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NON-REGULARITY OF MULTIPLICATIONS FOR GENERAL MEASURE ALGEBRAS

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ABSTRACT. Let $\mathfrak{M}(X)$ be the space of all finite regular Borel measures on X. A general measure algebra is a subspace of $\mathfrak{M}(X)$, which is an L-space and has a multiplication preserving the probability measures. Let $\mathcal{L} \subseteq \mathfrak{M}(X)$ be a general measure algebra on a locally compact space X. In this paper, we investigate the relation between Arens regularity of $\mathcal{L}$ and the topology of X. We find conditions under which the Arens regularity of $\mathfrak{L}$ implies the compactness of X. We show that these conditions are necessary. We also present some examples in showing that the new conditions are different from Theorem 3.1 of [7].

1. **Introduction**

One of the questions in abstract harmonic analysis which has drawn a considerable amount of attention has been the question of which properties the measure algebra $\mathfrak{M}(G)$ or $\mathfrak{L}^1(G)$ of a locally compact group G can determine the topology of G. One of the papers dealing with this question is due to Young (see [9]) who proved that the group algebra $L^1(G)$ is Arens regular if and only if G is finite. In [3], Dales, Ghahramani and Helemskii have shown that $\mathfrak{M}(G)$ is weakly amenable if and only if the group G is discrete.

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We consider the concept of Arens regularity for general measure algebras.

Let X be a locally compact Hausdorff space and $C_c(X), C_0(X)$ denote the normed spaces of all complex valued continuous functions on X with compact supports, vanishing at infinity, respectively. Let $\mathfrak{M}(X)$ be the space of all bounded regular Borel measures on X with its usual norm and let $\mathfrak{M}_p(X)$ be the subset of $\mathfrak{M}(X)$ consisting of all probability measures. The total variation $\mu \in \mathfrak{M}(X)$ is denoted by $|\mu|$ and

$$\|\mu\| = |\mu|(1) = \int_X d|\mu|.$$

We shall say that $\mathfrak{M}(X)$ has a *general measure multiplication* if there exists a bilinear associative map $m : \mathfrak{M}(X) \times \mathfrak{M}(X) \longrightarrow \mathfrak{M}(X)$ such that if $\mu, \nu \in \mathfrak{M}_p(X)$ then $m(\mu, \nu) \in \mathfrak{M}_p(X)$. We shall write $\mu\nu$ for $m(\mu, \nu)$. A general measure multiplication on $\mathfrak{M}(X)$ makes it a Banach algebra (see [7, Proposition 2.1]). A closed subspace $\mathfrak{L}$ of $\mathfrak{M}(X)$ is an L-space if $\mu \in \mathfrak{L}$ and $|\nu| \ll |\mu|$ implies that $\nu \in \mathfrak{L}$. A subalgebra $\mathfrak{L}$ of $\mathfrak{M}(X)$ which is an L-space will be called a *general measure algebra* on X.

For $\mu \in \mathfrak{M}(X)$, the support of μ, or $\operatorname{supp}\mu$, is the largest closed set $F \subseteq X$ such that $f \in C_c^+(X)$, $f(x) > 0$, for some $x \in F$, implies

$$\int_X f d|\mu| = |\mu|(f) > 0.$$

Also, the support of $\mathfrak{L}$ or supp $\mathfrak{L}$ is defined by $\operatorname{supp}\mathfrak{L} = cl(\bigcup_{\mu\in\mathfrak{L}} \operatorname{supp}\mu)$.

Let A, B, C be normed spaces. A^* and A^{**} will be the first and second topological dual of A. The natural duality between A^* and A will be denoted by $< a^*, a >$ for $a^* \in A^*$, $a \in A$. A is canonically embedded into A^{**}

Let $m : A \times B \longrightarrow C$ be a continuous bilinear map. Then we can define two bounded bilinear extensions $m_1, m_2 : A^{**} \times B^{**} \longrightarrow C^{**}$ by using iterated limits as follows. For $a \in A$, $b \in B$ and $a^{**} \in A^{**}$, $b^{**} \in B^{**}$, let

$$m_1(a^{**}, b^{**}) = w^* - \lim_{a\longrightarrow a^{**}} (w^* - \lim_{b\longrightarrow b^{**}} m(a, b))$$
$$m_2(a^{**}, b^{**}) = w^* - \lim_{b\longrightarrow b^{**}} (w^* - \lim_{a\longrightarrow a^{**}} m(a, b))$$

The bilinear map m is *Arens regular* (or briefly regular) if the equality

$$m_1(a^{**}, b^{**}) = m_2(a^{**}, b^{**}) \quad (a^{**} \in A^{**}, b^{**} \in B^{**})$$

holds. The details of the constructions may be found in many places, including the book [2] and the articles [4, 5, 6, 8].

Throughout this paper, the first (second) Arens multiplication is denoted by $\Box$ (respectively $\Diamond$). Thise multiplications can be defined on $\mathfrak{M}(X)^{**}$ by

$$F\Box G = w^* - \lim_n w^* - \lim_m(\mu_n\nu_m), \quad (F\Diamond G = w^* - \lim_m w^* - \lim_n(\mu_n\nu_m)),$$

where $(\mu_n), (\nu_m)$ are nets of elements of $\mathfrak{M}(X)$ such that $\mu_n \longrightarrow F$ and $\nu_m \longrightarrow G$ in the weak* topology.

2. The iterated limits method in Arens regular

The usuall criteria cited for Arens regularity involves the iterated limit conditions [See [6] and [7], Theorem 2.3]. In the following Theorem, we present a new condition for a general measure algebra which is different from the hypothesises of Theorem 3.1 in [7].

In the rest of the paper, we adopt the notation $\mathfrak{L}_p = \mathfrak{M}_p(X) \cap \mathfrak{L}$ where $\mathfrak{L}$ is a general measure algebra.

Lemma 2.1. *Let $\mathfrak{L}$ be a general measure algebra on a locally compact and non-compact Hausdorff space X. Suppose that for each positive measure μ in $\mathfrak{L}, 0 \in w^* - cl(\mu\mathfrak{L}_p)$. Then for each $\varepsilon > 0$ and each compact subset K of X and positive measures $\mu_1, \ldots, \mu_n$ in $\mathfrak{L}$, there exist $\nu \in \mathfrak{L}_p$ and $\psi \in C_c(X)$ such that $K \cap \operatorname{supp} \psi = \phi$ and $\mu_i.\nu(\psi) \geq \|\mu_i\| - \varepsilon$ for all i.*

Proof. By Urysohn's Lemma, we can find $\varphi \in C_c(X)$ with $0 \leq \varphi \leq 1$ and $\varphi(x) = 1$ for all $x \in K$. Since $0 \in w^* - cl(\mu_1 + \cdots + \mu_n)\mathfrak{L}_p$, we can take $\nu \in \mathfrak{L}_p$ such that

$$(\mu_1 + \cdots + \mu_n)\nu(\varphi) < \varepsilon.$$

Since ν and μ_i $(1 \leq i \leq n)$ are positive measures, for $1 \leq i \leq n$,

$$\|\mu_i\nu\| = (\mu_i\nu)(1) = \mu_i(1)\nu(1) = \mu_i(1) = \|\mu_i\|.$$

It follows that

$$\|(\mu_1 + \cdots + \mu_n)\nu\| = ((\mu_1 + \cdots + \mu_n)\nu)(1) = \|\mu_1\| + \cdots + \|\mu_n\|.$$

Now suppose that $U = \{x \in X : \varphi(x) < 1\}$. Then,

$$\begin{aligned} ((\mu_1 + \cdots + \mu_n)\nu)(U) &\geq ((\mu_1 + \cdots + \mu_n)\nu)(1 - \varphi) \\ &> \|\mu_1\| + \cdots + \|\mu_n\| - \varepsilon. \end{aligned}$$

We now choose a compact subset $H \subseteq U$ (so that $H \cap K = \phi$) with

$$((\mu_1 + \cdots + \mu_n)\nu)(H) > \|\mu_1\| + \cdots + \|\mu_n\| - \varepsilon$$

Therefore, since $\mu_1, \ldots, \mu_n$ are positive measures in $\mathfrak{L}$ and $\nu \in \mathfrak{L}_p$, for $1 \leq i \leq n$,

$$\begin{aligned}
&\mu_i\nu(H)\\
&= ((\mu_1 + \cdots + \mu_n)\nu)(H) - ((\mu_1 + \cdots + \mu_{i-1} + \mu_{i+1} + \ldots \mu_n)\nu)(H)\\
&> \|\mu_1\| + \cdots + \|\mu_n\| - \varepsilon - ((\mu_1 + \cdots + \mu_{i-1} + \mu_{i+1} + \cdots + \mu_n)\nu)(1)\\
&= \|\mu_i\| - \varepsilon
\end{aligned}$$

Since, H and K are disjoint compact subsets of X, we can find disjoint compact open subsets V and W of X such that $H \subset V$, $K \subseteq W$ and $V \cap \bar{W} = \bar{V} \cap W = \phi$. We apply Urysohn's lemma to obtain $\psi \in C_c(X)$ with $0 \leq \psi \leq 1$, $\psi = 1$ on H and $\psi = 0$ off V. Thus, for all $1 \leq i \leq n$,

$$\mu_i\nu(\psi) \geq \mu_i\nu(H) > \|\mu_i\| - \varepsilon$$

and

$$(\operatorname{supp}\psi) \cap K = \phi.$$

□

Theorem 2.2. *Let $\mathfrak{L}$ be a general measure algebra on a locally compact Hausdorff space X such that for each positive measure μ, $0 \in w^* - cl(\mu\mathfrak{L}_p)$ and $0 \in w^* - cl(\mathfrak{L}_p\mu)$. If $\mathfrak{L}_p$ is Arens regular then X is compact.*

Proof. Suppose that X is not compact, we must show that $\mathfrak{L}_p$ is not Arens regular.

We construct a sequence of measures by induction.

First step. We start with any $\mu_1 \in \mathfrak{L}_p$ and take a compact K such that $\mu_1(K) > 1 - \frac{1}{2}$. By Urysohn's Lemma, there exists $\psi_1 \in C_c(X)$ with $0 \leq \psi_1 \leq 1$ and $\psi_1 = 1$ on K. So,

$$\mu_1(\psi_1) \geq \mu_1(K) > 1 - \frac{1}{2}.$$

By Lemma 3.1, if $K_1 = \operatorname{supp}\psi_1$, there exist $\mu_2 \in \mathfrak{L}_p$ and $\psi_2 \in C_c(X)$ such that $0 \leq \psi_2 \leq 1$,

$$K_1 \cap (\operatorname{supp}\psi_2) = \phi \quad \text{and} \quad \mu_1\mu_2(\psi_2) > 1 - \frac{1}{2^2}.$$

n^{th} step. Suppose that $\mu_1, \ldots, \mu_{2n-1} \in \mathfrak{L}_p$ and $\psi_1, \ldots, \psi_{2n-1} \in C_c(X)$ with $(\operatorname{supp}\psi_i) \cap (\operatorname{supp}\psi_j) = \phi$, $1 \leq i \neq j \leq 2n-1$ and $0 \leq \psi_i \leq 1$, have

been chosen. Using Lemma 2.1, take $\mu_{2n} \in \mathfrak{L}_p$ and $\psi_{2n} \in C_c(X)$ such that $0 \leq \psi_{2n} \leq 1$ and,

$$((\operatorname{supp}\psi_1) \cup \cdots \cup (\operatorname{supp}\psi_{2n-1})) \cap (\operatorname{supp}\psi_{2n}) = \phi,$$

$$\mu_i\mu_{2n}(\psi_{2n}) > 1 - \frac{1}{2^{2n}} \quad \text{for} \quad 1 \leq i \leq 2n-1.$$

Then, using Lemma 2.1 again (but with left and right interchanged) we can find $\mu_{2n+1} \in \mathfrak{L}_p$ and $\psi_{2n+1} \in C_c(X)$ such that $0 \leq \psi_{2n+1} \leq 1$,

$$((\operatorname{supp}\psi_1) \cup \cdots \cup (\operatorname{supp}\psi_{2n})) \cap (\operatorname{supp}\psi_{2n+1}) = \phi,$$

$$\mu_{2n+1}\mu_i(\psi_{2n+1}) > 1 - \frac{1}{2^{2n+1}} \quad \text{for} \quad 1 \leq i \leq 2n.$$

We now write

$$\psi = \sum_{i=1}^{\infty}(-1)^i\psi_i = \lim \sum_{i=1}^{n}(-1)^i\psi_i,$$

where (ψ_i) is a sequence of elements of $C_c(X)$ with disjoint supports and $0 \leq \psi_i \leq 1$ for each i. Therefore,

$$|\psi| = \sum_{i=1}^{\infty}|(-1)^i\psi_i| = \sum_{i=1}^{\infty}\psi_i \leq 1.$$

Note that, for each n,

$$|\psi - (-1)^n\psi_n| = \sum_{i=1, i\neq n}^{\infty}|(-1)^i\psi_i| = |\psi| - \psi_n \leq 1 - \psi_n.$$

We shall consider the iterated limits of the double sequence with terms

$$\mu_{2m-1}\mu_{2n}(\psi).$$

First consider the case where $m > n$, we have $2m-1 > 2n$ and

$$1 \geq \mu_{2m-1}\mu_{2n}(\psi_{2m-1}) > 1 - \frac{1}{2^{2m-1}}.$$

Hence

$$|\mu_{2m-1}\mu_{2n}((-1)^{2m-1}\psi_{2m-1}) - (-1)| = \mu_{2m-1}\mu_{2n}(1 - \psi_{2m-1}) < \frac{1}{2^{2m-1}}.$$

So, when $m > n$,

$$\begin{aligned}|\mu_{2m-1}\mu_{2n}(\psi) - (-1)| &\leq |\mu_{2m-1}\mu_{2n}(\psi - (-1)^{2m-1}\psi_{2m-1})| \\ &\quad + |\mu_{2m-1}\mu_{2n}(-\psi_{2m-1}) + 1| \\ &\leq \mu_{2m-1}\mu_{2n}(1 - \psi_{2m-1}) + \mu_{2m-1}\mu_{2n}(1 - \psi_{2m-1}) \\ &< 2 \times \frac{1}{2^{2m-1}} = \frac{1}{2^{2(m-1)}}.\end{aligned}$$

Thus, for each n,

$$\lim_m \mu_{2m-1}\mu_{2n}(\psi) = -1,$$

or

$$\lim_n \lim_m \mu_{2m-1}\mu_{2n}(\psi) = -1.$$

Similarly, when $n > m$,

$$\begin{aligned}|\mu_{2m-1}\mu_{2n}(\psi) - 1| &\leq |\mu_{2m-1}\mu_{2n}(\psi - (-1)^{2n}\psi_{2n})| + |\mu_{2m-1}\mu_{2n}(\psi_{2n}) - 1| \\ &\leq \mu_{2m-1}\mu_{2n}(1 - \psi_{2n}) + \mu_{2m-1}\mu_{2n}(1 - \psi_{2n}) \\ &< 2 \times \frac{1}{2^{2n}} = \frac{1}{2^{2n-1}}.\end{aligned}$$

Thus

$$\lim_m \lim_n \mu_{2m-1}\mu_{2n}(\psi) = 1.$$

We conclude that $\mathfrak{L}_p$ is not Arens regular. □

Corollary 2.3. *Let $\mathfrak{L}$ be a general measure algebra on a locally compact Hausdorff space X. Suppose that for each compact subset K of X and each positive measure μ in $\mathfrak{L}$ we can find two positive measures λ and ν in $\mathfrak{L}$ such that*

$$\mathrm{supp}(\lambda\mu) \cap K = \phi = \mathrm{supp}(\mu\nu) \cap K.$$

If $\mathfrak{L}_p$ is Arens regular then X is compact.

Proof. It is enough to show that $0 \in w^* - cl(\mathfrak{L}_p\mu)$ and then, by Theorem 2.2, X is compact.

Given $\varepsilon > 0$, $\varphi \in C_0(X)$ and μ is a positive measure in $\mathfrak{L}$, we can choose a compact set K of X such that $|\varphi(x)| < \frac{\varepsilon}{\|\mu\|}$, whenever $x \notin K$. Taking $\lambda \in \mathfrak{L}_p$ such that $\mathrm{supp}(\lambda\mu) \cap K = \phi$. For $K_1 = \mathrm{supp}(\lambda\mu)$, we have

$$|\lambda\mu(\varphi)| = |\int_{K_1} \varphi(t)d(\lambda\mu)(t)| \leq \frac{\varepsilon}{\|\mu\|}\|\lambda\mu\| = \varepsilon.$$

Hence $0 \in w^* - cl(\mathfrak{L}_p\mu)$.

Similarly, for each positive measure μ in $\mathfrak{L}$, $0 \in w^* - cl(\mu\mathfrak{L}_p)$, and then conclusion follows. $\square$

3. Examples

Let $\mathfrak{L}$ be a general measure algebra and let $\mu \in \mathfrak{L}$ be a positive measure. Next, we show that, in the Theorem 2.2, the conditions $0 \in w^* - cl(\mu\mathfrak{L}_p)$ and $0 \in w^* - cl(\mathfrak{L}_p\mu)$ are necessary.

Example 3.1. *There exists a general measure algebra $\mathfrak{L}$ such that $0 \notin w^* - cl(\mu\mathfrak{L}_p)$ (for a positive measure μ in $\mathfrak{L}$) but $\mathfrak{L}$ is Arens regular without X being compact.*

Construction. Let X be a locally compact and non-compact Hausdorff space. Define a multiplication on $\mathfrak{M}(X)$ by

$$\mu\nu = \nu(1)\mu \quad (\mu, \nu \in \mathfrak{M}(X)).$$

For $\mu, \nu, \lambda \in \mathfrak{M}(X)$, we have

$$\mu(\nu\lambda) = (\lambda(1)\nu)(1)\mu = \lambda(1)(\nu(1)\mu) = (\mu\nu)\lambda.$$

If $\mu, \nu \in \mathfrak{M}_p(X)$ then

$$(\mu\nu)(1) = \nu(1)\nu(1) = 1.$$

So the multiplication is associative and it maps a pair of probability measures to a probability measure. Let $\mathfrak{L} \subseteq \mathfrak{M}(X)$ be a general measure algebra and let $\mu \in \mathfrak{L}$ be a positive measure. Take $\varphi \in C_0(X)$ with

$$0 \le \varphi \le 1, \quad \varphi = \frac{1}{\|\mu\|} \qquad (\text{on } \operatorname{supp} \mu).$$

Now, if $\nu \in \mathfrak{L}_p$ then

$$(\mu\nu)(\varphi) = \nu(1)\mu(\varphi) = 1.$$

Therefore, $0 \notin w^* - cl(\mu\mathfrak{L}_p)$.

Now we prove that $\mathfrak{L}$ is Arens regular. Let $F, G \in \mathfrak{L}^{**}$. Take two nets (μ_α) and (ν_β) of $\mathfrak{L}$ such that $F = w^* - \lim \hat{\mu}_\alpha$ and $G = w^* - \lim_\beta \hat{\nu}_\beta$. Then

$$\begin{aligned}
F \square G &= w^* - \lim_\alpha w^* - \lim_\beta (\mu_\alpha \nu_\beta)\hat{} \\
&= w^* - \lim_\alpha w^* - \lim_\beta \nu_\beta(1)\hat{\mu}_\alpha \\
&= w^* - \lim_\beta -w^* - \lim_\alpha (\mu_\alpha \nu_\beta)\hat{} \\
&= F \Diamond G,
\end{aligned}$$

and so $F\Box G = F\Diamond G$ which shows that $\mathfrak{L}$ is Arens regular.

Note that a general measure algebra which has separately weak* continuous multiplication and has an identity in its support is not Arens regular (See [7, Theorem 3.1]). But all the above properties are not necessary in general. Also, the following examples show that the Theorem 2.2 can not be derived as a consequence of [7, Theorem 3.1].

Example 3.2. *There exists a general measure algebra $\mathfrak{M}(X)$ which is neither Arens regular nor have the following properities:*

(i) *There is an identity for $\mathfrak{M}(X)$.*

(ii) *The multiplication in $\mathfrak{M}(X)$ is weak* separately continuous.*

Construction. Start with the order set $N = \{\dots, 3, 1, 2, 4, \dots\}$ of positive integers with discrete topology. Define an operation $(n, m) \mapsto n \circ m$, from $N \times N$ into N, by the following conditions:

$$n \circ m = \begin{cases} \max\{n, m\} & (n = 2k, m = 2k') \\ \min\{n, m\} & (n = 2k + 1, m = 2k' + 1) \\ mon = n & (n = 2k, m = 2k' + 1) \end{cases}$$

where $m, n, k, k' \in N$. It is easy to see that the operation is associative, commutative and jointly continuous (by discreteness).

Take a sequence (x_n) of distinct points of real numbers such that $x = \lim_n x_n$ exists. We put $X = \{\dots, x_3, x_1, x, x_2, x_4, \dots\}$ with new topology such that $\{x_{2n}\}$ is the only sequence in X which converges to x. A bace of this topology is a collection of all subset B of X which $B = \{x_n\}$ or $B = \{x_{2n}, x_{2(n+1)}, \cdots, x\}$. Then $\lim x_{2n} = x$ and

$$\mathfrak{M}(X) = \{a_0\delta_x + \sum_{n=1}^{\infty} a_n\delta_{x_n} : \sum_{n=0}^{\infty} |a_n| < \infty\}.$$

Define a multiplication (convolution) on $\mathfrak{M}(X)$ as follows:

$$\delta_y * \delta_z = \delta_z * \delta_y = \begin{cases} \delta_{x_{nom}} & (y = x_n, z = x_m) \\ \delta_y & (y = x_{2n}, z = x) or (y = x, z = x_{2m-1}) \\ \delta_x & (y = x = z) \end{cases}$$

It is easy to see that

$$\delta_y * (\delta_z * \delta_t) = (\delta_y * \delta_z) * \delta_t \quad (y, z, t \in X).$$

So, the multiplication is commutative and associative and it a pair of maps probability measures to a probability measure. Therefore, $\mathfrak{M}(X)$ is a general measure algebra.

Set $e_n = \delta_{x_{2n-1}}$. It is easy to check that

$$\lim_n e_n * \mu = \mu = \lim_n \mu * e_n \quad (\mu \in \mathfrak{M}(X)).$$

(i) $\mathfrak{M}(X)$ does not contain an identity. Since, if $e = \sum_{i\in I} a_i\delta_{x_i}$ is an identity (the set I is index set) then for each n,

$$e * \delta_{x_{2n-1}} = \delta_{x_{2n-1}}.$$

Hence, for each $k \in I$, $ko(2n+1) = \min\{k, 2n+1\} = 2n+1$. Thus, $k \geq 2n-1$ and it is impossible.

(ii) Since, $w^* - \lim_n \delta_{x_{2n}} = \delta_x$, then

$$w^* - \lim_n(\delta_{x_{2n}} * \delta_{x_2}) = w^* - \lim_n \delta_{x_{2n}} = \delta_x,$$

but $\delta_x * \delta_{x_2} = \delta_{x_2}$. Therefore,

$$w^* - \lim_n(\delta_{x_{2n}} * \delta_{x_2}) \neq (w^* - \lim_n \delta_{x_{2n}}) * \delta_{x_2}$$

Hence, the multiplication is not separately weak* continuous.

Finally we show that $\mathfrak{M}(X)$ is not Arens regular. Set $\mu_n = \delta_{x_n}$ write

$$\psi = \sum_{n=1}^{\infty}(-1)^n \chi_{\{x_{2n}\}}.$$

For each n, m, we have

$$\mu_{4n} * \mu_{4m+2}(\psi) = \begin{cases} \mu_{4m+2}(\psi) = -1 & m > n \\ \mu_{4n}(\psi) = 1 & m \leq n, \end{cases}$$

$$w^* - \lim_n \lim_m \mu_{4n} * \mu_{4m+2} \neq w^* - \lim_m \lim_n \mu_{4n} * \mu_{4m+2}.$$

Thus, $\mathfrak{M}(X)$ is not Arens regular.□

Example 3.3. By the same construction in the example 3.2, suppose that $Y = X\backslash\{x\}$, $\mathfrak{M}(Y)$ has not an identity, but $0 \in w^* - cl(\mu\mathfrak{M}_p(Y))$ for each positive measure μ in $\mathfrak{M}(Y)$. So, by Theorem 2.2, $\mathfrak{M}(Y)$ is not Arens regular. It follows that the Theorem 2.2, con not be derived as a consequecne of [7, Theorem 3.17].

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References

[1] R. Arens, The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.* **2** (1951), 839–848.
[2] H. G. Dales, Banach Aalgebras and Automatic Continuity, Clarendon Press, Oxford University Press, New York, 2000.
[3] H. G. Dales, F. Ghahramani and A. Ya. Helemskii, The amenability of measure algebras, *J. London Math. Soc.* **66** (2002), no. 2, 213–226.
[4] H. G. Dales, A. Rodriguez-Palacios and M. V. Velasco, The second transpose of a derivation, *J. London Math. Soc.* **64** (2001), no. 2, 707–721.
[5] M. Eshaghi Gorgji and M. Filali, Arens regularity of module actions, *Studia Math.* **181** (2007), no. 3, 237–254.
[6] J. Laali, Arens regularity of bilinear maps for Banach spaces, *Far East J. Math. Sci. (FJMS)* **12** (2004), no. 1, 89–103.
[7] J. Laali and J. Pym, Concepts of Arens regularity for general measure algebras, *Quart. J. Math. Oxford* **47** (1996), no. 2, 187–198.
[8] S. Mohammadzadeh and H. R. E. Vishki, Arens regularity of module actions and the second adjointg of a derivation, *Bull. Austral. Math. Soc.* **77** (2008), 465–476.
[9] N. J. Young, The irregularity of multiplication in group algebras, *Quart. J. Math. (Oxford)* **24** (1973), 59–62.

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