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POINTS AT RATIONAL DISTANCE FROM THE VERTICES OF A UNIT POLYGON

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Abstract. We investigate the existence of a point in the plane of a unit polygon that is at rational distance from each vertex of the polygon. A negative answer is obtained in almost all cases.

1. **Introduction**

If T is a unit equilateral triangle, then there are points in the plane of T, that are at rational distance from the vertices of T (any vertex will do). Further, as proved in [1] and [2], the set of such points is dense in the plane of T. Concerning the unit square S, it is not (yet) known whether there is a point in the plane of S that is at rational distance from the corners of S. Results as in [2] suggest a *negative* answer, but the problem remains open.

What about the *unit pentagon* P_5 (regular pentagon with unit side)? Is there a point in the plane of P_5 that is at rational distance from the vertices of P_5?

More generally, for $n \geq 3$, let P_n denote the *unit* $n-$*gon* (regular $n-$gon with unit side). Consider the following question:

(P1) Is there a point in the plane of P_n that is at rational distance from the vertices of P_n?

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As noted, the answer to (P1) is positive if $n = 3$, and it turns out that, for $n \geq 4$, the most difficult case is indeed the case $n = 4$. In this note, we focus on the cases $n \geq 5$ and we prove the result.

Theorem 1.1.
• *For $n = 5$, the answer to (P1) is NEGATIVE.*
• *For $n = 6$, the answer to (P1) is POSITIVE.*
• *For* all *$n \geq 7$, the answer to (P1) is NEGATIVE, except* perhaps *when $n \in \{8, 12, 24\}$.*

The key-tool lies in the following observation: When the answer to (P1) is positive for a given $n \geq 3$, then, an identity as

$$\frac{n}{4} \cot \frac{\pi}{n} = \sqrt{r_1} \pm \sqrt{r_2} \pm \cdots \pm \sqrt{r_n}$$

must occur, where the r_i are nonnegative rational numbers. But, such identity is *impossible* for $n = 5$ as well as for *all* $n \geq 7$, provided that $n \neq 8, 12, 24$.

2. Preliminaries

We start with a simple property.

Proposition 2.1. *Let d, m, n be positive integers with $d > 1$ and $n = dm$. Then, $\mathbb{Q}(\cot \frac{\pi}{d})$ and $\mathbb{Q}(\cos \frac{2\pi}{d})$ are subfields of $\mathbb{Q}(\cot \frac{\pi}{n})$.*

Proof. • Set $x = \frac{\pi}{n}$ and $y = \frac{\pi}{d}$. Then, $y = mx$. To see why $\mathbb{Q}(\cot y) \subset \mathbb{Q}(\cot x)$, or equivalently, $\cot y \in \mathbb{Q}(\cot x)$, use induction on $m \geq 1$ and the identity $\cot(m+1)x = \frac{\cot mx \cdot \cot x - 1}{\cot mx + \cot x}$. • Next, set $t = \cot \frac{\pi}{d}$. From $\cos \frac{2\pi}{d} = \frac{t^2-1}{t^2+1}$ and $t \in \mathbb{Q}(\cot \frac{\pi}{n})$, we get, $\cos \frac{2\pi}{d} \in \mathbb{Q}(\cot \frac{\pi}{n})$. Hence, $\mathbb{Q}(\cos \frac{2\pi}{d}) \subset \mathbb{Q}(\cot \frac{\pi}{n})$. □

Let us call a 2-group, a group in which every element has order 1 or 2. For convenience, we give the following definition.

Definition 2.2. We say that a real field F is "flat" if every subfield E of F satisfies:

The Galois group $G(E : \mathbb{Q})$ is a 2-group.

Remark 2.3. *Obviously, a subfield of a flat field is flat.*

Proposition 2.4. *Let $r_1, r_2, \dots, r_n$ be nonnegative rational numbers. Then,*

$$\mathbb{Q}(\sqrt{r_1} \pm \sqrt{r_2} \pm \cdots \pm \sqrt{r_n}) \text{ is a flat field.}$$

Proof. Due to Remark 2.3, it suffices to show that $F = \mathbb{Q}(\sqrt{r_1}, \sqrt{r_2}, \dots, \sqrt{r_n})$ is a *flat* field. As quickly seen, $F : \mathbb{Q}$ is a Galois extension (of degree 2^ν). We first show that $G = G(F : \mathbb{Q})$ is a 2-group. Let $\sigma \in G$. Then, $\sigma(\sqrt{r_i}) \in \{\pm\sqrt{r_i}\}$, and so $\sigma \circ \sigma(\sqrt{r_i}) = \sqrt{r_i}$. As an element, x in F has the form $f(\sqrt{r_1}, \sqrt{r_2}, \dots, \sqrt{r_n})$, where $f \in \mathbb{Q}[X_1, X_2, \dots, X_n]$. It follows easily that $\sigma \circ \sigma(x) = x$.

Since every 2-group is abelian, then, $F : \mathbb{Q}$ is an *abelian* extension. Now, let E be any subfield of F. Since $F : \mathbb{Q}$ is abelian, then, $E : \mathbb{Q}$ is a Galois extension and the group $G(E : \mathbb{Q})$ is isomorphic to a quotient of $G(F : \mathbb{Q})$. Since a quotient of a 2-group is a 2-group, we see that $G(E : \mathbb{Q})$ is a 2-group. □

Lemma 2.5. *Let p be a prime number. Suppose that the relation $a^2 = p(b^2 + c^2)$ holds for some positive rational numbers $a, band c$. Then, $\mathbb{Q}(\sqrt{a + b\sqrt{p}}) : \mathbb{Q}$ is a cyclic extension of degree 4.*

Proof. • $a + b\sqrt{p}$ is *NOT* a square in $\mathbb{Q}(\sqrt{p})$: Otherwise, for some $x, y \in \mathbb{Q}$, $a + b\sqrt{p} = (x + y\sqrt{p})^2$. Hence, $x^2 + py^2 = a$ and $2xy = b$. So, $x^2 + p\left(\frac{b}{2x}\right)^2 = a$ and x^2 is a zero of $X^2 - aX + \frac{1}{4}pb^2 = 0$. Since $\sqrt{a^2 - pb^2} = \sqrt{pc^2} = c\sqrt{p}$, it follows that x^2, and hence x, is irrational, giving a contradiction.

• Set $\theta = \sqrt{a + b\sqrt{p}}$. We just proved that $\theta \notin \mathbb{Q}(\sqrt{p})$. Moreover $\theta^2 \in \mathbb{Q}(\sqrt{p})$. It follows that θ has (algebraic) degree 2 over $\mathbb{Q}(\sqrt{p})$ and hence θ has degree 4 over $\mathbb{Q}$.

The irreducible polynomial of θ over $\mathbb{Q}$ is now clearly

$$f_0 = X^4 - 2aX^2 + (a - pb^2).$$

The conjugates of θ (over $\mathbb{Q}$) are: $\pm\theta$ and $\pm\mu$, where $\mu = \sqrt{a - b\sqrt{p}}$. Note that $\sqrt{p} = \frac{1}{b}(\theta^2 - a) \in \mathbb{Q}(\theta)$. Now, $\theta\mu = \sqrt{a^2 - pb^2} = c\sqrt{p} \in \mathbb{Q}(\theta)$. Hence, $\mu = \frac{c\sqrt{p}}{\theta} \in \mathbb{Q}(\theta)$. Therefore, $\mathbb{Q}(\theta) : \mathbb{Q}$ is a Galois extension of degree 4, and hence its Galois group $G = G(\mathbb{Q}(\theta) : \mathbb{Q})$ has order 4. Since

f_0 is irreducible over $\mathbb{Q}$, G acting on the roots of f_0 is a *transitive* group. In particular, for some $\sigma \in G$, we have,

$$\sigma(\theta) = \mu.$$

Claim: $\sigma(\sqrt{p}) = -\sqrt{p}$. Otherwise, we must have $\sigma(\sqrt{p}) = \sqrt{p}$. So, $\sigma(\theta^2) = \sigma(a + b\sqrt{p}) = a + b\sqrt{p} = \theta^2$ and $\sigma(\theta) = \pm\theta$, giving a contradiction. Now, $\sigma(\mu) = \sigma(\frac{c\sqrt{p}}{\theta}) = \frac{c\sigma(\sqrt{p})}{\sigma(\theta)} = \frac{-c\sqrt{p}}{\mu} = -\theta$. Finally, $\sigma(-\theta) = -\mu$ and $\sigma(-\mu) = \theta$. Hence, the action of σ on the roots of f_0 is the 4-cycle,

$$(\theta, \mu, -\theta, -\mu).$$

As G has order 4, we conclude that G is cyclic generated by σ. □

Proposition 2.6. *Each of* $\mathbb{Q}(\cot\frac{\pi}{5}) : \mathbb{Q}$ *and* $\mathbb{Q}(\cot\frac{\pi}{16}) : \mathbb{Q}$ *is a cyclic extension of degree 4.*

Proof. • We have $5\cot\frac{\pi}{5} = \sqrt{25 + 10\sqrt{5}}$. Apply Lemma 2.5 with $p = 5$ and $(a, b, c) = (25, 10, 5)$.

• We have $\cot\frac{\pi}{16} = 1 + \sqrt{2} + \sqrt{4 + 2\sqrt{2}}$. It is an exercise to check that $\mathbb{Q}(\cot\frac{\pi}{16}) = \mathbb{Q}(\sqrt{4 + 2\sqrt{2}})$. Apply Lemma 2.5 with $p = 2$ and $(a, b, c) = (4, 2, 2)$. □

Proposition 2.7. *Let* $p \geq 7$ *be a prime number. Then,* $\mathbb{Q}(\cos\frac{2\pi}{p}) : \mathbb{Q}$ *is a cyclic extension of degree* ≥ 3*. Furthermore,* $\mathbb{Q}(\cos\frac{2\pi}{9}) : \mathbb{Q}$ *is a cyclic extension of degree 3.*

Proof. • Set $\Omega = \mathbb{Q}(e^{i\frac{2\pi}{p}})$. It is well-known that $\Omega : \mathbb{Q}$ is a cyclic extension of degree $p - 1$. Now, $\mathbb{Q}(\cos\frac{2\pi}{p}) : \mathbb{Q}$ as a sub-extension of $\Omega : \mathbb{Q}$ is a cyclic extension, and it has degree $\frac{p-1}{2} \geq 3$.

• Set $\mathbb{Q}(e^{i\frac{2\pi}{9}})$. It is well-known that $\Omega : \mathbb{Q}$ is an *abelian* extension of degree $\varphi(9) = 6$. Now, $\mathbb{Q}(\cos\frac{2\pi}{9}) : \mathbb{Q}$ as a sub-extension of an abelian extension is a Galois extension, and so the order of its group must be equal to its degree, that is, to $\frac{1}{2}\varphi(9) = 3$. Since any group of order 3 is *cyclic*, the proof is complete. □

3. The relation $\frac{n}{4}\cot\frac{\pi}{n} = \sqrt{r_1} \pm \sqrt{r_2} \pm \cdots \pm \sqrt{r_n}$

Proposition 3.1. *Let $n \geq 5, n \neq 6$. Set $\Omega = \mathbb{Q}(\cot\frac{\pi}{n})$. Suppose that Ω is a flat field. Then, $n \in \{8, 12, 24\}$.*

Proof. • Suppose first that n is divisible by 5. By Proposition 2.1, $\mathbb{Q}(\cot\frac{\pi}{5})$ is a subfield of Ω, and, by Proposition 2.6, the Galois group of $\mathbb{Q}(\cot\frac{\pi}{5}) : \mathbb{Q}$ is a *cyclic group of order 4* (and hence is not a 2-group). Therefore, Ω is *NOT* flat.

• Suppose next that n is divisible by a prime $p \geq 7$. By Proposition 2.1, $\mathbb{Q}(\cos\frac{2\pi}{p})$ is a subfield of Ω, and, by Proposition 2.7, the Galois group of $\mathbb{Q}(\cos\frac{2\pi}{p}) : \mathbb{Q}$ is a *cyclic group of order* ≥ 3 (and hence is not a 2-group). Therefore, Ω is *NOT* flat.

• Suppose now that n is divisible by 16. By Proposition 2.1, $\mathbb{Q}(\cot\frac{\pi}{16})$ is a subfield of Ω, and, by Proposition 2.6, the Galois group of $\mathbb{Q}(\cot\frac{\pi}{16}) : \mathbb{Q}$ is a *cyclic group of order 4* (and hence is not a 2-group). Therefore, Ω is *NOT* flat.

• Suppose finally that n is divisible by 9. By Proposition 2.1, $\mathbb{Q}(\cos\frac{2\pi}{9})$ is a subfield of Ω, and, by Proposition 2.7, the Galois group of $\mathbb{Q}(\cos\frac{2\pi}{9}) : \mathbb{Q}$ is a *cyclic group of order 3* (and hence is not a 2-group). Therefore, Ω is *NOT* flat.

In conclusion, as long as we assume Ω to be flat, n cannot have a prime factor ≥ 5 and n cannot be divisible neither by 2^4 nor by 3^2. Hence, n must have the form $n = 2^\alpha 3^\beta$, with $\alpha \in \{0, 1, 2, 3\}$ and $\beta \in \{0, 1\}$. Furthermore, $n \geq 5$ and $n \neq 6$, and it remains that $n \in \{8, 12, 24\}$. □

Corollary 3.2. *Let $n = 5$ or $n \geq 7$, with $n \neq 8, 12, 24$. Then, the identity,*

$$\frac{n}{4}\cot\frac{\pi}{n} = \sqrt{r_1} \pm \sqrt{r_2} \pm \cdots \pm \sqrt{r_n},$$

where the r_i are nonnegative rational numbers, is impossible.

Proof. Otherwise, we would get $\mathbb{Q}(\sqrt{r_1} \pm \sqrt{r_2} \pm \cdots \pm \sqrt{r_n}) = \mathbb{Q}(\frac{n}{4}\cot\frac{\pi}{n}) = \mathbb{Q}(\cot\frac{\pi}{n})$. But, by Proposition 2.4, $\mathbb{Q}(\sqrt{r_1} \pm \sqrt{r_2} \pm \cdots \pm \sqrt{r_n})$ is a *flat* field, whereas by Proposition 3.1, $\mathbb{Q}(\cot\frac{\pi}{n})$ is *NOT* a flat field. We have a contradiction. □

4. **Proof of Theorem 1.1**

• For $n = 6$, the answer to (P1) is POSITIVE: The centroid of the unit hexagon P_6 is at distance one from each vertex.

• Let $n = 5$ or ≥ 7, with $n \neq 8, 12, 24$. We show that the answer to (P1) is NEGATIVE. For the purpose of gaining a contradiction, assume the existence of a point P in the plane of P_n that is at rational distance from the vertices $A_1, A_2, \ldots, A_n$ of P_n, written in cyclic order. Set $A_{n+1} = A_1$. Introduce the n triangles $T_i = PA_iA_{i+1}$, $i = 1, \ldots, n$ (note that, up to two triangles, T_i might be degenerated). Call "positive" a triangle T_i that intersects the interior of P_n, or equivalently, such that the intersection of T_i with P_n has a positive area (such triangle is non-degenerated). Otherwise, call T_i "negative". Note that there are always positive triangles T_i (if P is interior to P_n, then all the T_i are positive). Without loss of generality, we may assume that T_1 is positive. Now, observe the decisive properties:

(i) If we add the areas of all positive triangles T_i and then subtract the areas of all negative triangles T_i (if any), then we get *precisely* the area of P_n. In other words, we have the following relation:

$$\text{area}(P_n) = \text{area}T_1 \pm \text{area}T_2 \pm \cdots \pm \text{area}T_n.$$

(ii) Since every triangle T_i has rational sides, Heron's formula $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle shows that the area of every triangle T_i has the form $\sqrt{r_i}$, for some nonnegative rational number r_i (note that $\sqrt{r_i}$, which is at most an irrational number of degree 2, might be rational, even zero, if T_i is degenerated).

Combining *(i)* and *(ii)*, we get that $\text{area}(P_n) = \sqrt{r_1} \pm \sqrt{r_2} \pm \cdots \sqrt{r_n}$.

We leave it as an exercise to check that $\text{area}(P_n) = \frac{n}{4}\cot\frac{\pi}{n}$. Finally, we obtain:

$$\frac{n}{4}\cot\frac{\pi}{n} = \sqrt{r_1} \pm \sqrt{r_2} \pm \cdots \pm \sqrt{r_n},$$

in contradiction with Corollary 3.2. □

Remark 4.1. If P_n is not constructible by ruler and compasses ($\varphi(n)$ not a power of 2), then it can be shown that the (algebraic) degree of $\frac{n}{4}\cot\frac{\pi}{n}$ over $\mathbb{Q}$ contains an odd factor, while the degree of $\sqrt{r_1} \pm \sqrt{r_2} \pm \cdots \pm \sqrt{r_n}$ over $\mathbb{Q}$ is a power of 2. Thus, for such n, the answer to (P1) is negative. However, this will not shorten our general proof: No decisive information is obtained for the pentagon P_5, neither for P_{16} nor for P_{17}, etc. We even do not know if the constructible P_n are finite or infinite.

Open Problems.
(1) Solve Problem (P1) in the case $n = 8$ (respectively for $n = 12$ or $n = 24$).
(2) Are there points other than the centroid of the unit hexagon P_6 that are at rational distance from the vertices of P_6?

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