Bulletin of the Iranian Mathematical Society Vol. 38 *No. 1* (*2012*), *pp 241-252.*

ON THE EXISTENCE AND UNIQUENESS OF SOLUTION OF INITIAL VALUE PROBLEM FOR FRACTIONAL ORDER DIFFERENTIAL EQUATIONS ON TIME SCALES

A. AHMADKHANLU AND M. JAHANSHAHI*

Communicated by Mohammaad Asadzadeh

Abstract. In this paper, at first the concept of Caputo fractional derivative is generalized on time scales. Then the fractional order differential equations are introduced on time scales. Finally, sufficient and necessary conditions are presented for the existence and uniqueness of solution of initial value problem including fractional order differential equations.

1. Introduction

Mathematical models of some natural phenomena and physical problems have appeared as initial and boundary value problems including fractional order of ordinary and partial differential equations. A. Loukshin and J.Morove in 1985 on modeling of irrevocability of metals [13] and A. Nakhshev in the same year on modeling of liquids moving in underground layers encountered with fractional order differential equations [14]. Later, this kind of differential equations were used in electrochemistry, control and electromagnetic field theories [9, 11]. These important applications caused that this kind of differential equations were studied

MSC(2010): Primary: 34A12; Secondary: 34K37.
Keywords: Time scales, differential equations, initial value problem, fractional order derivative.
Received: 4 January 2010, Accepted: 10 October 2010.
*Corresponding author

by many mathematicians in recent years [7, 15, 16].
Hilger in 1990 introduced time scales to unify and extend the theory of differential equation, difference equations and other differential systems defined over nonempty closed subset of real line [12]. He proved the existence and uniqueness of initial value problems including differential equations on time scales. Some application of this kind of problems can be found in [2, 1, 4, 3].

In this paper we try to extend fractional order differential equations (dynamic equations) on time scales. For this, we need to define a fractional differential operator on time scales. This is done via Caputo differential operator. This generalization helps us to study relation between fractional difference equations and fractional differential equations. On the other hand, this extension also provides a background to study boundary value problems including fractional order difference and differential equations. One can use this relation for more investigations and solving fractional difference equations on time scales.

We consider the following initial value problem:

$$ {}^{c}\Delta^{\alpha} y(t) = f(t, y(t)), \qquad t \in [t_0, t_0 + a] = J \subseteq \mathbb{T}, \qquad 0 < \alpha < 1 \tag{1.1} $$

$$ y(t_0) = y_0 \tag{1.2} $$

where ${}^{c}\Delta^{\alpha}$ is Caputo fractional derivative operator and the function $f : J \times \mathbb{T} \to \mathbb{R}$ is a right-dense continuous function. Next we present sufficient and necessary conditions for the existence and uniqueness of the problem (1.1)-(1.2)

2. **Preliminaries**

In this section, some notations, definitions and lemmas which will be used in next section are recalled and introduced. At first, we use $C(J, \mathbb{R})$ for a Banach space of continuous functions with the norm

$$ \| y \|_{\infty} = \sup\{|y(t)|, t \in J\}, $$

where J is an interval.
A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The calculus on time scales was initiated by Aulbach and Hilger [12, 3] in order to create a theory that can unify and extend discrete and continuous analysis.

The real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the non-negative integers $\mathbb{N}_0$, the h-numbers ($h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$ is a fixed real number), and the q-numbers ($kq = q^{\mathbb{Z}} \cup \{0\} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$ is a fixed real number), are examples of time scales, as are $[0, 1] \cup [2, 3]$, $[0, 1] \cup \mathbb{N}$, and the Cantor set, where $[0, 1]$ and $[2, 3]$ are intervals of real numbers.
Aulbach and Hilger introduced also dynamic equations (Δ-differential equations) on time scales in order to unify and extend the theory of ordinary differential equations, difference (h-difference) equations, and q-difference equations. For a general introduction to the calculus of time scales we refer the reader to the textbooks by Bohner and Peterson [6]. Here we give only those notations and facts connected to time scales, which we need for our purpose in this paper.
Any time scale $\mathbb{T}$ is a complete metric space with the metric (distance) $d(t; s) = |t - s|$ for $t, s \in \mathbb{T}$. Consequently, according to the well-known theory of general metric spaces, we have for $\mathbb{T}$ the fundamental concepts such as open balls (intervals), neighborhoods of points, open sets, closed sets, compact sets, and so on. In particular, for a given number $N > 0$, the N-neighborhood $U_\delta(t)$ of a given point $t \in \mathbb{T}$ is the set of all points $s \in \mathbb{T}$ such that $d(t, s) < N$: By a neighborhood of a point $t \in \mathbb{T}$ it is meant an arbitrary set in $\mathbb{T}$ containing a N-neighborhood of the point t. Also we have for functions $f : \mathbb{T} \to \mathbb{R}$ the concepts of limit, continuity, and the properties of continuous functions on general complete metric spaces (note that, in particular, any function $f : \mathbb{Z} \to \mathbb{R}$ is continuous at each point of $\mathbb{Z}$). The main task is to introduce and investigate the concept of derivative for functions $f : \mathbb{Z} \to \mathbb{R}$. This proves to be possible due to the special structure of the metric space $\mathbb{T}$. In the definition of the derivative an important role is played by the so-called forward and backward jump operators [4].

Definition 2.1. *For $t \in \mathbb{T}$, define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

In this definition, in addition we put $\sigma(\max \mathbb{T}) = \max \mathbb{T}$, if there exists a finite $\max \mathbb{T}$, and $\rho(\min \mathbb{T}) = \min \mathbb{T}$, if there exists a finite $\min \mathbb{T}$.

Obviously both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because of our assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$. Let $t \in \mathbb{T}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Also, if $t < \max \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \min \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-scattered and left-scattered at the same time are called isolated.

Definition 2.2. (*Delta Derivative*) *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}$. Then the delta derivative (or Δ-derivative) of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood U of t in $\mathbb{T}$ such that*

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U$$

Definition 2.3. (*Delta Integral*) *Let $[a, b]$ be a closed bounded interval in $\mathbb{T}$. A function $F : [a, b] \to \mathbb{R}$ is called a delta antiderivative of a function $f : [a; b) \to \mathbb{R}$ provided that F is continuous on $[a, b]$ and delta differentiable on $[a, b)$; and $F^{\Delta}(t) = f(t)$ for all $t \in [a, b)$. Then we define the Δ-integral from a to b of f by*

$$\int_a^b f(t)\Delta t = F(b) - F(a)$$

Definition 2.4. *A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (or rd-continuous) provided that it is continuous at all right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$: The set of all right-dense continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$. Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous provided that it is continuous at all left-dense points of $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points of $\mathbb{T}$. The set of all left-dense continuous functions on $\mathbb{T}$ is denoted by $C_{ld}(\mathbb{T})$.*

All rd-continuous bounded functions on $[a, b)$ are delta integrable from a to b. For a more general treatment of the delta integral on time scales (Riemann and Lebesgue integration on time scales), see [10] and [4].

Proposition 2.5. *Suppose $a, b \in \mathbb{T}$, $a < b$ and $f(t)$ is continuous on$[a, b]$, then we have*

$$\int_a^b f(t)\Delta t = [\sigma(a) - a]f(a) + \int_{\sigma(a)}^b f(t)\Delta t$$

Proof. refer to [4] □

Proposition 2.6. *Suppose $\mathbb{T}$ is a time scale and $[a, b] \subset \mathbb{T}$, f is increasing continuous function on $[a, b]$. If the extension of f is given in the following form:*

$$F(s) = \begin{cases} f(s) & ; s \in \mathbb{T} \\ f(t) & ; s \in (t, \sigma(t)) \notin \mathbb{T} \end{cases}$$

Then we have

$$\int_a^b f(t)\Delta t \leq \int_a^b F(t)dt$$

Proof. Let $r \in [a, b]$ be a right-scattered point, then by making use of proposition 2.5 we have

$$\int_r^{\sigma(r)} f(t)\Delta t = [\sigma(r) - r]f(r).$$

Since f is a increasing function, consequently its extension F will be an increasing continuous function. Therefore, applying the mean value theorem for integrals implies

$$[\sigma(r) - r]F(r) \leq \int_r^{\sigma(r)} F(t)dt \leq [\sigma(r) - r]F(\sigma(r))$$

and

$$[\sigma(r) - r]f(r) \leq \int_r^{\sigma(r)} F(t)dt \leq [\sigma(r) - r]f(\sigma(r))$$

therefore

$$\int_r^{\sigma(r)} f(t)\Delta t \leq \int_r^{\sigma(r)} F(t)dt.$$

Now if $[a,b]$ has only one right-scattered point s then by proposition 2.5 and previous definitions, we have

$$\begin{aligned}\int_a^b f(t)\Delta t &= \int_a^s f(t)\Delta t + \int_s^{\sigma(s)} f(t)\Delta t + \int_{\sigma(s)}^b f(t)\Delta t \\ &\leq \int_a^s F(t)dt + \int_s^{\sigma(s)} F(t)dt + \int_{\sigma(s)}^b F(t)dt.\end{aligned}$$

If the same proof is applied for n right-scattered points in $[a,b]$ we obtain:

$$\int_a^b f(t)\Delta t \leq \int_a^b F(t)dt.$$

□

Definition 2.7. *Suppose $\mathbb{T}$ is a time scale, $[a,b] \subseteq \mathbb{T}$ and the function $h(x)$ is an integrable function on $[a,b]$, then Δ-fractional integral of h is defined by the following relation*

$$^{\Delta}I_{a+}^{\alpha}h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\Delta s,$$

where $\Gamma(\alpha)$ is the Euler Gamma function.

Definition 2.8. *Let $h : \mathbb{T} \to \mathbb{R}$ be a function. The Caputo Δ-fractional derivative of h is defined by:*

$$^{c}\Delta_{a+}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{\Delta^n}(s)\Delta s \tag{2.1}$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

It is easy to see that $^{\Delta}I_{a+}^{\alpha}{}^{\Delta}I_{a+}^{\beta}h(t) = {}^{\Delta}I_{a+}^{\alpha+\beta}h(t)$.

Remark 2.9. *For $\mathbb{T} = \mathbb{R}$, the differential operator which was defined in definition 2.8 is the same as the Caputo fractional derivative. For this kind of operators some results about the existence and uniqueness of solutions of initial and boundary value problems have been given in* [5].

3. **Initial Value Problems on Time Scales**

In this section we are going to give the solution of the initial value problem including fractional order differential equations on time scales. For this, suppose $\mathbb{T}$ is a time scale and $J = [t_0, t_0 + a] \subseteq \mathbb{T}$, then the function $y \in C(J, \mathbb{R})$ is a solution of problem (1.1)-(1.2), if we have

$$\begin{aligned} {}^c\Delta^\alpha y(t) &= f(t, y) \qquad ;\text{on } J \\ y(t_0) &= y_0 \end{aligned}$$

For establishing this solution, we need to prove the following lemma and theorem.

Lemma 3.1. *Let $0 < \alpha < 1$, $J \subseteq \mathbb{T}$, and let $f : J \times \mathbb{R} \to \mathbb{R}$ be the function. Then the function $y(t)$ is a solution of problem*(1.1)-(1.2) *if and only if this function is a solution of the following integral equation:*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} f(s, y(s))\Delta s$$

Proof. For $y(t)$ from (2.1) we have:

$${}^c\Delta_{t_0}^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t - s)^{-\alpha} y^\Delta(s)\Delta s =^\Delta I_{t_0}^{1-\alpha} y^\Delta(t).$$

Then the proof can be concluded from the relations:

$${}^\Delta I_{t_0}^{\alpha c} \Delta_{t_0}^\alpha y(t) =^\Delta I_{t_0}^{\alpha \Delta} I_{t_0}^{1-\alpha} y^\Delta(t) = y(t) - y(0)$$

and

$$y(t) = y_0 +^\Delta I_{t_0}^{\alpha c} \Delta_{t_0}^\alpha y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} f(s, y(s))\Delta s.$$

□

Our first result is based on the Banach fixed point theorem [8].

Theorem 3.2. *Suppose $J = [t_0, t_0 + a] \subseteq \mathbb{T}$. Then the initial value problem* (1.1)-(1.2) *has a unique solution on J if the function $f(t, y(t))$ is a right-dense continuous bounded function such that there exists $M > 0$; $|f(t, y(t)| < M$ on J and the Lipshitz condition*

$$\exists L > 0;\ \forall t \in J \,\text{and}\ x, y \in \mathbb{R}, \qquad ||f(t, x) - f(t, y)|| \le L||x - y||$$

is holds.

Proof. Let S be the set of rd-continuous functions and $J \subseteq \mathbb{T}$. For $y \in S$, define

$$||y|| = \sup_{t \in J} ||y(t)||.$$

It is easy to see that S is a Banach space with this norm. The subset of $S(\rho)$ and the operator T are defined by

$$S(\rho) = \{X \in S : ||X_s|| \leq \rho\},$$

and

$$T(y) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, y(s)) \Delta s.$$

According to proposition 2.5 we have:

$$\begin{aligned} |T(y(t))| &\leq ||y_0|| + \tfrac{1}{\Gamma(\alpha)} \textstyle\int_{t_0}^{t} (t-s)^{\alpha-1} M \Delta s \\ &\leq ||y_0|| + \tfrac{M}{\Gamma(\alpha)} \textstyle\int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s. \end{aligned}$$

Since the $(t-s)^{\alpha-1}$ is an increasing monotone function, by using proposition 2.5 we can write:

$$\int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s \leq \int_{t_0}^{t} (t-s)^{\alpha-1} ds.$$

Consequently,

$$|T(y(t))| \leq ||y_0|| + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \leq ||y_0|| + \frac{M}{\Gamma(\alpha)} \frac{a^\alpha}{\alpha}.$$

By considering

$$\rho = ||y_0|| + \frac{M a^\alpha}{\Gamma(\alpha+1)},$$

we conclude that T is an operator from $S(\rho)$ to $S(\rho)$. Moreover, for $x, y \in S(\rho)$ we have:

$$\begin{aligned} ||T(x) - T(y)|| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| \Delta s \\ &\leq \frac{L||x-y||_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s \\ &\leq \frac{L||x-y||_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \\ &\leq \frac{L}{\Gamma(\alpha)} \frac{a^\alpha}{\alpha} = \frac{L a^\alpha}{\Gamma(\alpha+1)} ||x-y||_\infty. \end{aligned}$$

If $\frac{La^{\alpha}}{\Gamma(\alpha+1)} < 1$, then this will be a contraction map. This implies the existence and uniqueness of solution of problem (1.1)-(1.2). $\square$

Remark 3.3. *If we let $\alpha = 1$, then this problem is reduced to an initial value problem on time scales for which the existence and uniqueness of its solution has been studied in* [12].

Theorem 3.4. *Suppose $f : J \times \mathbb{R} \to \mathbb{R}$ is a rd-continuous bounded function such that there exists $M > 0$ with $|f(t,y)| \leq M$ for all $t \in J, y \in \mathbb{R}$. Then the problem* (1.1)-(1.2) *has a solution on J.*

Proof. We shall use Schauder's fixed point theorem [8], to prove that T defined by (2.1) has a fixed point. The proof will be given in several steps.

Step 1: T is continuous.
Let y_n be a sequence such that $y_n \to y$ in $C(J,\mathbb{R})$. Then for each $t \in J$,

$$\begin{aligned}
&|T(y_n)(t) - T(y)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |f(s,y_n(s)) - f(s,y(s))| \Delta s \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \sup_{s\in J} |f(s,y_n(s)) - f(s,y(s))| \Delta s \\
&\leq \frac{||f(.,y_n(.)) - f(.,y(.))||_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s \\
&\leq \frac{||f(.,y_n(.)) - f(.,y(.))||_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \\
&\leq \frac{||f(.,y_n(.)) - f(.,y(.))||_\infty}{\Gamma(\alpha)} \frac{a^{\alpha}}{\alpha} \\
&\leq \frac{a^{\alpha}||f(.,y_n(.)) - f(.,y(.))||_\infty}{\Gamma(\alpha+1)}.
\end{aligned}$$

Since f is a continuous function, we have

$$||T(y)(t) - T(y)(t)||_\infty \leq \frac{a^{\alpha}}{\Gamma(\alpha+1)} ||f(.,y_n(.)) - f(.,y(.))||_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: The mao T sends bounded sets into bounded sets in $C(J,\mathbb{R})$. Indeed, it is enough to show that for any ρ, there exists a positive constant l, such that for each $y \in B_\rho = \{y \in C(J,\mathbb{R}) : ||y||_\infty \leq \rho\}$, we have $||T(y)||_\infty \leq l$.

By hypothesis for each $t \in J$ we have,

$$\begin{aligned}|T(y)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|f(s,y(s))|\Delta s\\ &\leq \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\Delta s \leq \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}ds\\ &\leq \frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)} = \frac{Ma^{\alpha}}{\Gamma(\alpha+1)} = l.\end{aligned}$$

Step 3: The map T sends bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.
Let $t_1, t_2 \in J, t_1 < t_2$ and let B_ρ be a bounded set of $C(J,\mathbb{R})$ as in Step 2, and let $y \in B_\rho$. Then

$$\begin{aligned}&|T(y)(t_2) - T(y)(t_1)|\\ &\leq \tfrac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))\Delta s - \int_{t_0}^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))\Delta s\Big|\\ &\leq \tfrac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^{t_1}((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} + (t_2-s)^{\alpha-1})f(s,y(s))\Delta s -\\ &\qquad \int_{t_0}^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))\Delta s\Big|\\ &\leq \tfrac{M}{\Gamma(\alpha)}\Big|\int_{t_0}^{t_1}((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})\Delta s + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Delta s\Big|\\ &\leq \tfrac{M}{\Gamma(\alpha)}\Big|\int_{t_0}^{t_1}((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\Big|\\ &\leq \tfrac{M}{\Gamma(\alpha+1)}[(t_2-t_1)^{\alpha} + (t_1-t_0)^{\alpha} - (t_2-t_0)^{\alpha}] + \tfrac{M}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}\\ &= \tfrac{2M}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} + \tfrac{M}{\Gamma(\alpha+1)}[(t_1-t_0)^{\alpha} - (t_2-t_0)^{\alpha}]\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is continuous and completely continuous.

Step 4: A priori bounds.
Now it remains to show that the set

$$\Omega = \{y \in C(J,\mathbb{R}) : y = \lambda T(y), 0 < \lambda < 1\}$$

is bounded set.
Let $y \in \Omega$, then $y = \lambda T(y)$ for some $0 < \lambda < 1$: Thus, for each $t \in J$ we have

$$y(t) = \lambda \left[\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, y(s)) \Delta s \right]$$

We can complete this step by considering the estimation in step 2. As a consequence of Schauder's fixed point theorem, we conclude that T has a fixed point which is a solution of the problem (1.1)-(1.2). $\square$

Acknowledgments

The authors are extremely grateful to the referee for useful suggestions that improved the content of the paper.

References

[1] R. P. Agarwal and M. Boner, Basic calculus on time scales and some of its applications, *Result. Math.* **35** (1999), 3–22.
[2] R. P. Agarwal, M. Bohner, A. Peterson and D. O'Regan, Dynamic equations on time scales, *J. Comput. Appl. Math.* **141** (2002), no. 1-2, 1–26.
[3] B. Aulbach and S. Hilger, Linear dynamic processes with inhomogeneous time scale, in: G.A. Leonov, V. Reitmann, W. Timmermann (Eds.), *Nonlinear Dynamics and Quantum Dynamical Systems*, Math. Research. Bd. Akademie-Verlag, Berlin. **59** (1990), 9–20.
[4] R. P. Agarwal, M. Bohner, A. Peterson and D. O'Regan, *Advances in Dynamic Equations on Time Scales*, Birkhaurser, Boston, 2003.
[5] M. Benchohra, S. Hamani and S. K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1–12.
[6] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäuser Boston, Inc., Boston, MA, 2003.
[7] A. Carpinteri and F. Mainardi, Eds, Fractional calculus: Some basic problems in continuum and statistical mechanics, in *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien. (1997), 291–348.
[8] J. Cronin, Differential Equations, Introduction and Qualitative Theory, Marcel Dekker, Inc., New York, 1994.
[9] K. Diethelm and A.D. Freed, Fractional calculus in biomechanics: A 3D viscoelastic model using regularized fractional-derivative kernels with application to the human calcaneal fat pad. *Biomech. Model. Mechanobiology.* **5** (2006), 203–215.
[10] G. Sh. Guseinov, Integration on time scales, *J. Math. Anal. Appl.* **285** (2003), 107–127.

[11] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Inc., River Edge, NJ, 2000.

[12] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, *Result. Math.* **18** (1990), 18–56.

[13] A. A. Lokshin, Yu. V. Suvorova, The Mathematical Theory of Wave Propagation in Media with Memory, Moskov. Gos. Univ., Moscow, 1982.

[14] A. M. Nakhashev, BVPs with non-local conditions and relation it to fractional derivative equation, *J. Diff. Equation* **21** (1985), no. 1, 1–10.

[15] K. B. Oldham and J. Spanier, The Fractional Calculus, Mathematics in Science and Engineering, Vol. 111. Academic Press, New York-London, 1974.

[16] I. Podlubny, Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Mathematics in Science and Engineering, 198, Academic Press, Inc., San Diego, CA, 1999.

Asghar Ahmadkhanlu
Department of Mathematics, Azarbayjan University of Tarbiat Moallem, 35 Km Tabriz-Maragheh Road, P.O. Box 53714-161, Tabriz, Iran
Email: `s.a.ahmadkhanlu@azaruniv.edu`

Mohamad Jahanshahi
Department of Mathematics, Azarbayjan University of Tarbiat Moallem, 35 Km Tabriz-Maragheh Road, P.O. Box 53714-161, Tabriz, Iran
Email: `jahanshahi@azaruniv.edu`