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ON THE STABILITY OF GENERALIZED DERIVATIONS ON BANACH ALGEBRAS

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ABSTRACT. We investigate the stability of generalized derivations on Banach algebras with a bounded central approximate identity. We show that every approximate generalized derivation in the sense of Rassias, is an exact generalized derivation. Also the stability problem of generalized derivations on the faithful Banach algebras is investigated.

1. Introduction

The stability of functional equations appeared at first by Ulam in 1940 [11], where in 1941, Hyers studied a version of this problem in [5].

In 1978, Th.M.Rassias [9] extended the result of Hyers as follows: *Suppose that E_1, E_2 are Banach spaces and $f : E_1 \to E_2$ is a mapping for which there exist $\epsilon > 0$ and $0 \leq p < 1$ such that $\|f(x+y)-f(x)-f(y)\| < \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p$ for all $x \in E_1$.*

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Gavruta [4] generalized the theorem of Rassias as follows:
Let G be an abelian group and E be a Banach space. Denote by $\phi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\phi}(x, y) := \frac{1}{2} \sum_{k=0}^{\infty} 2^{-k} \phi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to E$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \phi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to E$ such that $\|f(x) - T(x)\| \leq \widetilde{\phi}(x, x)$ for all $x \in G$.

Since then, several stability problems of various functional equations have been investigated by many mathematicians (see [2, 10]).

In this paper, we consider the stability of generalized derivations. Park in [7] and [8] has studied the stability of derivations. Moslehian in [6], extended the results of [7] to generalized derivations from unital Banach algebra A to a unit linked Banach A-bimodule. In the present paper, we prove the generalized Hyers-Ulam-Rassias stability of generalized derivations on Banach algebras which have a bounded central approximate identity. In particular, we show that every approximate generalized derivation in the sense of Rassias, is an exact generalized derivation (Theorem 2.3, Corollary 2.4).
Also we investigate the stability problem of generalized derivations on the faithful Banach algebras.

Let A be a Banach algebra and M be a Banach A-bimodule. A linear mapping $\mu : A \to M$ is called a generalized derivation if there exists a derivation $\delta : A \to M$ such that $\mu(ab) = a\mu(b) + \delta(a)b$ for all $a, b \in A$. For example, for fixed arbitrary elements $x, y \in M$, the mapping $\mu_{x,y}(a) = xa - ay$ is a generalized derivation (it is called the inner generalized derivation).

We recall that an approximate identity $(e_\lambda)_{\lambda \in \Lambda}$ in Banach algebra A is central if $\{e_\lambda : \lambda \in \Lambda\} \subseteq Z(A)$. [3]
Also Banach algebra A is called left faithful (right faithful), if

$$\{a \in A : aA = 0\} = 0 \quad (\{a \in A : Aa = 0\} = 0).$$

2. The stability of generalized derivations on Banach algebras with a bounded central approximate identity

In this section, we suppose that A is a Banach algebra with a bounded left approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ and M is a Banach A-bimodule such that $e_\lambda m = me_\lambda$ for all $m \in M$ and $\lambda \in \Lambda$. Moreover, $e_\lambda m - m \to 0$ in M for all $m \in M$. Our final result of the section appears in Corollary 2.4, but at first we prove a generalized version of Hyers-Ulam-Rassias stability problem on generalized derivations from A to M.

Lemma 2.1. *Let A be a Banach algebra, M be a Banach A-bimodule and $(e_\lambda)_{\lambda\in\Lambda}$ be a bounded net in A such that $me_\lambda \to m$ for all $m \in M$. Let also (b_n) be a sequence in M such that $\lim_{n\to\infty}(b_n x) = 0$ uniformly on bounded subsets of A. Then $\lim_{n\to\infty} b_n = 0$.*

Proof. Let $\epsilon > 0$. Since $(e_\lambda)_{\lambda\in\Lambda}$ is bounded, $\lim_{n\to\infty}(b_n x) = 0$ uniformly on $\{e_\lambda : \lambda \in \Lambda\}$. Then there exists $N \in \mathbb{N}$ such that $\|b_n e_\lambda\| < \frac{\epsilon}{2}$ for all $n \geq N$ and all $\lambda \in \Lambda$.
Fix $n \geq N$. There is $\lambda_n \in \Lambda$ such that $\|b_n - b_n e_{\lambda_n}\| < \frac{\epsilon}{2}$. Hence we have

$$\|b_n\| \leq \|b_n - b_n e_{\lambda_n}\| + \|b_n e_{\lambda_n}\| < \epsilon.$$

Therefore, $\lim_{n\to\infty} b_n = 0$. □

Theorem 2.2. *Suppose $f : A \to M$ is a mapping with $f(0) = 0$ and $\phi : A \times A \times A \times A \to [0, \infty)$ is a function such that*

$$\lim_{n\to\infty} 2^{-n}\phi(2^n a, 2^n b, 2^n c, 2^n d) = 0 \tag{2.1}$$

for all $a, b, c, d \in A$, where the convergence is uniformly on $\{0\} \times \{0\} \times \{c_0\} \times E$ for every $c_0 \in A$ and every bounded subset E of A. We also suppose

$$\widetilde{\phi}(a) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\phi(2^n a, 2^n a, 0, 0) \tag{2.2}$$

converges uniformly on every bounded subset of A. Moreover, we assume the existence of a map $g : A \to M$ such that

$$\|f(\lambda a + \lambda b + cd) - \lambda f(a) - \lambda f(b) - cf(d) - g(c)d\| \leq \phi(a, b, c, d) \tag{2.3}$$

for all $\lambda \in T = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ *and* $a, b, c, d \in A$. *Then there exists a unique generalized derivation* $\mu : A \to M$ *such that*

$$\|f(a) - \mu(a)\| \leq \widetilde{\phi}(a) \tag{2.4}$$

for all $a \in A$.

Proof. Put $c = d = 0$ and $b = a$ and $\lambda = 1$ in (2.3) to get

$$\|f(2a) - 2f(a)\| \leq \phi(a, a, 0, 0)$$

for all $a \in A$. One can use the induction on n to show that

$$\left\|\frac{f(2^n a)}{2^n} - f(a)\right\| \leq \frac{1}{2}\sum_{k=0}^{n-1} 2^{-k}\phi(2^k a, 2^k a, 0, 0) \tag{2.5}$$

for all $n \in \mathbb{N}$ and all $a \in A$. Replacing a by $2^m a$ in (2.5), we get

$$\left\|\frac{f(2^{n+m} a)}{2^{n+m}} - \frac{f(2^m a)}{2^m}\right\| \leq \frac{1}{2}\sum_{k=m}^{n+m-1} 2^{-k}\phi(2^k a, 2^k a, 0, 0) \tag{2.6}$$

for all $n, m \in \mathbb{N}$ and all $a \in A$.
Let E be a bounded subset of A. Since $\sum_{n=0}^{\infty} 2^{-n-1}\phi(2^n a, 2^n a, 0, 0)$ converges uniformly on E, the inequality (2.6) implies that the sequence $(\frac{f(2^n a)}{2^n})$ is uniformly Cauchy on E. It follows that this sequence converges uniformly on E, since M is complete. Set

$$\mu(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n}. \tag{2.7}$$

By a similar method to the proof of [6, Theorem 2.1], one can easily show that μ is linear. It follows from (2.5) that $\|\mu(a) - f(a)\| \leq \widetilde{\phi}(a)$ for all $a \in A$.

Now we prove that μ is a generalized derivation.
Let $a = b = 0$ and replace c, d by $2^n c, 2^n d$, respectively, in (2.3). We have

$$\|f(2^{2n} cd) - 2^n c f(2^n d) - 2^n g(2^n c)d\| \leq \phi(0, 0, 2^n c, 2^n d),$$

and so

$$\|2^{-2n} f(2^{2n} cd) - 2^{-n} c f(2^n d) - 2^{-n} g(2^n c)d\| \leq 2^{-2n}\phi(0, 0, 2^n c, 2^n d)$$

for all $c, d \in A$ and all $n \in \mathbb{N}$.
Let E be a bounded subset of A, $c \in A$ and $\epsilon > 0$. Since $\frac{1}{2^{2n}}\phi(0, 0, 2^n c, 2^n x)$

converges uniformly on E to 0, there exists $N_1 \in \mathbb{N}$ such that

$$\|2^{-2n}f(2^{2n}cx) - 2^{-n}cf(2^n x) - 2^{-n}g(2^n c)x\| < \frac{\epsilon}{3} \tag{2.8}$$

for all $n \geq N_1$ and all $x \in E$.
By (2.7), $(\frac{f(2^n x)}{2^n})$ converges uniformly on bounded sets E and cE to $\mu(x)$, so there exists $N_2 \in \mathbb{N}$ such that

$$\|2^{-2n}f(2^{2n}cx) - \mu(cx)\| < \frac{\epsilon}{3}, \tag{2.9}$$

and

$$\|2^{-n}cf(2^n x) - c\mu(x)\| < \frac{\epsilon}{3} \tag{2.10}$$

for all $n \geq N_2$ and all $x \in E$.
By (2.8),(2.9) and (2.10), we have

$$\begin{aligned} &\|2^{-n}g(2^n c)x - \mu(cx) + c\mu(x)\| \\ &\leq \|2^{-n}g(2^n c)x - 2^{-2n}f(2^{2n}cx) + 2^{-n}cf(2^n x)\| \\ &+\|2^{-2n}f(2^{2n}cx) - \mu(cx)\| + \|2^{-n}cf(2^n x) - c\mu(x)\| < \epsilon \end{aligned}$$

for all $n \geq \max\{N_1, N_2\}$ and all $x \in E$. Hence the sequence $(2^{-n}g(2^n c)x)$ converges uniformly on E and we have

$$\lim_{n\to\infty}(2^{-n}g(2^n c)d) = \mu(cd) - c\mu(d) \tag{2.11}$$

for all $c, d \in A$.
Let $c_1, c_2, x \in A$. Then

$$\begin{aligned} &\lim_{n\to\infty}(2^{-n}g(2^n c_1c_2)x) = \mu(c_1c_2x) - c_1c_2\mu(x) \\ &= \lim_{n\to\infty}(2^{-n}g(2^n c_1)c_2x) + c_1\mu(c_2x) - c_1c_2\mu(x) \\ &= \lim_{n\to\infty}(2^{-n}g(2^n c_1)c_2x) + c_1\lim_{n\to\infty}(2^{-n}g(2^n c_2)x), \end{aligned}$$

and so

$$\lim_{n\to\infty}((2^{-n}g(2^n c_1c_2) - 2^{-n}g(2^n c_1)c_2 - 2^{-n}c_1g(2^n c_2))x) = 0$$

uniformly on bounded subsets. Since A has a bounded left approximate identity (e_λ) such that $me_\lambda \to m$ for all $m \in M$, by Lemma 2.1 we obtain

$$\lim_{n\to\infty}(2^{-n}g(2^n c_1c_2) - 2^{-n}g(2^n c_1)c_2 - 2^{-n}c_1g(2^n c_2)) = 0 \tag{2.12}$$

for all $c_1, c_2 \in A$.

Now we prove that the sequence $(2^{-n}g(2^n c))$ converges for all $c \in A$. Let $c \in A$. By the Cohen factorization theorem, there exist $c_1, c_2 \in A$ such that $c = c_1c_2$. By the proof of Cohen factorization theorem (see [1]), there exist $0 < \gamma < 1$ and a sequence (e_n) in $\{e_\lambda : \lambda \in \Lambda\}$ such

that $c_1 = \sum_{k=1}^{\infty} \gamma(1-\gamma)^{k-1} e_k$. Since $e_\lambda m = m e_\lambda$ for all $m \in M$ and all $\lambda \in \Lambda$, we have $c_1 m = m c_1$ for all $m \in M$.
By (2.12) we have

$$\lim_{n\to\infty} (2^{-n} g(2^n c) - 2^{-n} g(2^n c_1) c_2 - 2^{-n} c_1 g(2^n c_2)) = 0. \tag{2.13}$$

Now $2^{-n} c_1 g(2^n c_2) = 2^{-n} g(2^n c_2) c_1$ for all $n \in \mathbb{N}$ and $(2^{-n} g(2^n x) y)$ converges for all $x, y \in A$. Therefore, by (2.13) the sequence $(2^{-n} g(2^n c))$ converges. Set $\delta(c) := \lim_{n\to\infty} 2^{-n} g(2^n c)$ for all $c \in A$.
By (2.11) and (2.12) we have

$$\mu(cd) = c\mu(d) + \delta(c) d \tag{2.14}$$

and

$$\delta(c_1 c_2) = \delta(c_1) c_2 + c_1 \delta(c_2)$$

for all $c_1, c_2, c, d \in A$. Since μ is linear, by (2.14) we get

$$\delta(\alpha c_1 + \beta c_2) e_\lambda = (\alpha \delta(c_1) + \beta \delta(c_2)) e_\lambda$$

for all $c_1, c_2 \in A$ and $\alpha, \beta \in \mathbb{C}$ and $\lambda \in \Lambda$. Thus δ is a linear derivation. It then follows from (2.14) that μ is a generalized derivation.
It is easy to see that the additive mapping μ satisfying (2.4) is unique. □

The following theorem that extend Corollary 2.4 of [6], states that every approximate generalized derivation in the sense of Rassias, is an exact generalized derivation.

Theorem 2.3. *Suppose that $f : A \to M$ is a mapping with $f(0) = 0$ for which there exist a map $g : A \to M$ and constants $\beta > 0$ and $0 \le p < 1$ such that*

$$\|f(\lambda a + \lambda b + cd) - \lambda f(a) - \lambda f(b) - cf(d) - g(c)d\| \le$$

$$\beta(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p) \tag{2.15}$$

for all $\lambda \in T$ and all $a, b, c, d \in A$. Then f is a generalized derivation and g is a derivation.

Proof. Put $\phi(a, b, c, d) = \beta(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ in Theorem 2.2. Since the series $\sum_{n=0}^{\infty} \beta 2^{n(p-1)} \|x\|^p$ converges uniformly on every bounded subset of A, ϕ and $\widetilde{\phi}$ satisfy in (2.1) and (2.2). Therefore, there exists a unique generalized derivation $\mu : A \to M$ such that $\|f(a) - \mu(a)\| \le \frac{\beta\|a\|^p}{1-2^{p-1}}$.

Also by the proof of Theorem 2.2, the sequence $(2^{-n}g(2^n a))$ converges for all $a \in A$ and the mapping δ, defined by $\delta(a) := \lim_{n\to\infty} \frac{g(2^n a)}{2^n}$, is a derivation.
Putting $a = b = 0$ and replacing c by $2^n c$ in (2.15), we get

$$\|f(2^n cd) - 2^n cf(d) - g(2^n c)d\| \leq \beta(\|2^n c\|^p + \|d\|^p)$$

and hence

$$\|\frac{f(2^n cd)}{2^n} - cf(d) - \frac{g(2^n c)}{2^n}d\| \leq \frac{\beta}{2^n}(\|2^n c\|^p + \|d\|^p).$$

Taking the limit as $n \to \infty$ we obtain

$$\mu(cd) = cf(d) + \delta(c)d \quad (c, d \in A). \tag{2.16}$$

Now we have

$$cf(2^n d) + 2^n\delta(c)d = \mu(2^n cd) = 2^n cf(d) + \delta(2^n c)d = 2^n cf(d) + 2^n\delta(c)d,$$

and so $cf(d) = 2^{-n}cf(2^n d)$ for all $c, d \in A$ and $n \in \mathbb{N}$. By the limit process, we get $cf(d) = c\mu(d)$ for all $c, d \in A$. Thus $f = \mu$, since $e_\lambda m \to m$ for every $m \in M$.
Similarly, putting $a = b = 0$ and replacing d by $2^n d$ in (2.15), we obtain

$$\left\|\frac{f(2^n cd)}{2^n} - c\frac{f(2^n d)}{2^n} - g(c)d\right\| \leq \frac{\beta}{2^n}(\|c\|^p + \|2^n d\|^p).$$

Taking the limit as $n \to \infty$, we have $\mu(cd) - c\mu(d) - g(c)d = 0$. Therefore, by (2.16), we obtain $g(c)d = \delta(c)d$ for all $c, d \in A$. Then $g = \delta$ which means f is a generalized derivation and g is a derivation. □

Corollary 2.4. *Let A be a Banach algebra with a bounded central approximate identity (e_λ). The Theorems 2.2 and 2.3 remain true if we replace M by A.*

Definition 2.5. *The mapping $f : A \to A$ is called an approximately generalized derivation if $f(0) = 0$ and there exist a positive number ϵ and a mapping $g : A \to A$ such that*

$$\|f(\lambda a + \lambda b + cd) - \lambda f(a) - \lambda f(b) - cf(d) - g(c)d\| \leq \epsilon$$

for all $\lambda \in T$ and all $a, b, c, d \in A$. [6]

The next corollary immediately follows by Theorem 2.3.

Corollary 2.6. *Let A be a Banach algebra with a bounded central approximate identity. Suppose that $f : A \to A$ is an approximately generalized derivation with the corresponding mapping g. Then f is a generalized derivation and g is a derivation.*

Proof. Put $p = 0$ in Theorem 2.3. □

3. Stability of generalized derivations on faithful algebras

The significance of theorems, given in section 2, is that we do not need any additional functional inequality on g for existence of derivation δ. In this section, we suppose that g is an approximately linear function and prove the generalized Hyers-Ulam-Rassias stability of generalized derivations on faithful Banach algebras.

Theorem 3.1. *Let A be a left faithful Banach algebra. Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist a map $g : A \to A$ with $g(0) = 0$ and a function $\phi : A^6 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} 2^{-n}\phi(2^n a_1, 2^n b_1, 2^n c, 2^n d, 2^n a_2, 2^n b_2) = 0, \tag{3.1}$$

$$\widetilde{\phi}(a_1, a_2) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\phi(2^n a_1, 2^n a_1, 0, 0, 2^n a_2, 2^n a_2) < \infty, \tag{3.2}$$

and

$$\|f(\lambda a_1+\lambda b_1+cd)+g(\lambda a_2+\lambda b_2)-\lambda(f(a_1)+f(b_1)+g(a_2)+g(b_2))-cf(d)-g(c)d\|$$
$$\le \phi(a_1, b_1, c, d, a_2, b_2) \tag{3.3}$$

for all $\lambda \in T$ and all $a_1, a_2, b_1, b_2, c, d \in A$. Then there exist a unique derivation $\delta : A \to A$ and a unique generalized derivation $\mu : A \to A$ such that

$$\|\delta(a) - g(a)\| \le \widetilde{\phi}(0, a) \text{ and } \|\mu(a) - f(a)\| \le \widetilde{\phi}(a, 0)$$

for all $a \in A$.

Proof. We use the Rassias method to show that the sequences $(\frac{f(2^n a)}{2^n})$ and $(\frac{g(2^n a)}{2^n})$ are convergent (see [4, 6]).

Let $a_1, a_2 \in A$ and $c = d = 0$ and $b_1 = a_1, b_2 = a_2$ and $\lambda = 1$ in (3.3). Then

$$\|f(2a_1) + g(2a_2) - 2f(a_1) - 2g(a_2)\| \leq \phi(a_1, a_1, 0, 0, a_2, a_2). \tag{3.4}$$

By induction on n, we get

$$\left\|\frac{f(2^n a_1)}{2^n} - f(a_1) + \frac{g(2^n a_2)}{2^n} - g(a_2)\right\|$$

$$\leq \frac{1}{2}\sum_{k=0}^{n-1} 2^{-k}\phi(2^k a_1, 2^k a_1, 0, 0, 2^k a_2, 2^k a_2) \tag{3.5}$$

for all $n \in \mathbb{N}$.
Also by (3.4) and induction on n, we obtain

$$\|2^{-n}f(2^n a_1) - 2^{-m}f(2^m a_1) + 2^{-n}g(2^n a_2) - 2^{-m}g(2^m a_2)\|$$

$$\leq \frac{1}{2}\sum_{k=m}^{n-1} 2^{-k}\phi(2^k a_1, 2^k a_1, 0, 0, 2^k a_2, 2^k a_2) \tag{3.6}$$

for all $n > m$.

Fix $a \in A$. Let $a_2 = 0$ and $a_1 = a$ in (3.6). It follows from the convergence of series (3.2) and inequality (3.6) that the sequence $(\frac{f(2^n a)}{2^n})$ is Cauchy and so converges.
Again, putting $a_1 = 0$ and $a_2 = a$ in (3.6), we see that the sequence $(\frac{g(2^n a)}{2^n})$ converges. Set

$$\mu(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n} \quad , \quad \delta(a) := \lim_{n\to\infty} \frac{g(2^n a)}{2^n}.$$

Putting $a_2 = 0$ and $a_1 = a$ (respectively, $a_1 = 0$ and $a_2 = a$) in (3.5) and taking the limit as $n \to \infty$ we obtain, respectively,

$$\|\mu(a) - f(a)\| \leq \widetilde{\phi}(a, 0) \textit{ and } \|\delta(a) - g(a)\| \leq \widetilde{\phi}(0, a) \tag{3.7}$$

for all $a \in A$.

By the same reasoning as in the proof of [6, Theorem 2.1], one can show that μ and δ are linear mappings.

Now let $a_1 = b_1 = a_2 = b_2 = 0$ and replace c, d by $2^n c, 2^n d$, respectively, in (3.3). We obtain

$$\|f(2^{2n}cd) - 2^n cf(2^n d) - 2^n g(2^n c)d\| \leq \phi(0, 0, 2^n c, 2^n d, 0, 0)$$

and so

$$\|2^{-2n}f(2^{2n}cd) - 2^{-n}cf(2^n d) - 2^{-n}g(2^n c)d\| \leq 2^{-2n}\phi(0, 0, 2^n c, 2^n d, 0, 0).$$

By taking the limit as $n \to \infty$ we get $\mu(cd) = c\mu(d) + \delta(c)d$ for all $c, d \in A$.

It is sufficient to prove that δ is a derivation.
Let $c_1, c_2, x \in A$. We have
$\delta(c_1c_2)x = \mu(c_1c_2x) - c_1c_2\mu(x)$
$= c_1\mu(c_2x) + \delta(c_1)c_2x - c_1c_2\mu(x)$
$= c_1(\mu(c_2x) - c_2\mu(x)) + \delta(c_1)c_2x$
$= (c_1\delta(c_2) + \delta(c_1)c_2)x.$
Therefore, $\{\delta(c_1c_2) - c_1\delta(c_2) - \delta(c_1)c_2\}A = \{0\}$ and so $\delta(c_1c_2) = c_1\delta(c_2) + \delta(c_1)c_2$, since A is left faithful.
Thus δ is a derivation and μ is a generalized derivation.
It is easy to see that the additive mappings μ and δ satisfying (3.7) are unique. □

By Theorem 3.1 and by a similar method to the proof of Theorem 2.3, one can prove the following theorem.

Theorem 3.2. *Let A be a left and right faithful Banach algebra. Suppose that $f : A \to A$ and $g : A \to A$ are mappings with $f(0) = 0$ and $g(0) = 0$. Suppose that there exist the constants $\beta > 0$ and $p < 1$ such that*

$$\|f(\lambda a_1+\lambda b_1+cd)+g(\lambda a_2+\lambda b_2)-\lambda(f(a_1)+f(b_1)+g(a_2)+g(b_2))-cf(d)-g(c)d\|$$
$$\leq \beta(\|a_1\|^p + \|a_2\|^p + \|b_1\|^p + \|b_2\|^p + \|c\|^p + \|d\|^p)$$

for all $\lambda \in T$ and all $a_1, a_2, b_1, b_2, c, d \in A$. Then f is a generalized derivation and g is a derivation.

Proof. Put $\phi(a_1, b_1, c, d, a_2, b_2) = \beta(\|a_1\|^p + \|b_1\|^p + \|c\|^p + \|d\|^p + \|a_2\|^p + \|b_2\|^p)$ in Theorem 3.1. Then there exist a derivation $\delta : A \to A$ and a generalized derivation $\mu : A \to A$, defined by

$$\mu(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n} \quad , \quad \delta(a) := \lim_{n\to\infty} \frac{g(2^n a)}{2^n}.$$

By the same reasoning as in the proof of Theorem 2.3, we obtain $cf(d) = c\mu(d)$ and $g(c)d = \delta(c)d$ for all $c, d \in A$. Since A is left and right faithful, we have $f = \mu$ and $g = \delta$. □

Putting $p = 0$ in Theorem 3.2, we obtain a result similar to Corollary 2.6 for left and right faithful Banach algebras.

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