{x_1} \times \Omega \subset R^3$, $\Omega = \{(0, x_2, x_3)\}$, are described by the following system of partial differential equations:

$$\sum_{k=1}^{3} \frac{\partial \sigma_{ik}(v)}{\partial x_k} = \rho \frac{\partial^2 v_i}{\partial t^2}, \quad i = 1, 2, 3, \tag{1.1}$$

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where $v(t, x_1, x_2, x_3) = (v_1, v_2, v_3)$ and $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary, and

$$\sigma(v) = (\sigma_{ik}(v))_{i,k=1}^3, \; \sigma_{ik}(v) = \lambda\delta_{ik}\mathrm{div}v + \mu\Big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\Big)$$

is the stress tensor. Here $\lambda$, $\mu > 0$ are the Lame constants, $\rho = \rho(x_2, x_3) \in C(\bar{\Omega})$, and $\rho > 0$ is the density of the medium. We consider a fixed boundary condition on the lateral surface $\Gamma = R^+_{x_1} \times \partial\Omega$:

$$v|_\Gamma = 0. \tag{1.2}$$

In this paper, all solutions of differential equations are obtained by separation of variables. That is, we look for weak solutions $v(t, x_1, x_2, x_3)$ of the form $e^{iwt}W(x_1, x_2, x_3)$. By substituting this into equations (1.1) and by using (1.2), we obtain steady-state oscillations of the semi-cylinder $Q$:

$$\sum_{k=1}^{3} \frac{\partial\sigma_{ik}(W)}{\partial x_k} + \rho w^2 W_i = 0, \quad i = 1, 2, 3, \tag{1.3}$$

with the condition on the lateral surface

$$W(x_1, x_2, x_3)|_\Gamma = 0.$$

Now we look for a solution to equation (1.3) of the form $W(x_1, x_2, x_3) = e^{-ikx_1}u(x_2, x_3)$. This assumption yields a two-parameter spectral problem in the space of vector-valued functions $L_2^3(\Omega) := L_2(\Omega) \oplus L_2(\Omega) \oplus L_2(\Omega)$:

$$\begin{aligned} L(k, w)u := (A + kB + k^2C - w^2R)u = 0, \\ u|_{\partial\Omega} = 0, \end{aligned} \tag{1.4}$$

where

$$A = -\begin{pmatrix} \mu\Delta & 0 & 0 \\ 0 & \mu\Delta + (\lambda+\mu)D_2^2 & (\lambda+\mu)D_2D_3 \\ 0 & (\lambda+\mu)D_2D_3 & \mu\Delta + (\lambda+\mu)D_3^2 \end{pmatrix},$$

$$B = -i(\lambda+\mu)\begin{pmatrix} 0 & D_2 & D_3 \\ D_2 & 0 & 0 \\ D_3 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \lambda+2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

$$R = \begin{pmatrix} \rho(x_2, x_3) & 0 & 0 \\ 0 & \rho(x_2, x_3) & 0 \\ 0 & 0 & \rho(x_2, x_3) \end{pmatrix},$$

$\Delta = D_2^2 + D_3^2$; $D_k = \frac{\partial}{\partial x_k}$, $k = 2, 3$ and $L_2(\Omega) = \{f \,\big|\, \int_\Omega |f|^2 \, d\mu < \infty\}$. By applying the boundary condition $u|_{\partial\Omega} = 0$, we obtain the domain $\mathcal{D}(L)$ of the unbounded operator pencil $L$. This domain, sometimes called the energetic space of the problem, is denoted by $\mathcal{H} := \mathcal{D}(L)$. Clearly, $\mathcal{H} = W_0^{1,2}(\Omega) \oplus W_0^{1,2}(\Omega) \oplus W_0^{1,2}(\Omega)$ (for facts on Sobolev spaces we cite [3]).

Now we define some spectral sets. Let $k$ and $w$ be complex numbers. We say that a complex number $k$ is an eigenvalue (or wave number) and $w$ is a frequency (or eigen-frequency) if there is a $0 \neq u \in L_2^3(\Omega)$ such that

$$Au + kBu + k^2 Cu = w^2 Ru.$$

The spectrum of the operator pencil $L(k, w)$ is defined as $\sigma(L) := \{(k, w) |\, 0 \in \sigma(L(k, w))\}$, where $L(k, w)$ denotes the value of $L$ at the point $(k, w)$. Define $\sigma_w(L) := \left\{k \,\middle|\, (k, w) \in \sigma(L)\right\}$ and $\sigma_k(L) := \left\{w \,\middle|\, (k, w) \in \sigma(L)\right\}$. In this paper, we investigate the variational theory of eigenvalue spectra for the two parameter operator pencils $L(k, w)$ defined in equation (1.4).

Many interesting and related problems have been investigated in the literature. Kostyuchenko and Orazov studied the structures of dispersion curves $k(w)$, how the motion of wave numbers $k$ depends on $w$, and completeness problems for eigenvectors and associate vectors [10] (see also [8] for basis problems). A. A. Shkalikov recently studied similar questions for dissipative operator functions [11]. More general results in the abstract theory of regular waveguides may be found in [15]. However, the questions of this paper are distinct from those examined in [10, 11] and [15]. Similar analyses of definite-type eigenvalues for abstract waveguides can be found in [1] and [2]. The main differences between the results of this paper and those given in [1, 2] are as follows.

i) We are interested in a concrete, real-valued mechanical problem and its associated two-parameter operator pencils.

ii) We set the problem in a real Hilbert space, which allows us to obtain deeper results in some cases.

iii) In the literature on nonlinear spectral problems, including [1] and [2], no spectral problems are studied in the mixed spectral zone

$[\delta_-(w), \delta_+(w)]$ (see Figure 1 below). For the first time in the literature, we apply Ljusternik-Schnirelman critical point theory to obtain some results concerning this zone.

The main concern of this paper is propagating waves. In addition to finding solutions to the wave equation, we derive existence criteria and results concerning wave localization. The problem of finding frequencies $w$ for a fixed wave number $k$ is an overdamped problem, so we use a classical Poincaré-Ritz type variational principle to describe the waves. However, the problem of finding wave numbers $k$ for a fixed frequency $w$ in a definite spectral zone is not overdamped. In this case we use other variational principles, such as that of Voss and Werner [13] and [7]. To study eigenvalues in a mixed spectral zone, we apply Ljusternik-Schnirelman critical point theory. There is currently no other method for studying eigenvalues in a mixed spectral zone, except for linear operator pencils of the form $L(\lambda) = \lambda A - B$ (see [4]).

Propagating waves are solutions of equation (1.1) in the form $v(t, x_1, x_2, x_3) = u(x_2, x_3)e^{i(wt-kx_1)}$, where $w$ and $k$ are real numbers. On the other hand $v(t, x_1, x_2, x_3) = u(x_2, x_3)e^{i(wt-kx_1)}$ is a solution of (1.1) if and only if the triple $(k, w, u)$ is a solution of the eigenvalue problem (1.4). Particularly, we are interested in neutral (resonance) pairs $(k, u)$ for a fixed $w$. A pair $(k, u)$, for a fixed $w$ is said to be a neutral pair if $0 \neq u \in \mathcal{H}$,

$$Au + kBu + k^2Cu = w^2Ru \text{ and } (L'(k)u, u) = 0, \tag{1.5}$$

where $L(k) := Au + kBu + k^2Cu$.

We start by deriving some facts about the operators $A$, $B$, $C$ and $R$ defined above, which are very important to the investigation. The original space for the problem is $L_2^3(\Omega)$, and the energetic space where we define the solution is $\mathcal{H} := W_0^{1,2}(\Omega) \bigoplus W_0^{1,2} \bigoplus W_0^{1,2}$ or $\mathcal{H} = D(A^{\frac{1}{2}})$ for Drichlet problems (see [10]). The following Proposition shows that the operators $A, B, C$ and $R$ are well defined on $\mathcal{H}$. We extend the operator $A$ by $(Au, u) := (A^{\frac{1}{2}}u, A^{\frac{1}{2}}u)$. In what follows, the notations $(u, v)$, $\|u\|$ and $\Big((u, v)_{\mathcal{H}}, \|u\|_{\mathcal{H}}\Big)$ denote the scalar product and norm in the space $L_2^3(\Omega)$ $(\mathcal{H})$, respectively.

**Proposition 1.1. I)** *The operator $A$ is a self-adjoint, positive-definite; i.e., $(Au, u) \geq \delta(u, u)$, for some $\delta > 0$ and all $u \in D(A)$ with the discrete spectrum,*

**II)** *The operators $C$ and $R$ are bounded and positive definite in $L_2^3(\Omega)$.*

**III)** *The operator $B$ is symmetric in $L_2^3(\Omega)$ and $|(Bu,v)| \leq \|u\|_{\mathcal{H}}\|v\| + \|v\|_{\mathcal{H}}\|u\|$, for all $u, v \in \mathcal{H}$. In particular, this statement means that $D(A^{\frac{1}{2}}) \subset D(B)$.*

**IV)** *(the coercivity condition) $(L(k)u,u) := (Au,u) + k(Bu,u) + k^2(Cu,u) \geq \|u\|_{\mathcal{H}}$, for all $u \in \mathcal{H}$.*

*Proof.* **I)** The operator $A$ is defined by

$$A = -\begin{pmatrix} \mu\Delta & 0 & 0 \\ 0 & \mu\Delta + (\lambda+\mu)D_2^2 & (\lambda+\mu)D_2D_3 \\ 0 & (\lambda+\mu)D_2D_3 & \mu\Delta + (\lambda+\mu)D_3^2 \end{pmatrix}.$$

Let $u(x_2,x_3) = (u_1(x_2,x_3), u_2(x_2,x_3), u_3(x_2,x_3)) \in \mathcal{H} \subset L_2^3(\Omega)$. Then $Au = -\big(\mu\Delta u_1, (\mu\Delta + (\lambda+\mu)D_2^2)u_2 + (\lambda+\mu)D_2D_3u_3, (\lambda+\mu)D_2D_3u_2 + (\mu\Delta + (\lambda+\mu)D_3^2)u_3\big)$. Hence,

$$(Au,u) = -\mu\int_\Omega \Delta u_1.\bar{u}_1\,dx - \mu\int_\Omega \Delta u_2.\bar{u}_2\,dx_2dx_3 -$$
$$(\lambda+\mu)\int_\Omega \frac{\partial^2 u_2}{\partial x_2^2}\bar{u}_2 dx_2dx_3 - (\lambda+\mu)\int_\Omega \frac{\partial^2 u_3}{\partial x_2\partial x_3}\bar{u}_2 dx_2dx_3 -$$
$$(\lambda+\mu)\int_\Omega \frac{\partial^2 u_2}{\partial x_2\partial x_3}\bar{u}_3 dx_2dx_3 - \mu\int_\Omega \Delta u_3.\bar{u}_3\,dx_2dx_3 -$$
$$(\lambda+\mu)\int_\Omega \frac{\partial^2 u_3}{\partial x_3^2}\bar{u}_3 dx_2dx_3 = \mu\int_\Omega |\nabla u_1|^2\,dx_2dx_3 +$$
$$\mu\int_\Omega |\nabla u_2|^2\,dx_2dx_3 + \mu\int_\Omega |\nabla u_3|^2\,dx_2dx_3 + (\lambda+\mu)\Big[\int_\Omega \Big|\frac{\partial u_2}{\partial x_2}\Big|^2\,dx_2dx_3 +$$
$$\int_\Omega \Big|\frac{\partial u_3}{\partial x_3}\Big|^2\,dx_2dx_3 + \int_\Omega \frac{\partial u_3}{\partial x_3}\frac{\partial \bar{u}_2}{\partial x_2}\,dx_2dx_3 + \int_\Omega \frac{\partial u_2}{\partial x_2}\frac{\partial \bar{u}_3}{\partial x_3}\,dx_2dx_3\Big].$$

Now by using Hölder's inequality we obtain

$$\int_\Omega \frac{\partial u_3}{\partial x_3}\frac{\partial \bar{u}_2}{\partial x_2}\,dx_2dx_3 + \int_\Omega \frac{\partial u_2}{\partial x_2}\frac{\partial \bar{u}_3}{\partial x_3}\,dx_2dx_3 = 2Re\int_\Omega \frac{\partial u_3}{\partial x_3}\frac{\partial \bar{u}_2}{\partial x_2}\,dx_2dx_3 \leq$$
$$2\Big[\int_\Omega \Big|\frac{\partial u_3}{\partial x_3}\Big|^2\,dx_2dx_3\Big]^{\frac{1}{2}}\Big[\int_\Omega \Big|\frac{\partial u_2}{\partial x_2}\Big|^2\,dx_2dx_3\Big]^{\frac{1}{2}} \leq$$
$$\int_\Omega \Big|\frac{\partial u_2}{\partial x_2}\Big|^2\,dx_2dx_3 + \int_\Omega \Big|\frac{\partial u_3}{\partial x_3}\Big|^2\,dx_2dx_3.$$

On the other hand Poincaré inequality yields $\int_\Omega |\nabla u|^2 dx_2dx_3 \geq \lambda_1(\Delta)\int_\Omega |u|^2 dx_2dx_3$. Consequently, $(Au,u) \geq 3\mu\lambda_1(\Delta)\|u\|^2$, i.e., $A$ is positive definite. Finally, it is known that the embedding $W_0^{1,2}(\Omega) \hookrightarrow$

$L_2(\Omega)$ is compact (see [3]). This means that the spectrum of $A$ is compact.

**II)** This property follows immediately from the definitions of operators $C$ and $R$.

**III)** The operator $B$ is defined by

$$B = -i(\lambda+\mu)\begin{pmatrix} 0 & D_2 & D_3 \\ D_2 & 0 & 0 \\ D_3 & 0 & 0 \end{pmatrix}.$$

Now setting $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ we can write

$$(Bu, v) = -i(\lambda+\mu)\Big[\int_\Omega (\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3})\bar{v}_1\, dx_2 dx_3 + \int_\Omega \frac{\partial u_1}{\partial x_2}\bar{v}_2\, dx_2 dx_3 + \int_\Omega \frac{\partial u_1}{\partial x_3}\bar{v}_3\, dx_2 dx_3\Big] = i(\lambda+\mu)\Big[\int_\Omega \frac{\partial \bar{v}_1}{\partial x_2}u_2 + \frac{\partial \bar{v}_1}{\partial x_3}u_3\, dx_2 dx_3\Big] - i(\lambda+\mu)\Big[\int_\Omega \frac{\partial u_1}{\partial x_2}\bar{v}_2\, dx_2 dx_3 + \int_\Omega \frac{\partial u_1}{\partial x_3}\bar{v}_3\, dx_2 dx_3\Big].$$

From this formula by using Hölder's inequality we obtain that $|(Bu, v)| \le \|u\|_{\mathcal{H}}\|v\| + \|v\|_{\mathcal{H}}\|u\|$.

**IV)** This conclusion is standard in the spectral theory of the regular waveguides (see [15], p. 44, for general hyperbolic equations in a cylinder). □

**Remark 1.2.** *It follows from Condition III) that the operator $B$ is $A$-compact; i.e., $A^{\frac{1}{2}}BA^{\frac{1}{2}}$ is compact (see [9]). Condition IV) is often replaced by the so-called energetic stability condition: $(Au, u) + k(Bu, u) + k(Cu, u) \ge (c_0k^2 + \zeta)(Ru, u)$ for some $c_0 > 0$, $\zeta > 0$, which is stronger than the coercivity condition. In our case, this condition is also satisfied.*

Proposition 1.1 means that the operator pencil $L(k, w)$ is a pencil of the waveguide type (w.g.t.). A good resource on this subject is the book *Spectral Theory of Guided Waves* by A. Silbergleit and Yu. Kopilevich [15], which discusses a wide variety of waveguides. However, this book does not include any variational problems. The following theorem makes use of results given in Chapter 11 of [15].

**Theorem 1.3.** *1) The spectra $\sigma_w(L)$ and $\sigma_k(L)$ are discrete for a fixed complex $w$ and complex $k$ respectively,*

*2) The set $\sigma_{\mathbb{R}}(L) := \{(k, w)\,\big|\,(k, w) \in \sigma(L),\ k \in \mathbb{R},\ w \in \mathbb{R}\}$ lies inside the hyperbola $w^2 - c_0^2k^2 \ge \zeta$,* *3)There are no real eigenvalues if* $w < \sqrt{\zeta}$,

*4) If* $w^2 = \zeta$*, then the only real eigenvalue permitted is* $k = 0$*. Zero will be an eigenvalue only if* $\zeta$ *is an eigenvalue of the problem* $Au = \lambda Ru$,

*5)* $(k, w) \in \sigma(L)$ *and* $Imk = 0$ *imply* $Imw = 0$.

For more information about localization problems for general operator functions, see [5].

## 2. **Propagating waves: existence criteria, classification, and methods of finding them**

As mentioned in the previous section,
$v(t, x, x_2, x_3) = u(x_2, x_3)e^{i(wt-kx)}$ is a propagating wave if and only if $w$, $k$ are real and the equation

$$Au + kBu + k^2Cu = w^2Ru, \tag{2.1}$$

has a nontrivial solution $u \in \mathcal{H}$. Notice that $u \in \mathcal{H}$ is a solution of (2.1) if and only if

$$(A^{\frac{1}{2}}u, A^{\frac{1}{2}}v) + k(Bu, v) + k^2(Cu, v) = w^2(Ru, v)$$

holds for all $v \in \mathcal{H}$.

To classify all propagating waves for a fixed real $w$ we need the following definition.

**Definition 2.1.** *We say that an eigenvalue* $k$ *of the problem* (2.1) (*for a fixed real* $w$) *is of positive* (*negative*) *type if* $(L'(k)u, u) > 0$ $((L'(k)u, u) < 0)$ *for all non-zero eigenvectors corresponding to* $k$*, respectively.*

Neutral pairs $(k, w)$ are defined by (1.5). Let us fix $w$ and define

$$p_{\pm}(u, w) = \frac{-(Bu, u) \pm \sqrt{d(u, w)}}{2(Cu, u)},$$

where $d(u, w) = (Bu, u)^2 - 4((A - w^2R)u, u)(Cu, u)$. The functionals $p_{\pm}(u, w)$ are the roots of the equation $(L_w(k)u, u) = 0$, $u \neq 0$. Clearly, if $k \in \mathbb{R}$ is an eigenvalue and $u \neq 0$ is its corresponding eigenvector, then either $p_+(u, w) = k$ or $p_-(u, w) = k$. Moreover, at each eigenvector we have $d(u, w) \geq 0$. Now we define two conic subsets, which will play an important role in finding the real eigenvalues by variational principles: $G(w) = \{u \in \mathcal{H} \mid d(u, w) > 0\}$ and $G'(w) = \{u \in \mathcal{H}\backslash\{0\} \mid d(u, w) \geq 0\}$. The set $G'(w) = \{u \in \mathcal{H}\backslash\{0\} \mid$

$d(u,w) \geq 0\}$ will also be the domain of the functionals $p_{\pm}(u,w)$. Let $W_{p_{\pm}}(w) = \{p_{\pm}(u,w) | u \in G(w)\}$ be the numerical range of $p_{\pm}(u,w)$ on $G(w)$. Now define the bounds of the ranges of $p_{\pm}(x,w)$ on $G(w)$ and $G'(w)$ as $k_-(w) = \inf W_{p_-}(w)$, $\quad k'_-(w) = \inf_{u \in G'(w)} p_-(u,w)$, $k_+(w) = \sup W_{p_+}(w)$, $\quad k'_+(w) = \sup_{u \in G'(w)} p_+(u,w)$, $\delta_-(w) = \inf W_{p_+}(w)$, $\delta_+(w) = \sup W_{p_-}(w)$. Notice that the inequalities $k'_-(w) \leq k_-(w) \leq \delta_-(w) \leq \delta_+(w) \leq k_+(w) \leq k'_+(w)$ hold for all two-parameter operator pencils of waveguide type Only one inequality is not obvious: $\delta_-(w) \leq \delta_+(w)$. Its proof for one-parameter, bounded operator pencils of waveguide type may be found in [2] (p. 1279). These inequalities will lead us to the distribution and a classification of real eigenvalues as in Figure 1.

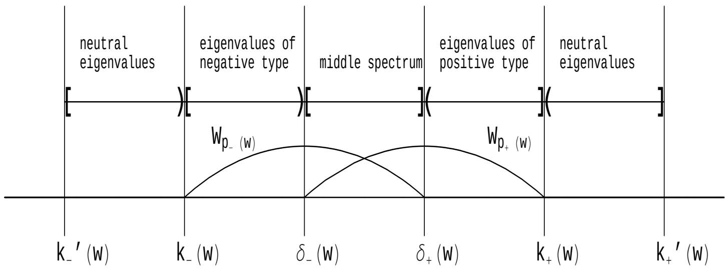


Fig 1. The distribution of the real eigenvalues.

Evidently, all mixed-type eigenvalues belong to the middle spectrum. The end intervals $[k'_-(w), k_-(w))$ and $(k_+(w), k'_+(w)]$ contain only resonance waves. All waves in $[k_-(w), \delta_-(w))$ are outgoing, and all waves in $(\delta_+(w), k_+(w)]$ are incoming. The middle interval $[\delta_-(w), \delta_+(w)]$ may contain any kind of wave. The following theorem is about the structure of frequencies for a fixed real $k$.

**Theorem 2.2.** *Let us fix $k \in \mathbb{R}$. Then the following facts hold:*

*1) There are a countable number of real frequencies denoted by $w_n^{\pm}(k)$, which lie in the intervals $[\sqrt{\zeta + c_0^2 k^2}, +\infty)$ and $(-\infty, -\sqrt{\zeta + c_0^2 k^2}]$.*

*2) All real frequencies are described by*

$$w_n^2(k) = \min_{\substack{L \subset \mathcal{H} \\ \dim L = n}} \max_{\substack{u \in L \\ u \neq 0}} \frac{(L(k)u,u)}{(Ru,u)}, \; n = 1, 2, ..., \tag{2.2}$$

*where $L(k) := A + kB + k^2 C$.*

*3)* $w_n^{\pm}(k) \to \pm\infty$ *for* $k \to \pm\infty$*, and* $\frac{w_n^{+}(k)}{\lambda_n} \to 1$ *for* $n \to \infty$*. The numbers* $\lambda_n$*,* $n = 1, 2, ...,$ *are eigenvalues of the generalized eigenvalue problem* $Au = \lambda Ru$*.*

*Proof.* 1) It follows from the statement 2) of Theorem 1.1.

2) In this case, eigenvalue problem (2.1) reduces to the generalized eigenvalue problem $L(k)u = w^2 Ru$. Therefore (2.2) is the Poincaré-Ritz principle for this problem, taking into account the fact that the domain of the operator pencil $L(k)$ is the Sobolev space $\mathcal{H}$.

3) We have $L(k,w) \geq (c_0^2 k^2 + \zeta)R$ (see Remark 1.1). Then $\frac{(L(k,w)u,u)}{(Ru,u)} \geq (c_0^2 k^2 + \zeta)$, and it follows from this inequality that $w_n^{\pm}(k) \to \pm\infty$ for $k \to \pm\infty$. Finally, by Proposition 1.1 we obtain that the spectra of the operator pencils $L(k,w)$ and $A^{-\frac{1}{2}} L(k,w) A^{-\frac{1}{2}}$ coincide. On the other hand, the pencil $A^{-\frac{1}{2}} L(k,w) A^{-\frac{1}{2}}$ is a compact perturbation of the operator $A$, and that is why $\frac{w_n^{+}(k)}{\lambda_n} \to 1$ for $n \to \infty$. (See [6, Chapter V], [15, Theorem 13.1] and [9]). □

We remark that Figure 1 characterizes the distribution of real eigenvalues for *all regular waveguides*, not just the half-cylinder (see [15] for similar results for abstract waveguides). However, for a specific problem one can obtain more results. For example, in our case all operators $A, C$ and $R$ transform real-valued functions to real-valued functions. But operator $B$ maps real-valued functions to complex-valued functions. Using this fact we can simplify the problem as follows.

Let us consider a propagating wave $v(t, x, x_2, x_3) = u(x_2, x_3)e^{i(wt - kx_1)}$. There are two possible cases:

A) The function $u(x_2, x_3)$ is a real-valued function in $\mathcal{H}$;

B) The function $u(x_2, x_3)$ a complex-valued.

For case A), it follows from the eigenvalue problem (2.1) that

$$\begin{cases} Au + k^2 Cu = w^2 Ru \\ Bu = 0, \end{cases} \tag{2.3}$$

where $u \in \mathcal{H}$ and $k, w \in \mathbb{R}$. One can therefore pose the following problem in the real-valued space $\mathcal{H}$: find solutions to equation (1.1) of the form $v(t, x, x_2, x_3) = u(x_2, x_3)\cos(wt - kx_1)$ and $v(t, x, x_2, x_3) = u(x_2, x_3)\sin(wt - kx_1)$, where $u \in \mathcal{H}$ and $k, w \in \mathbb{R}$. This problem is equivalent to that posed in (2.3).

Our first observation for The case A), is given in the following theorem.

**Theorem 2.3.** *a)* $(Bu, u) = 0$ *for an arbitrary real-valued function* $u \in \mathcal{H}$*,*

*b)* $\delta_+(u, w) = \delta_-(u, w) = 0$*;*

*c) there are only a finite number of real eigenvalues for a fixed real* $w$*; all positive (negative) eigenvalues are of positive type (negative type), and arranged as*

$$\pm k_1^{\pm}(w) \geq \pm k_2^{\pm}(w) \geq ... \geq \pm k_n^{\pm}(w),$$

*respectively.*

*d)* $k'_-(w) = k_-(w)$ *and* $k'_+(w) = k_+(w)$*; i.e.,* $[k'_-(w), k_-(w)) = \emptyset$ *and* $(k_+(w), k'_+(w)] = \emptyset$*;*

*e) The only neutral (resonance) eigenvalue permitted is* $k = 0$*. Zero will be a neutral eigenvalue only if* $w^2$ *is an eigenvalue of the generalized eigenvalue problem* $Au = w^2 Ru$*.*

*Proof.* a) Using the integration by-parts in the space $\mathcal{H} = W_0^{1,2}(\Omega) \oplus W_0^{1,2}(\Omega) \oplus W_0^{1,2}(\Omega)$ yields

$$(Bu, u) = -i(\lambda + \mu)\Big[\int_\Omega (\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}) u_1 \, dx_2 dx_3 + \int_\Omega \frac{\partial u_1}{\partial x_2} u_2 \, dx_2 dx_3 + \int_\Omega \frac{\partial u_1}{\partial x_3} u_3 \, dx_2 dx_3\Big] = -i(\lambda + \mu)\Big[\int_\Omega -\frac{\partial u_1}{\partial x_2} u_2 \, dx_2 dx_3 - \frac{\partial u_1}{\partial x_3} u_3 \, dx_2 dx_3 + \int_\Omega \frac{\partial u_1}{\partial x_2} u_2 \, dx_2 dx_3 + \int_\Omega \frac{\partial u_1}{\partial x_3} u_3 \, dx_2 dx_3\Big] = 0.$$

b) It follows from a) that

$$p_\pm(u, w) = \pm\sqrt{\frac{((w^2 R - A)u, u)}{(Cu, u)}}.$$

Hence, $p_+(u, w) \geq 0$ and $p_-(u, w) \leq 0$. By the definition of $\delta_\pm(w)$, we obtain

$$\delta_+(w) = \sup W_{p_-} = \sup\{p_-(u, w) | u \in G(w)\} \leq 0$$

and

$$\delta_-(w) = \inf W_{p_+} = \inf\{p_+(u, w) | u \in G(w)\} \geq 0.$$

On the other hand, $\delta_-(w) \leq \delta_+(w)$. Consequently, $\delta_-(w) = \delta_+(w) = 0$.

c) By Theorem 1.1, all eigenvalues of problem (2.3) belong to the interval $\Big[-\frac{\sqrt{w^2 - \zeta}}{c_0}, \frac{\sqrt{w^2 + \zeta}}{c_0}\Big]$ and $\sigma_w(L)$ is discrete. These facts mean that there is no finite concentration point of the spectrum $\sigma_w(L)$. Consequently, there are only a finite number of eigenvalues in $\Big[-\frac{\sqrt{w^2 - \zeta}}{c_0}, \frac{\sqrt{w^2 + \zeta}}{c_0}\Big]$.

The fact that all positive (negative) eigenvalues are of positive type (negative type), respectively, follows from $(L'(k)u,u) = (Bu,u)+2k(Cu,u) = 2k(Cu,u)$.

d) According to Figure 1 we have $L_w(k) \geq 0$ for $k \in [k'_-(w), k_-(w)) \cup (k_+(w), k'_+(w)]$. Thus, the inequality

$$\|L_w(k)u\|^2 \leq (L_w(k)u,u)\|L_w(k)\|$$

holds. It follows from this inequality that all values of the functionals $p_\pm(u,w)$ in the intervals $[k'_-(w), k_-(w))$ and $(k_+(w), k'_+(w)]$ consist of neutral eigenvalues. On the other hand, $(L'_w(k)u,u) = 2k(Cu,u) \neq 0$ for $k \neq 0$. Hence, there are no neutral eigenvalues on $(-\infty, 0)$ and $(0, +\infty)$. This proves d).

e) This statement has already been proved in c). □

More generally, considering cases A) and B) together, we obtain the following result.

**Theorem 2.4.** *a) All eigenvalues in $[k_-(w), \delta_-(w))$, arranged in an increasing order, are described by the following two principles*

$$k^i_-(w) = \min_{\substack{L\subset G_-\\ \dim L=i}} \max_{u\in L} p_-(u,w), \qquad i = 1,2,...,n.$$

$$k^i_-(w) = \max_{\substack{L\subset H\\ \dim L=i-1}} \inf_{\substack{u\in G_-\\ u\perp L}} p_-(u,w) \qquad i = 1,2,...,n,$$

*where $G_-(w) = \{u|u \in G(w),\ p_-(u,w) < \delta_-(w)\}$ . Similar formulas hold for the eigenvalues in $(k_+(w), k'_+(w)]$.*

*b) For a fixed $w \in \mathbb{R}$, the spectral set $\sigma_w(L)$ is nonempty and $k_\pm(w) \in \sigma_w(L)$. Thus, at least two propagating waves always exist.*

*Proof.* a) By Proposition 1.1, the pencil we are studying is a waveguide type operator pencil. Variational principles for real eigenvalues in the intervals $[k_-(w), \delta_-(w))$ and $(k_+(w), k'_+(w)]$ have already been studied for this case in [1, 2], and [7]. Thus, a) holds.

b) Let us show that $k_-(w)$ is an eigenvalue. By definition $k_-(w) = \inf W_{p_-}(w)$, there is a $u^-_n \in G(w)$, $u^-_n = 1$ such that $p_-(u^-_n, w) \to k_-(w)$. Since $\|u^-_n\| = 1$, there is a weakly convergent subsequence $u^-_{n_m}$ of $u^-_n$. Thus, $u^-_{n_m} \rightharpoonup u$ for $m \to \infty$. Using the definition of $p_-(u,w)$, we get $0 \leq \big(L_w(k_-(w))u^-_{n_m}, u^-_{n_m}\big) = \big((L_w(k_-(w)) - L_w(p_-(u^-_{n_m})))u^-_{n_m}, u^-_{n_m}\big) \leq \|L_w(k_-(w)) - L_w(p_-(u^-_{n_m}, w))\|$. Hence, $\big(L_w(k_-(w))u^-_{n_m}, u^-_{n_m}\big) \to 0$ for

$m \to \infty$ by continuity of $L_w(k)$. On the other hand, since $L_w(k_-(w)) \geq 0$, we obtain

$$\|L_w(k_-(w))u^-_{n_m}\|^2 \leq \big(L_w(k_-(w))u^-_{n_m}, u^-_{n_m}\big)\|L_w(k_-(w))\|.$$

Taking the limit for $m \to \infty$, we find that $L_w(k_-(w))u^-_{n_m} \to 0$ and consequently $L_w(k_-(w))u = 0$. Thus, $\exists u^-_{n_m}$, $\|u^-_{n_m}\| = 1$, $u^-_{n_m} \rightharpoonup u$ and $L_w(k_-(w))u = 0$. Evidently, $u \neq 0$; otherwise, $k_-(w)$ would be a point of continuous spectrum of the operator pencil $L_w(k)$. But by Theorem 1.1, the spectrum $\sigma_w(L)$ is discrete. This contradiction means that $u \neq 0$ and $L_w(k_-(w))u = 0$; i.e., $k_-(w)$ is an eigenvalue. By the same argument, one can establish that $k_+(w)$ is an eigenvalue. □

## 3. Eigenvalues in the interval $[\delta_-(w), \delta_+(w)]$: A critical point approach

In this section, we study eigenvalues in the interval $[\delta_-(w), \delta_+(w)]$. This part of the spectrum is known as the middle spectrum. We are not aware of any prior results from the literature in this spectral zone, which contains all mixed-type eigenvalues (but not only this type). First, we note that classical variational theory does not work in the zone $[\delta_-(w), \delta_+(w)]$. We suggest two methods. The first is to transform this problem into an eigenvalue problem for a self-adjoint operator in a Krein space. Then, by using the methods given in [4], some eigenvalues in $[\delta_-(w), \delta_+(w)]$ may be found. The second way is to transform problem (2.1) into an eigenvalue problem for a nonlinear operator in a real Hilbert space, and then apply Ljusternik- Schnirelman critical point theory (see [14]) to find neutral pairs in $[\delta_-(w), \delta_+(w)]$. In this paper, we take the second approach.

Recall that for a fixed $w$, a pair $(k, u)$ is said to be a neutral pair if

(3.1) $$Au + kBu + k^2Cu = w^2Ru \text{ and } (L'(k)u, u) = 0, \;\; 0 \neq u \in \mathcal{H},$$

where $L(k) := Au + kBu + k^2Cu$.

First, we are going to set the problem in a real-valued Hilbert space. Let $(k, w) \in \mathbb{R} \times \mathbb{R}$ and $0 \neq u \in \mathcal{H}$ satisfy (3.1). Setting $u(x_2, x_3) = u_1(x_2, x_3) + iu_2(x_2, x_3)$ and $B = iB_0$ into (3.1), where

$$B_0 = -(\lambda + \mu)\begin{pmatrix} 0 & D_2 & D_3 \\ D_2 & 0 & 0 \\ D_3 & 0 & 0 \end{pmatrix},$$

we obtain

$$A(u_1 + iu_2) + ikB_0(u_1 + iu_2) + k^2(Cu_1 + iCu_2) = w^2(Ru_1 + iu_2).$$

Consequently,

$$\begin{aligned} Au_1 - kB_0u_2 + k^2Cu_1 &= w^2Ru_1 \\ Au_2 + kB_0u_1 + k^2Cu_2 &= w^2Ru_2. \end{aligned}$$

This system may be written in the following operator-matrix form:

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + k \begin{pmatrix} 0 & -B_0 \\ B_0 & 0 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + k^2 \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$= w^2 \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Put $\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} 0 & -B_0 \\ B_0 & 0 \end{pmatrix}$, $\tilde{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ and $\tilde{R} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ and $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. We then obtain a two-parameter spectral problem in the form

$$\tilde{A}u + k\tilde{B}u + k^2\tilde{C} = w^2\tilde{R}u, \tag{3.2}$$

where $u \in \mathcal{H} \oplus \mathcal{H}$. In what follows, we assume that $\mathcal{H}$ consists of real-valued functions. We therefore have $\tilde{A}^* = \tilde{A}$, $\tilde{C}^* = \tilde{C}$, and $\tilde{R}^* = \tilde{R}$. Further, $\tilde{B}$ is a symmetric operator, since $B_0^* = -B_0$. All statements (I-IV) given in Proposition 1.1 hold. Thus, we have a two-parameter, waveguide-type eigenvalue problem in the real Hilbert space $X = \mathcal{H} \oplus \mathcal{H}$.

Let $(k, w)$ be a neutral eigen-pair for problem (3.2), corresponding to an eigenvector $u$. It follows from $(\tilde{L}'(k)u, u) = 0$ that $k = -\frac{(\tilde{B}u,u)}{2(\tilde{C}u,u)}$. Substituting this value into (3.2), we arrive at the nonlinear generalized eigenvalue problem in the form

$$\frac{(\tilde{B}u, u)^2}{4(\tilde{C}u, u)^2}\tilde{C}u - \frac{(\tilde{B}u, u)}{2(\tilde{C}u, u)}\tilde{B}u + \tilde{A}u = w^2\tilde{R}u. \tag{3.3}$$

Clearly, the F-derivative of the functional $G(u) = \frac{1}{2}[(\tilde{A}u, u) - \frac{(\tilde{B}u,u)^2}{4(\tilde{C}u,u)}]$ is $G'(u) = \frac{(\tilde{B}u,u)^2}{4(\tilde{C}u,u)^2}\tilde{C}u - \frac{(\tilde{B}u,u)}{2(\tilde{C}u,u)}\tilde{B}u + \tilde{A}u$. We also have $F'(u) = \tilde{R}u$, where $F(u) = \frac{(\tilde{R}u,u)}{2}$. Thus, problem (3.3) is equivalent to the problem of finding the critical points of the functional $F(u)$ restricted to the level set $N_1 = \{x \big| G(u) = 1\}$. More precisely, we obtain the eigenvalue problem

$$F'(u) = \frac{1}{w^2} G'(u), \ \ u \in N_1 \tag{3.4}$$

or

$$F'(u) = \lambda G'(u), \ \ u \in N_1, \tag{3.5}$$

where $\lambda = \frac{1}{w^2}$.

Our next step is based on the Ljusternik- Schnirelman critical point theory (see [14], Chapter 44 and Theorem 44.A). We are now going to give a simple description of this theory from the point of view of our problem. We start with the following definition.

**Definition 3.1.** *Let $X$ and $Y$ be Banach spaces and $A : X \to Y$ be an operator (linear or nonlinear). We say that*

*i) The operator $A$ is strongly continuous iff $u_n \rightharpoonup u$ implies $Au_n \to Au$, where $u_n \rightharpoonup u$ denotes weak convergence in $X$, and*

*ii) The operator $A$ is bounded if $A$ maps bounded sets into bounded sets.*

Now consider $F, G : X \to \mathbb{R}$ and $F', G' : X \to X^*$, where $X^*$ denotes the adjoint of $X$. We give some basic properties of the functionals and their F-derivatives.

**Proposition 3.2.** *The functionals $F$ and $G$ belong to $C^1(X, \mathbb{R})$; $F$ and $F'$ are strongly continuous; $G'$ is bounded and uniformly continuous on bounded sets in $X$.*

*Proof.* All of these properties can easily be checked using the properties of the operators $A$, $B$, $C$ and $R$ given in Proposition 1.1, Definition 3.1, and the fact that the embedding $W_0^{1,2}(\Omega) \hookrightarrow L_2(\Omega)$ is compact. □

By $\mathcal{K}_m$ denote the class of all compact, symmetric subsets $K$ of $N_1$ such that $genK \geq m$. Here $genK$ is defined as the smallest natural number $n \geq 1$ for which there exists an odd and continuous function $f : K \to \mathbb{R}^n \setminus \{0\}$ (see [14] and [12]). Let

$$c_m = \sup_{L \subset \mathcal{K}_m} \inf_{u \in L} F(u) \ \ m = 1, 2, ...$$

Now we use the Ljusternik-Schnirelman theory to show that $c_m > 0, \ \ m = 1, 2, ...$ and that all these numbers are eigenvalues of problem (3.5), i.e, that there exists $u_m \in N_1$ such that

$$F'(u_m) = c_m G'(u_m).$$

Moreover, we have $F(u_m) = c_m$, which means that the numbers $c_m$ are critical levels for the functional $F$ on $N_1$.

Set $crit_{N_1,c}F := \{u \in N_1 \Big| F'u = c\,G'u \text{ and } F(u) = c\}$.

We start with the Palais-Smale condition, which is crucial to this theory. First define the tangential mapping $TF_{N_1}(u)$ at the vector $u \in N_1$ by $\langle TF(u), h\rangle := \langle F'(u), h\rangle$ for all $h \in TN_1(u)$, where $TN_1(u)$ denotes the set of all vectors $h$ tangent to $N_1$ at the point $u \in N_1$.

**Definition 3.3.** *Let $F, G \in C^1(X, \mathbb{R})$ and $N_1 = \{u \Big| G(u) = 1\}$. We say that the functional $F$ satisfies the Palais-Smale condition with respect to $N_1$ at a point $c \in \mathbb{R}$ if $u_n \in N_1$, $F(u_n) \to c$ and $\|TF(u_n)\| \to 0$ implies that $u_n$ has a convergent subsequence in $X$.*

It is known that (see [14], p.289, Theorem 43.C) under the conditions given in Proposition 3.1, we have $TN_1(u) = Ker(G'(u))$. Now we are ready to prove the following Proposition.

**Proposition 3.4.** *The functional $F(u) = \frac{(\tilde{R}u,u)}{2}$ satisfies the Palais-Smale condition with respect to $N_1$ at each $0 \neq c \in \mathbb{R}$.*

*Proof.* By definition $TF(u)$ is a functional, which is defined only on $TN_1(u)$. We extend this functional from $TN_1(u)$ to the whole space $X$ by

$$Du := F'(u) - F(u)G'(u).$$

Evidently, $\langle Du, h\rangle = \langle F'(u), h\rangle = \langle TF(u), h\rangle$, $h \in TN_1(u)$; i.e., $Du$ is an extension of $TF(u)$. Here, $Du \in X^*$ and $\langle Du, h\rangle$ (or $\langle F'(u), h\rangle$, $\langle TF(u), h\rangle$) denotes the value of functional $Du$ (or $F'(u)$, $TF(u)$) at $h \in X$. In particular, this fact means that $\|TF(u)\| \leq \|Du\|$. We can also prove that $\|TF(u)\| = |Du\|$. Indeed, $\langle Du, u\rangle = 0$, $u \in N_1$. Let $P$ be the orthogonal projection from $X$ to $G'(u)$. Then a vector $x \in X$ may be written in the form $x = cu + Px$. Therefore, $\big|\langle Du, x\rangle\big| = \big|\langle Du, Px\rangle\big| = \big|\langle TF(u), Px\rangle\big| \leq \|TF(u)\|.\|Px\| \leq \|TF(u)\|.\|x\|$, *i, e.*, $\|Du\| \leq \|TF(u)\|$ and consequently $\|Du\| = \|TF(u)\|$. Now let $u_n \in N_1$, $Du_n \to 0$, and $F(u_n) \to c$ for $c \neq 0$. We need to prove that $u_n$ has a convergent subsequence in $X$. By Proposition 1.1, we have $(L(k)u, u) \geq c\|u\|^2_X$ for all $u \in X$ and $k \in \mathbb{R}$. On the other hand, $\Big(L\big(\frac{(Bu,u)}{-2(Cu,u)}\big)u, u\Big) = \langle G'(u), u\rangle = 2G(u)$. Thus, $G(u) \geq c\|u\|^2_X$ for all $u \in X$. It immediately follows from this inequality that $u_n$ is a bounded sequence in $X$. Since $X$ is a reflexive Banach space, we find that $u_n$ has a weakly convergent subsequence: $u_{n_k} \rightharpoonup u$ for some $u \in \bar{co}N_1$, the convex hull of $N_1$. Now it follows from the strong continuity of $F'$ that $F'(u_{n_k}) \to F'(u)$ as $k \to \infty$. By Proposition 3.1, $G'(u_n)$ is also bounded. It follows from $Du_n = F'(u_n) - F(u_n)G'(u_n)$ that

$$G'(u_n) = \frac{F'(u_n) - D(u_n)}{F(u_n)} \to \frac{F'(u)}{c}.$$

Hence, we have $u_{n_k} \rightharpoonup u$ and $G'(u_{n_k}) \to \frac{F'(u)}{c}$. To prove that $u_{n_k} \to u$, it is enough to show that $\|u_{n_k}\| \to \|u\|$. For the sake of simplicity, we check it for the case $u = 0$. Thus, $u_{n_k} \rightharpoonup 0$ and $G'(u_{n_k}) \to \frac{F'(0)}{c} = 0$. Again by Proposition 1.1, $(L(k)u, u) \geq c\|u\|_X^2$ for all $u \in X$ and $k \in \mathbb{R}$. On the other hand, $\Big(L\big(\frac{(Bu,u)}{-2(Cu,u)}\big)u, u\Big) = \langle G'(u), u\rangle$. This yields $\langle G'(u), u\rangle \geq c\|u\|_X^2$ for all $u \in X$. Now, putting into this inequality $u = u_{n_k}$, we get $\langle G'(u_{n_k}), u_{n_k}\rangle \geq c\|u_{n_k}\|_X^2$. Finally, taking the limit as $k \to \infty$, we get $\|u_{n_k}\|_X \to 0$. □

The next step in nonlinear eigenvalue problems is to prove the existence of "deformations".

**Definition 3.5.** *A set $M \subset X$ allows Ljusternik-Schnirelman deformations with respect to $F$ and $c \in \mathbb{R}$ iff for each open set $U$ in $X$ such that $crit_{N_1,c}F \subset U$, there exists a number $\varepsilon(U) > 0$ and a contiuous mapping $d : M \times [0,1] \to M$ such that*
*i) $d(u,0) = u,\ u \in M$, and*
*ii) $F(u) \geq c - \varepsilon,\ u \in M \setminus U$ implies $F(d(u,1)) \geq c + \varepsilon$.*

It is known that the Palais-Smale condition for $c \neq 0$ guarantees this fact. Thus as a consequence of Proposition 3.2 (see [14], p. 331, Corollary 44.30), we have

**Corollary 3.6.** *For each $c \neq 0$, the level set $N_1$ allows Ljusternik-Schnirelman deformations $d(u,t) : N_1 \times [0,1] \to N_1$ with respect to $c$ and $F$. Further, $d$ is odd; i.e., $d(-u,t) = -d(u,t)$.*

Another important corollary of Palais-Smale condition is the following fact.

**Corollary 3.7.** *The set $crit_{N_1,c}F$ is compact for all $c > 0$.*

*Proof.* By definition $crit_{N_1,c}F := \{u \in N_1 \big| F'(u) = c\,G'(u)$ and $F(u) = c\}$. Let $u_n \in crit_{N_1,c}F$. Then we have $F(u_n) = c$ and $TF(u_n) = 0$. It follows from the Palais-Smale condition that $u_n$ has a convergent subsequence; i.e., that $crit_{N_1,c}F$ is compact. □

To end the paper, we present our main theorem.

**Theorem 3.8.** *Let* $c_m = \sup_{L\subset\mathcal{K}_m} \inf_{u\in L} F(u),\ \ m = 1, 2, \cdots$ . *Then:*
*i)* $crit_{N_1,c_m} F \neq \emptyset$*; i.e., there exists* $u_m \in N_1$ *and* $\lambda_m \in \mathbb{R}$ *such that* $F'(u_m) = \lambda_m G'(u_m)$ $m = 1, 2, ...,$
*ii)* $\lambda_m = c_m,\ m = 1, 2, \cdots$ .

*Proof.* Let $crit_{N_1,c_m} F = \emptyset$. Clearly, $c_m > 0$, $m = 1, 2, ...,$ since $F(u) > 0$, $u \in N_1$. Then by Proposition 3.2, $F$ satisfies the Palais-Smale condition with respect to $N_1$ and $c_m$. By Corollary 3.1, $N_1$ allows continuous and odd deformations $d : N_1 \times [0, 1] \to \mathbb{R}$. We can choose $U = \emptyset$ in Definition 3.3, because we assume $crit_{N_1,c_m} F = \emptyset$. Then by Corollary 3.1, there exists an odd and continuous function $d : N_1 \times [0, 1] \to \mathbb{R}$ such that $F(u) \geq c_m - \varepsilon,\ u \in N_1$ implies $F(d(u, 1) \geq c_m + \varepsilon$. By definition $c_m = \sup_{L\subset\mathcal{K}_m} \inf_{u\in L} F(u),\ \ m = 1, 2, ....$ Thus, there exists $K \subset \mathcal{K}_m$, such that $F(u) \geq c_m - \varepsilon$, $u \in K$. By statement ii) of Definition 3.3 and Corollary 3.1, we get $F(d(K, 1)) \geq c_m + \varepsilon$ and $d(K, 1) \in \mathcal{K}_m$ because $d$ is odd and continuous. Let $d(K, 1) = K_1$. Then we have $\inf_{K_1} F(u) \geq c_m + \varepsilon$. This fact means that $\sup_{L\subset\mathcal{K}_m} \inf_{u\in L} F(u) \geq c_m + \varepsilon$, which contradicts $c_m = \sup_{L\subset\mathcal{K}_m} \inf_{u\in L} F(u),\ \ m = 1, 2, \cdots$ .

ii) We have $F'(u_m) = \lambda_m G'(u_m)$ and $u_m \in N_1$. Then $\langle F'(u_m), u_m\rangle = \lambda_m\langle G'(u_m), u_m\rangle$. Consequently, $F(u_m) = \lambda_m G(u_m)$. Since $F(u_m) = c_m$ and $G(u_m) = 1$, we obtain $c_m = \lambda_m$. □

Finally, recall that we have concentrated our attention on the problem

$$F'(u) = \lambda G'(u),\ \ u \in N_1.$$

But the problem of finding of neutral pairs $(k, u)$ for an operator pencil $\tilde{L}_w(k)$ at fixed $w \in \mathbb{R}$ is described by the equation

$$\frac{(\tilde{B}u, u)^2}{4(\tilde{C}u, u)^2}\tilde{C}u - \frac{(\tilde{B}u, u)}{2(\tilde{C}u, u)}\tilde{B}u + \tilde{A}u = w^2\tilde{R}u. \tag{3.6}$$

This is the same as

$$F'(u) = \frac{1}{w^2}G'(u),\ \ u \in N_1.$$

Using the connection $\lambda = \frac{1}{w^2}$ and (3.4), we can state the following.

**Corollary 3.9.** *Problem* (3.6) *has infinitely many eigenvalues* $w_m^2$ *which correspond to eigenvectors* $u_m$ *such that*
*a)* $\frac{1}{w_m^2} = c_m = \sup_{L\subset\mathcal{K}_m} \inf_{u\in L} F(u),$
*b)* $F(u_m) = \frac{1}{w_m^2},$

*c)* $w_m^2 \to \infty$.

*d)* $\left(-\frac{(\tilde{B}u_m, u_m)}{2(\tilde{C}u_m, u_m)}, u_m\right)$ *will be a neutral eigen-pair for problem 3.2 at* $w = w_m$.

## References

[1] Y. Abramov, Two parameter operator pencils of waveguide type, *Dokl. Math.* **286** (1986), no. 4, 777–781.

[2] Y. Abramov, Pencils of waveguide type and related extremal problems, *J. Soviet Math.* **64** (1993), no. 6, 1278–1289.

[3] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, Elsevier/Academic Press, Amesterdam, 2003.

[4] P. Binding, R. Hryniv and B. Najman, A variational principle in Krein space II. *Integral Equations Operator Theory* **51** (2005), no. 4, 477–500.

[5] M. I. Gil, *Operator Functions and Localization of Spectra,* Lecture Notes in Mathematics, 1830. Springer-Verlag, Berlin, 2003.

[6] I.Gohberg and M. Krein, Introduction to the Theory of Linear Non-Selfadjoint Operators in Hilbert Space, Izdat, Moscow, 1965.

[7] M. Hasanov, Spectral problems for operator pencils in non-separated root zones, *Turk. J. Math.* **31** (2007) 43-52.

[8] M. Hasanov, Basis problems for matrix valued functions, *J. Math. Anal. Appl.* **342** (2008) 766-772.

[9] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1995.

[10] A. G. Kostyuchenko and M. B. Orazov, The problem of oscillations of an elastic half cylinder and related selfadjoint quadratic pencils, *J. Soviet Math.* **33** (1986) 1025–1065.

[11] A. A. Shkalikov, On perturbation theory for points of the discrete spectrum of dissipative operator functions, *Differ. Equ.* **45**, (2009), no. 4, 580-590.

[12] M. Struwe, Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Fourth Edition, Springer-Verlag, 2008.

[13] H. Voss and B. Werner, A minimax principle for nonlinear eigenvalue problems with applications to nonoverdamped systems, *Math. Methods Appl. Sci.* **4** (1982), 415–424.

[14] E. Zeidler, Nonlinear Functional Analysis and its Applications, Vol III., Variational Methods and Optimization, Springer-Verlag, 1985.

[15] A. Zilbergleit and Y. Kopilevich, Spectral Theory of Guided Waves, IOP Publishing, Bristol, 1996.

**Mahir Hasansoy**
Department of Mathematics, Dogus University, Acibadem/Kadiköy, Istanbul, Turkey
`Email: mhasansoy@dogus.edu.tr`