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PROJECTIVE MAXIMAL SUBMODULES OF EXTENDING REGULAR MODULES

E. MOMTAHAN

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Abstract. We show that a projective maximal submodule of a finitely generated, regular, extending module is a direct summand. Hence, every finitely generated, regular, extending module with projective maximal submodules is semisimple. As a consequence, we observe that every regular, hereditary, extending module is semisimple. This generalizes and simplifies a result of Dung and Smith. As another consequence, we observe that every right continuous ring, whose maximal right ideals are projective, is semisimple Artinian. This generalizes some results of Osofsky and Karamzadeh. We also observe that four classes of rings, namely right $\aleph_0$-continuous rings, right continuous rings, right $\aleph_0$-continuous regular rings and right continuous regular rings are not axiomatizable.

1. Introduction

Here, R is an associative ring with unity and M is a right R-module. By a regular ring we mean von Neumann regular. A module M is called regular if every finitely generated submodule of M is a direct summand. Every projective module over a regular ring is regular. A module is called hereditary if its submodules are projective. We mention that projective regular modules were studied in [20] and [22] by Ware and Zelmanowitz, respectively, besides those who touched the concept of general regular

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modules (i.e., without the projectivity condition) subject to their works on purities (see [21]). Ggeneral regular modules were also studied in [13] by Karamzadeh.

We say that a submodule N of M is essential in M, denoted by $N \trianglelefteq M$, if N intersects each nonzero submodule of M nontrivially. If N is a proper submodule which is essential in M, we denote it by $N \triangleleft M$. For any right R-module M, the singular submodule $Z(M)$ is defined as $Z(M) = \{m \in M : \mathrm{Ann}_R(m) \trianglelefteq R\}$. The module M is called singular if $Z(M) = M$ and nonsingular if $Z(M) = 0$. It is well-known that $Z(M)$ is an invariant submodule, i.e., $\mathrm{Hom}_R(M, M)Z(M) \subseteq Z(M)$, and every submodule of a singular (nonsingular) module is singular(nonsingular). A module M is called ($\aleph_0$–)CS or ($\aleph_0$–)extending if every (countably generated) submodule is essential in a direct summand of M. If, in addition, every submodule of M which is isomorphic to a summand of M is itself a summand, then M is called a continuous module. It is well-known that a direct summand of an extending (a continuous) module is extending (continuous) (see [15, Proposition 2.7]). A ring is called right continuous if R_R is a right continuous module. When R is a regular ring, R is a right ($\aleph_0$–)continuous ring if and only if it is a right ($\aleph_0$-) extending ring, because R_R always has C_2, i.e., every submodule which is isomorphic to a summand is itself a summand, for regular rings. A ring is called right ($\aleph_0$-)self-injective if for every (countably generated) right ideal I and every right R-module homomorphism $\phi : I \longrightarrow R_R$, there exists a right R-module homomorphism $\theta : R \longrightarrow R$ such that $\theta|_I = \phi$. It is well-known that right self-injective rings are right continuous. However, neither right $\aleph_0$-self-injectivity nor right $\aleph_0$-continuity implies the other one (see Section 3). Let M be a right R-module. By the socle of M, we mean the sum (or direct sum) of all simple submodules of M. The category of all right R-modules is denoted by R-Mod. A right ideal I in a ring R is said to be small if $I + K = R$ implies that $K = R$, where K is a right ideal of R. A ring R is called semiperfect if $R/Jac(R)$ is semisimple Artinian and idempotents lift modulo the Jacobson radical.

In [17] and [18], Osofsky has shown that a ring with the property that all its cyclic right modules are injective is semisimple Artinian. She has also proved that any right hereditary right self-injective ring is semisimple Artinian. This interesting fact is quoted by several authors (see, for example, [12], [3] and [5]). Karamzadeh [12, Corollary 3] has improved this result by replacing "hereditary", by "maximal right ideals are projective". In [5, Theorem 6], Dung and Smith have remarkably generalized Osofsky's observation as follows:

> ([5, Theorem 6]) Let R be any ring and M be a right R-module such that M is a hereditary extending module. Then, M is a direct sum of Noetherian uniform modules.

Their proof is based on the fundamental observation of Osofsky (see [6, Lemma 10.3]), in which she used a set theoretic result of Tarsky. As a consequence of Theorem 6, they observed that every regular, hereditary, extending module is semisimple. In Proposition 2.2, we observe that every finitely generated, regular, extending module in which every maximal submodule is projective, is semisimple. As a corollary, we prove that every hereditary, regular, extending module is semisimple and this simplifies and generalizes their observation (see [5], page 177, a paragraph before Proposition 13). The reader is reminded that their main result ([5, Theorem 6]) remains intact by our observation.

Dung and Smith have also shown (as a consequence of their main Theorem) that: (i) Any right hereditary right extending ring is right noetherian and (ii) with R being a ring which is either commutative or semiprime, any hereditary continuous right R-module is semisimple (see [5, Corollary 7 and Theorem 11]). They also proved that ([5, Example 12]) there are hereditary injective Noetherian R-modules which are not semisimple, where R is neither commutative nor semiprime. Nevertheless, in Proposition 3.4, we observe (based on Proposition 2.2) that *a right continuous ring, in which maximal right ideals are projective, is semisimple Artinian.* This shows that if in (i) we also assume R is right continuous, then we obtain semisimple Artinian rings and furthermore, if we restrict ourselves only to rings, it is possible to drop the conditions "commutative" and "semiprime" in (ii).

2. Regular extending modules

It is well-known that for a countably generated module N, these two statements are equivalent: (a) N is a direct sum of finitely generated modules; (b) every finitely generated submodule of N is contained in a finitely generated direct summand (e.g., [21, (8.9)]). Based on these facts, the following key lemma has been proved in [13]. Here, we give a proof for the sake of completeness.

Lemma 2.1. *Let M be a regular module. Then, every countably generated submodule of M is a direct sum of finitely generated submodules.*

Proof. Let P be a countably generated submodule of M. Then, every finitely generated submodule of P is a direct summand of M, and hence by the modular law a direct summand of P. Now, by the aforementioned fact the proof is complete. $\square$

Karamzadeh has observed that in a right self-injective ring, every projective maximal right ideal is a direct summand. This was our motivation to prove the following (see [12, Proposition 1] and [14, Corollary 1.7]).

Proposition 2.2. *Let M be a finitely generated regular extending module and P be a maximal submodule of M which is projective. Then, P is a direct summand of M.*

Proof. By the Kaplansky theorem (see [11]), each projective module is a direct sum of countably generated modules. By the Lemma 2.1, we can write $P = \sum_{i\in I} \oplus P_i$, where each P_i is a finitely generated submodule. If I is infinite, then $I = A \cup B$, where $A \cap B = \emptyset$ and $|A| = |B| = \infty$. Set $P_1 = \sum_{i\in A} \oplus P_i$ and $P_2 = \sum_{i\in B} \oplus P_i$. Now, there exist direct summands, K_1 and K_2, such that $P_i \trianglelefteq K_i$, for $i = 1, 2$. That $K_1 \bigcap K_2 = (0)$ is an immediate consequence of the fact that $P_1 \bigcap P_2 = (0)$. We claim that either $P_1 = K_1$ or $P_2 = K_2$. Otherwise, $P = P_1 \oplus P_2 \lhd K_1 \oplus K_2 = M$, and then

$$\frac{M}{P} = \frac{K_1 \oplus K_2}{P_1 \oplus P_2} \cong \frac{K_1}{P_1} \oplus \frac{K_2}{P_2},$$

but, $\frac{M}{P}$ is a simple module, giving a contradiction. Hence, either $P_1 = K_1$ or $P_2 = K_2$. But, this means that either A or B is finite, because M is finitely generated, a contradiction again. Hence, I is finite. $\square$

In [5], Dung and Smith, as a corollary of their main theorem, observed that every regular hereditary extending module is semisimple. The next corollary is theirs.

Corollary 2.3. *Let M be a right hereditary regular extending module. Then, M is semisimple.*

Proof. By Kaplansky's theorem, M is a direct sum of countably generated projective submodules, and by Lemma 2.1, M is indeed a direct sum of finitely generated submodules, i.e., $M = \sum_{i\in I} \oplus M_i$, where each M_i is finitely generated, extending (by [15, Proposition 2.7]) and regular. Since, by Proposition 2.2, every maximal submodule of each M_i is a direct summand, we know that the M_i are semisimple. This implies that M is semisimple.

□

Remark 2.4. *Let M be a finitely generated $\aleph_0$-extending regular module. Then, every countably generated maximal submodule of M is a direct summand. Therefore, essential maximal submodules of M are uncountably generated. In particular, if M is countable, then M is Artin semisimple.*

3. **Continuous rings**

Proposition 2.2 gives the following fact immediately.

Proposition 3.1. *Let R be a right continuous regular ring. Then, every projective maximal right ideal of R is a direct summand.*

Proof. By Proposition 2.2, the verification is immediate. □

Corollary 3.2. *Let R be a commutative regular continuous ring. Then, every projective prime ideal of R is a direct summand.*

The following lemma is a modification of a result by Güngöroglu [9].

Lemma 3.3. *Let R be a right continuous ring and P be a projective maximal right ideal. Then, $P = eR + Jac(R)$, where $e = e^2$.*

Proof. Let P be a right maximal ideal of R which is projective. Then, P/J is a maximal right ideal of R/J, where $J = Jac(R)$. By [15, Propositions 3.11], R/J is a right continuous regular ring. On the other hand, for every $f : R \longrightarrow R$, we have $f(J) \subseteq J$ (J is small). Now, by the dual basis lemma, we conclude that P/J is a projective maximal right ideal of R/J. By Proposition 3.1, it follows that P/J is a direct summand of R/J. Since idempotents of R/J lift to R by [15, Lemma 3.7], there is an idempotent $e \in R$ such that $P = eR + J$. □

Fact: Let R be a ring and N be a sigular right R-mdule. Then, no non-zero submodule inside N is projective. In fact, in R-Mod, every singular right R-module has the form F/K, where F is a free right R-module and K is an essential submodule of F. Now, if F/K is projective, then the following short exact sequence splits: $0 \to K \to F \to F/K \to 0$, i.e., K is a direct summand, but K is essential (and not equal to F); a contradiction.

Let R be a ring. The above observation shows that $Z(R_R)$ contains no non-zero projective right subideal; in particular, $Z(R_R)$ is not projective.

Since in a right continuous ring, $Jac(R) = Z(R_R)$ ([15, Proposition 3.5]), we conclude that in a right continuous ring, $Jac(R)$ contains no non-zero projective right ideal. As a generalization of Osofsky's observation, Karamzadeh proved that a right self-injective ring, in which maximal right ideals are projective, is Artin semisimple ([12, Corollary 3]). He also observed that the converse may not be true (see [14]). The next proposition generalizes Karamzadeh's result (and hence Osofsky's one).

Proposition 3.4. *Let R be a right continuous ring. Then, a maximal right ideal of R is projective if and only if it is a direct summand. In particular, a right continuous ring, in which maximal right ideals are projective (e.g., a hereditary right continuous ring), is Artin semisimple.*

Proof. We have already shown that every maximal right ideal P is of the form $P = eR + J$, where $J = Jac(R)$ (Lemma 3.3). Hence, $(1-e)P \subseteq J = Z(R_R)$, and $P = eP \oplus (1-e)P$, i.e., $(1-e)P$ is projective, and hence is zero. This means that $P = eP = eR$. The converse is trivial. □

Corollary 3.5. *Let R be a right continuous regular ring which is not a division ring. Then, $S = soc(R)$ is not a maximal right ideal and each maximal right ideal of R/S is essential.*

Proof. If $S = 0$ and a maximal right ideal, then R is a division ring, giving a contradiction. Now, suppose that $S \neq 0$ and is a maximal right ideal. Since S is projective, $R = S \oplus m$. Now, m is a minimal right ideal, i.e., $m \subset S$, a contradiction. Suppose P/S is a maximal right ideal in R/S and $R/S = P/S \oplus K/S$. Since P/S is principal, we can write $P = xR \oplus D$, where $D \subseteq S$, and so P is projective, and hence $R = P \oplus m$, but then $m \subseteq S \subseteq P$; a contradiction. □

In [5, Corollary 14], it has been observed that: *every right continuous ring, in which every right ideal is countably generated, is a semiperfect ring.* The next corollary is a generalization of this observation.

Corollary 3.6. *Let R be a right continuous ring such that every maximal right ideal of R is countably generated. Then, R is semiperfect. In particular, any countable right continuous ring is semiperfect.*

Proof. Let J denote the Jacobson radical of R. Then, R/J is a von Neumann regular ring and idempotents can be lifted (see [15, Corollary 3.9 and Theorem 3.11]). Since every maximal right ideal of R/J is countably generated, using regularity, every maximal right ideal of R/J is projective. Hence, by Proposition 3.1, every maximal right ideal of

R/J is a direct summand, i.e., R/J is semisimple artinian. Hence, R is semiperfect. □

It is well-known that there exist prime right $\aleph_0$-continuous regular rings which are not right $\aleph_0$-self-injective (see [8, Example 14.9]). On the other hand, there exist prime unit regular right and left $\aleph_0$-self-injective rings which are neither right nor left $\aleph_0$-continuous (see [8, Example 14.21]). But, the next result shows that right $\aleph_0$-continuous regular rings and right $\aleph_0$-self-injective regular rings share this property that the essential maximal right ideals are uncountably generated.

Proposition 3.7. *Let R be either a right $\aleph_0$-self-injective regular or a right $\aleph_0$-extending regular ring. Then every countably generated right maximal ideal is a direct summand.*

Proof. For R being a right $\aleph_0$-self-injective regular, the fact has been shown in [16], by the author.

Now, let R be a right $\aleph_0$-continuous regular ring and P be a countably generated right ideal. By a slight modification of Proposition 3.1, and this fact that countably generated right ideals of a regular ring are projective, we conclude that P is a direct summand.

□

One may conclude from [8, Lemma 14.18] that in a right $\aleph_0$-continuous regular ring, every countably generated right ideal is a right subideal of a countably generated essential right ideal. This implies that in right $\aleph_0$-continuous regular rings, there are always countably generated essential right ideals. Now, by the above proposition we may conclude that a right $\aleph_0$-continuous regular ring contains countably generated essential right ideals which are not maximal.

Corollary 3.8. *Let R be a right $\aleph_0$-continuous regular ring. If R contains an infinite set of orthogonal idempotents, then R contains a countably generated essential right ideal which is not maximal.*

In the sequel, by X we mean a Tychonoff space and by $C(X)$ we mean the ring of all real valued continuous functions on X. The reader is referred to [7] for undefined terms and notions about $C(X)$.

Example 3.9. *In view of Corollary 3.8, it is worth to mention that, in $C(X)$, where X is an infinite space, there is always an essential ideal which is not a prime ideal (see* [1, Proposition 4.1]*).*

Example 3.10. *Let X be a Tychonoff space and $C(X)$ be the ring of real valued continuous functions. Brookshear has shown that each projective prime ideal of $C(X)$ is generated by an idempotent and therefore is a direct summand (see* [4]*). On the other hand, in* [2]*, it has been shown that (i) X is extremally disconnected (i.e., the closure of any open set is open) if and only if $C(X)$ is extending and (ii) $C(X)$ is $\aleph_0$-extending if and only if X is basically disconnected (i.e., the closure of any cozero-set is open). Now, one may ask: "Is a commutative regular ring with all projective prime ideals direct summand, a continuous ring ?". To get a counter example, choose a P-space which is not an exteremally disconnected space. Since every P-space is basically disconnected, $C(X)$ is an $\aleph_0$-continuous regular ring, and by Brookshear's observation its projective prime ideals are direct summands, but $C(X)$ is not a continuous ring by (i). However, in a commutative $\aleph_0$-continuous regular ring, every countably generated prime ideal is a direct summand.*

4. On axiomatizability of continuous rings

Now, we turn to model theoretic aspects of continuous and $\aleph_0$-continuo -us rings and prove that four classes of rings, namely right $\aleph_0$-continuous rings, right continuous rings, right $\aleph_0$-continuous regular rings and right continuous regular rings are not axiomatizable. We use definitions and terminologies of [10]. A class $\mathcal{C}$ of rings is called *axiomatizable* if there exists a family of first order sentences in the corresponding language such that $\mathcal{C}$ consists exactly of the rings satisfying these first order sentences. We say that two rings are *elementary equivalent*, and denote this by $R \equiv S$, if R and S satisfy the same first order sentences in the corresponding language. A class $\mathcal{C}$ of rings is called elementarily closed if $R \equiv S$ and $R \in \mathcal{C}$ implies $S \in \mathcal{C}$. It is well-known that a class $\mathcal{C}$ is axiomatizable if and only if $\mathcal{C}$ is closed under elementary equivalence and under formation of ultraproducts (see [10, Theorem 2.12]). In the sequel, we use the downward Löwenheim-Skolem theorem. Briefly, this states that if a countable infinite set of sentences formalized within a first order predicate calculus has a model, then the sentences have a countable submodel. We also need the following result from [10].

Proposition 4.1. *Let R be a semiperfect ring. Then, R is stable under elementary descent and under elementary equivalence.*

Proof. See [10, Proposition 10.10]. □

It is well-known that the class of self-injective rings is not closed under elementary equivalence and under formation of ultraproducts. In [16], it has been observed that the class of right $\aleph_0$-self-injective (regular) rings is not axiomatizable, but it is closed under formation of ultraproducts. Along this line, we study the axiomatizability of the following classes of rings.

Proposition 4.2. *The following classes of rings are not closed under elementary equivalence and hence are not axiomatizable:*

(1) *the class of right continuous regular rings;*
(2) *the class of right $\aleph_0$-continuous rings;*
(3) *the class of right continuous rings;*
(4) *the class of right $\aleph_0$-continuous regular rings.*

Proof. Let $\mathcal{C}$ be any of the four classes mentioned above. Suppose that this class is closed under elementary equivalence. Let $R \in \mathcal{C}$. Then, by the downward Löwenheim-Skolem theorem, R has a countable substructure (subring) S, which is elementarily equivalent with R, i.e., $S \equiv R$. By the hypothesis, S belongs to any of the four classes. And by Corollary 3.6, S is semiperfect. Hence, by Proposition 4.1, R is also semiperfect. But, there are non-semiperfect rings which belong to any of the four classes (for example, an infinite direct product of a field belongs to each of the above classes and is not semiperfect yet); a contradiction.

□

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Ehsan Momtahan Department of Mathematics, College of Science, Yasouj University, P.O. Box 75918-74831, Yasouj, Iran

Email: e_momtahan@mail.yu.ac.ir