FINITE GROUPS WITH THREE RELATIVE COMMUTATIVITY DEGREES

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Communicated by Ali Reza Ashrafi

ABSTRACT. For a finite group G and a subgroup H of G, the relative commutativity degree of H in G, denoted by $d(H, G)$, is the probability that an element of H commutes with an element of G. Let $\mathcal{D}(G) = \{d(H, G) : H \leq G\}$ be the set of all relative commutativity degrees of subgroups of G. It is shown that a finite group G admits three relative commutativity degrees if and only if $G/Z(G)$ is a non-cyclic group of order pq, where p and q are primes. Moreover, we determine all the relative commutativity degrees of some known groups.

1. **Introduction**

If G is a finite group, then the *commutativity degree* of G, denoted by $d(G)$, is the probability that two randomly chosen elements of G commute. The commutativity degree first studied by Gustafson [4] and it was shown that $d(G) \leq \frac{5}{8}$ for every non-abelian finite group G and equality holds precisely when $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let H be a subgroup of G. Erfanian et al. in [1] generalized the commutativity degree of G by considering the *relative commutativity*

degree of H in G, denoted by $d(H, G)$, as the probability that an element of H commutes with an element of G that is

$$d(H, G) = \frac{\{(h, g) \in H \times G : [h, g] = 1\}}{|H||G|}.$$

Also they have given several lower and upper bounds for the relative commutativity degree $d(H, G)$. We refer the reader to [1, 4, 5] for more details.

Now let $\mathcal{D}(G) = \{d(H, G) : H \leq G\}$. It is obvious that $|\mathcal{D}(G)| = 1$ if and only if G is an abelian group. Also, it is easy to see that there is no finite group G with $|\mathcal{D}(G)| = 2$ (see Lemma 2.4). We intend to study the set $\mathcal{D}(G)$ and classify all finite groups G with three commutativity degrees. We will show that if G is a finite group, then $|\mathcal{D}(G)| = 3$ if and only if $G/Z(G)$ is a group of order pq for some primes p and q. Moreover, the number of relative commutativity degrees will be computed for some classes of finite groups including dihedral groups, generalized quaternion groups and quasi-dihedral groups. The motivation to the research is the Farrokhi's classification of finite groups with two subgroup normality degrees in [3]. For further details on this topic, we refer the interested reader to [6].

2. **Preliminary results**

We begin with some basic lemmas.

Lemma 2.1. *Let G be a finite group and $H \leq K \leq G$. Then $d(K, G) \leq d(H, G)$ and the equality holds if and only if $K = HC_K(g)$ for all $g \in G$.*

Proof. Since $g^H \subseteq g^K$, we have $\frac{|C_K(g)|}{|K|} \leq \frac{|C_H(g)|}{|H|}$ for each $g \in G$. Hence

$$d(K, G) = \frac{1}{|G|} \sum_{g \in G} \frac{|C_K(g)|}{|K|} \leq \frac{1}{|G|} \sum_{g \in G} \frac{|C_H(g)|}{|H|} = d(H, G).$$

Also $d(K, G) = d(H, G)$ if and only if $\frac{|C_K(g)|}{|K|} = \frac{|C_H(g)|}{|H|}$ for all $g \in G$, which is equivalent to $K = HC_K(g)$ for all $g \in G$. $\square$

Note that for any subgroup $H \leq G$, we have

$$d(H, G) = \frac{1}{|G|} \sum_{g \in G} \frac{|C_H(g)|}{|H|}.$$

Lemma 2.2. *Let G be a non-abelian finite group and $x \in G \setminus Z(G)$. Then $d(\langle x \rangle, G) \neq 1, d(G)$.*

Proof. Clearly $d(\langle x \rangle, G) \neq 1$ because x is not central. Suppose that $d(\langle x \rangle, G) = d(G)$, then by Lemma 2.1, $G = \langle x \rangle C_G(y)$ for all $y \in G$. In particular $G = \langle x \rangle C_G(x) = C_G(x)$, which implies that $x \in Z(G)$, a contradiction. $\qquad\square$

From the above lemma the following result can be obtained immediately.

Corollary 2.3. *If G is a non-abelian finite group, then $|\mathcal{D}(G)| \neq 2$.*

In the sequel we shall discuss finite groups with three relative commutativity degrees.

Lemma 2.4. *Let G be a non-abelian finite group and suppose that $\mathcal{D}(G) = \{1, d, d(G)\}$. If H is a subgroup of G such that $d(H, G) = d$, then H is abelian.*

Proof. Let $h \in H \setminus Z(H)$, then by Lemma 2.2, $d(\langle h \rangle, G) \neq d(G)$. Thus $d(\langle h \rangle, G) = d(H, G)$ and by Lemma 2.1, $H = \langle h \rangle C_H(g)$ for all $g \in G$. Now by replacing g by h, we conclude that $h \in Z(H)$, a contradiction. $\qquad\square$

Lemma 2.5. *Let G be a finite group with $|\mathcal{D}(G)| = 3$. Then $C_G(x)$ is an abelian maximal subgroup of G, for all $x \in G \setminus Z(G)$.*

Proof. Let $x \in G \setminus Z(G)$. If $C_G(x)$ is a non-abelian group, then by Lemmas 2.4 and 2.1, we have $G = C_G(x)C_G(g)$ for all $g \in G$. In particular, $G = C_G(x)C_G(x) = C_G(x)$ and hence $x \in Z(G)$, which is a contradiction. Now let M be a maximal subgroup of G containing $C_G(x)$. If M is non-abelian, then by Lemmas 2.4 and 2.1, $G = MC_G(x)$ so $G = M$, a contradiction. Thus M is abelian and consequently $M = C_G(x)$, as required. $\qquad\square$

Lemma 2.6. *If H is a subgroup of G, then $d(HZ(G), G) = d(H, G)$.*

Proof. The proof is straightforward. $\qquad\square$

Lemma 2.7. *Let G be a finite group with $|\mathcal{D}(G)| = 3$. Then the following assertions are true:*

   (i) *if x is a p-element, then $x^p \in Z(G)$;*
   (ii) *if $x, y \in G \setminus Z(G)$ are p-element and q-element, respectively, then*

$$(p - q)|G| = q(p - 1)|C_G(x)| - p(q - 1)|C_G(y)|.$$

   In particular, $|C_G(x)| = |C_G(y)|$ if and only if $p = q$.

Proof. (i) Let $x \in G \setminus Z(G)$ be a p-element of order p^n. We proceed by induction on n. Cearly the result holds for $n = 0, 1$. Suppose that the result holds for $n - 1$. Then $|x^p| = p^{n-1}$ and by hypothesis $x^{p^2} \in Z(G)$. If $x^p \notin Z(G)$, then by Lemma 2.2, $d(\langle x \rangle, G) = d(\langle x^p \rangle, G)$. On the other hand,

$$d(\langle x \rangle, G) = \frac{1}{p^n |G|}(p^{n-2}|G| + (p^{n-1} - p^{n-2})|C_G(x^p)| + (p^n - p^{n-1})|C_G(x)|)$$

and

$$d(\langle x^p \rangle, G) = \frac{1}{p^{n-1}|G|}(p^{n-2}|G| + (p^{n-1} - p^{n-2})|C_G(x^p)|),$$

from which we obtain

$$p = [G : C_G(x)] + (p - 1)[C_G(x^p) : C_G(x)].$$

Hence $[G : C_G(x)] = [C_G(x^p) : C_G(x)] = 1$ that is $x \in Z(G)$, contradicting the hypothesis.

(ii) By Lemma 2.2, $d(\langle x \rangle, G) = d(\langle y \rangle, G)$. By part (i) we have

$$d(\langle x \rangle, G) = \frac{1}{p^m |G|}(p^{m-1}|G| + (p^m - p^{m-1})|C_G(x)|)$$

and

$$d(\langle y \rangle, G) = \frac{1}{q^n |G|}(q^{n-1}|G| + (q^n - q^{n-1})|C_G(y)|),$$

from which the result follows. The rest of the proof is a direct consequence of the last two equations. □

3. Main results

We are now in a position to give the main theorems. The nilpotent and non-nilpotent cases are discussed separately.

Theorem 3.1. *Let G be a finite nilpotent group. Then $|\mathcal{D}(G)| = 3$ if and only if $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. In particular, $\mathcal{D}(G) = \{1, \frac{2p-1}{p^2}, \frac{p^2+p-1}{p^3}\}$.*

Proof. Since G is nilpotent, so $G = P_1 \times P_2 \times \cdots \times P_n$, where P_i is the Sylow p_i-subgroup of G. Clearly

$$\mathcal{D}(G) = \mathcal{D}(P_1)\mathcal{D}(P_2) \cdots \mathcal{D}(P_n) \supseteq \mathcal{D}(P_1) \cup \mathcal{D}(P_2) \cup \cdots \cup \mathcal{D}(P_n).$$

Since $|\mathcal{D}(G)| = 3$, there exists a Sylow p_i-subgroup $P = P_i$ such that $\mathcal{D}(P) = \mathcal{D}(G)$ and P_j is abelian for each $j \neq i$. Let $x, y \in P$ such that $xy \neq yx$, $M = C_P(x)$ and $N = C_P(y)$. Then by Lemma 2.5(i),

M and N are abelian maximal subgroups of P. Clearly $P = MN$ and $M \cap N = Z(P)$. Hence

$$\left| \frac{P}{Z(P)} \right| = \left| \frac{MN}{M \cap N} \right| = \frac{|M|}{|M \cap N|} \cdot \frac{|N|}{|M \cap N|} = p^2.$$

Conversely, let $H \leq G$ be a proper non-central subgroup of G such that $d(H, G) \neq d(G)$. By Lemma 2.6, $d(H, G) = d(HZ(G), G)$ and we may assume that $Z(G) \subseteq H$. Then $H/Z(G) \cong \mathbb{Z}_p$ and we have

$$d(H, G) = \frac{1}{|H||G|} \sum_{h \in H} |C_G(h)| = \frac{1}{|H|} \sum_{h \in H} \frac{1}{|h^G|}$$
$$= \frac{1}{p|Z(G)|} \left(|Z(G)| + (p-1)|Z(G)| \frac{1}{p} \right)$$
$$= \frac{2p-1}{p^2}.$$

Similarly, it can be easily shown that $d(G) = \frac{p^2+p-1}{p^3}$ and consequently $\mathcal{D}(G) = \{1, \frac{2p-1}{p^2}, \frac{p^2+p-1}{p^3}\}$. □

Theorem 3.2. *Let G be a finite non-nilpotent group. Then $|\mathcal{D}(G)| = 3$ if and only if $G/Z(G)$ is a non-cyclic group of order pq, where p and q are distinct primes. In particular, $\mathcal{D}(G) = \{1, \frac{1}{p} + \frac{1}{q} - \frac{1}{pq}, \frac{1}{p} + \frac{1}{q^2} - \frac{1}{pq^2}\}$, whenever $p > q$.*

Proof. Since G is not nilpotent, there exist a p-element x and a q-element y ($p \neq q$) such that $xy \neq yx$. Clearly we may assume that q is the smallest prime dividing $|G/Z(G)|$. By Lemma 2.5, $M = C_G(x)$ and $N = C_G(y)$ are different abelian maximal subgroups of G. Moreover, $M \cap N = Z(G)$ and by Lemma 2.7

(3.1) $(p-q)|G| = q(p-1)|M| - p(q-1)|N|.$

Note that by Lemma 2.7(ii), $|M| \neq |N|$ so that M and N are not conjugate.

Let $^{-} : G \to G/Z(G)$ be the natural homomorphism. If M and N are non-normal subgroups of G, then $N_G(M) = M$, $N_G(N) = N$ and the

conjugates of $\overline{M}$ and $\overline{N}$ all have trivial intersection. Thus

$$
\begin{aligned}
|\overline{G}| &\geq \left| \bigcup_{\overline{g}\in\overline{G}} \overline{M}^{\overline{g}} \cup \bigcup_{\overline{g}\in\overline{G}} \overline{N}^{\overline{g}} \right| \\
&= 1 + [\overline{G}:\overline{M}](|\overline{M}|-1) + [\overline{G}:\overline{N}](|\overline{N}|-1) \\
&\geq 1 + \frac{|\overline{G}|}{2} + \frac{|\overline{G}|}{2} > |\overline{G}|,
\end{aligned}
$$

which is a contradiction. Thus M or N is a normal subgroup of G, which implies that $G = MN$. If $M, N \trianglelefteq G$, then $\overline{G} \cong \overline{M} \times \overline{N}$ is abelian and hence G is nilpotent, a contradiction. Therefore $\overline{G}$ is a Frobenius group with $\overline{M}$ and $\overline{N}$ as its Frobenius kernel and complement in some order. Moreover, $\gcd(|\overline{M}|, |\overline{N}|) = 1$.

From the equation (3.1) it follows that $|\overline{M}|$ divides $p(q-1)|\overline{N}|$, which implies that $|\overline{M}|$ divides p. Hence $|\overline{M}| = p$ and, using the equation (3.1) once more, we obtain $|\overline{N}| = q$. Therefore $|\overline{G}| = pq$, as required.

Conversely, suppose that G is a finite non-nilpotent group such that $|G/Z(G)| = pq$ for some primes. Clearly $G/Z(G)$ is non-abelian, $p \neq q$ and we may assume that $p > q$. Let H be a proper non-central subgroup of G. Then by Lemma 2.6, we may assume that $Z(G) \subseteq H$. Thus $H/Z(G) \cong \mathbb{Z}_p$ or $\mathbb{Z}_q$. In the first case

$$
d(H, G) = \frac{1}{|H|}\left(|Z(G)| + (p-1)|Z(G)|\frac{1}{q} \right) = \frac{1}{p} + \frac{1}{q} - \frac{1}{pq}
$$

and in the second case

$$
d(H, G) = \frac{1}{|H|}\left(|Z(G)| + (q-1)|Z(G)|\frac{1}{p} \right) = \frac{1}{p} + \frac{1}{q} - \frac{1}{pq}.
$$

On the other hand $G/Z(G)$ is a Frobenius group with Frobenius kernel and complement isomorphic to cyclic groups of orders p and q, respectively. Thus

$$
\begin{aligned}
d(G) &= \frac{1}{|G|}\left(|Z(G)| + (p-1)|Z(G)|\frac{1}{q} + (pq-p)|Z(G)|\frac{1}{p} \right) \\
&= \frac{1}{p} + \frac{1}{q^2} - \frac{1}{pq^2}.
\end{aligned}
$$

Therefore $\mathcal{D}(G) = \{1, \frac{1}{p} + \frac{1}{q} - \frac{1}{pq}, \frac{1}{p} + \frac{1}{q^2} - \frac{1}{pq^2}\}$ and the proof is complete. $\qquad\square$

4. **Examples**

This subsection is devoted to the determination of the set of all relative commutativity degrees of some classes of isoclinic finite groups. First, we compute the set of all relative commutativity degrees of dihedral groups, then we apply isoclinism between groups to obtain the relative commutativity degrees for some other classes of groups.

Theorem 4.1. *Let* $G = D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1} \rangle$ *be the dihedral group of order* $2n$. *Then*

$$|\mathcal{D}(G)| = \begin{cases} 2\tau(n) - 1, & n \ odd, \\ 2k\tau(m) - 1, & n = 2^k m, \ k \geq 1 \ and \ m \ odd. \end{cases}$$

where $\tau(m)$ *is the number of divisors of* m.

Proof. It is known that an arbitrary subgroup of $G = D_{2n}$ has the form $\langle a^k \rangle$, $\langle a^t b \rangle$ or $\langle a^k, a^l b \rangle$, where $t = 0, \cdots, n-1$, $k|n$ and $l = 0, 1, 2, \ldots, \frac{n}{k} - 1$. We proceed in two steps.

First suppose that n is odd. Then

$$d(\langle a^k \rangle, G) = \frac{1}{|\langle a^k \rangle|} \sum_{h \in \langle a^k \rangle} \frac{1}{|h^G|} = \frac{k}{n} \left(1 + \sum_{i=1}^{\frac{n}{k}-1} \frac{1}{|(a^{ki})^G|} \right)$$

$$= \frac{k}{n} \left(1 + \frac{1}{2}(\frac{n}{k} - 1) \right) = \frac{1}{2} + \frac{k}{2n},$$

$$d(\langle a^t b \rangle, G) = \frac{1}{|\langle a^t b \rangle|} \sum_{h \in \langle a^l b \rangle} \frac{1}{|h^G|} = \frac{1}{2} \left(1 + \frac{1}{n} \right) = \frac{1}{2} + \frac{1}{2n},$$

$$d(\langle a^k, a^l b \rangle = \frac{1}{|\langle a^k, a^l b \rangle|} \sum_{h \in \langle a^k, a^l b \rangle} \frac{1}{|h^G|}$$

$$= \frac{k}{2n} \left(1 + \sum_{i=1}^{\frac{n}{k}-1} \frac{1}{|(a^{ki})^G|} + \sum_{i=1}^{\frac{n}{k}} \frac{1}{|(a^{ki+l}b)^G|} \right)$$

$$= \frac{k}{2n} \left(1 + \frac{1}{2} \left(\frac{n}{k} - 1 \right) + \frac{1}{n} \left(\frac{2n}{k} - \frac{n}{k} \right) \right) = \frac{1}{4} + \frac{1}{2n} + \frac{k}{4n},$$

where $k|n$ and $l = 0, 1, 2, \ldots, \frac{n}{k} - 1$.

Thus $\mathcal{D}(G) = \{\frac{1}{2} + \frac{k}{2n}, \frac{1}{4} + \frac{1}{2n} + \frac{k'}{4n} : k, k'|n\}$ and a simple verification shows that $|\mathcal{D}(G)| = 2\tau(n) - 1$.

Now suppose that $n = 2^k m$ is even, where $k \geq 1$ and m is odd. Then

$$d(\langle a^k \rangle, G) = \frac{1}{|\langle a^k \rangle|} \sum_{h \in \langle a^k \rangle} \frac{1}{|h^G|} = \begin{cases} \frac{k}{n}\left(1 + \frac{1}{2}(\frac{n}{k} - 1)\right) = \frac{1}{2} + \frac{k}{2n}, & \frac{n}{k} \ odd, \\[2mm] \frac{k}{n}\left(2 + \frac{1}{2}(\frac{n}{k} - 2)\right) = \frac{1}{2} + \frac{k}{n}, & \frac{n}{k} \ even. \end{cases}$$

Since $(a^l b)^G = \{a^{l+2i}b : 0 \leq i \leq \frac{n}{2}\}$ we have $d(\langle a^l b \rangle, G) = \frac{1}{2}(1 + \frac{1}{\frac{n}{2}}) = \frac{1}{2} + \frac{1}{n}$. Also

$$d(\langle a^k, a^l b \rangle, G) = \begin{cases} \frac{1}{4} + \frac{k}{4n} + \frac{1}{n}, & \frac{n}{k} \ odd, \\[2mm] \frac{1}{4} + \frac{k}{2n} + \frac{1}{n}, & \frac{n}{k} \ even. \end{cases}$$

Therefore

$$\mathcal{D}(D_{2n}) = \left\{ \frac{1}{2} + \frac{k_1}{n}, \frac{1}{2} + \frac{k_2}{2n}, \frac{1}{4} + \frac{1}{n} + \frac{k_3}{2n}, \frac{1}{4} + \frac{1}{n} + \frac{k_4}{4n} : \right.$$

$$\left. k_1, k_2, k_3, k_4 | n, \frac{n}{k_1}, \frac{n}{k_3} \ \text{are even and} \ \frac{n}{k_2}, \frac{n}{k_4} \ \text{are odd} \right\},$$

where $k_3, k_4 \neq n$. We consider the following cases:

(1) If $\frac{1}{2} + \frac{k_1}{n} = \frac{1}{4} + \frac{1}{n} + \frac{k_3}{2n}$, then $\frac{n}{2} = k_3 + 2(1-k_1)$ and $k_3 + 2(1-k_1) < 0$ if $k_1 > 1$. Thus $k_1 = 1$ and $k_3 = \frac{n}{2}$.

(2) If $\frac{1}{2} + \frac{k_1}{n} = \frac{1}{4} + \frac{1}{n} + \frac{k_4}{4n}$, then $n = k_4 + 4(1-k_1)$ and $k_4 + 2(1-k_1) < 0$ if $k_1 > 1$. Thus $k_1 = 1$ and $k_4 = n$, which is a contradiction.

(3) If $\frac{1}{2} + \frac{k_2}{2n} = \frac{1}{4} + \frac{1}{n} + \frac{k_3}{2n}$, then $\frac{n}{2} = k_3 + 2 - k_2$ and $k_3 + 2 - k_2 < 0$ if $k_2 > 2$. Thus $k_2 \leq 2$. Since $\frac{n}{k_2}$ is odd we should have $k_2 = 2$, hence $k_3 = \frac{n}{2}$.

(4) If $\frac{1}{2} + \frac{k_2}{2n} = \frac{1}{4} + \frac{1}{n} + \frac{k_4}{4n}$, then $n = k_4 + 4 - 2k_2$ and $k_4 + 4 - 2k_2 < 0$ if $k_2 > 2$. Thus $k_2 \leq 2$, which is a contradiction.

(5) If $\frac{1}{2} + \frac{k_1}{n} = \frac{1}{2} + \frac{k_2}{2n}$, then $k_2 = 2k_1$.

(6) If $\frac{1}{4} + \frac{1}{n} + \frac{k_3}{2n} = \frac{1}{4} + \frac{1}{n} + \frac{k_4}{4n}$, then $k_4 = 2k_3$.

Now, by utilizing the cases (1)-(6), the result follows. □

Corollary 4.2. $|\mathcal{D}(D_{2n})| = 3$ if and only if $n = p$ or $2p$, where p is a prime.

Definition 4.3. Let G_1 and G_2 be two groups and H_1 and H_2 be sub-group of G_1 and G_2, respectively. Suppose that α is an isomorphism from $G_1/Z(G_1)$ to $G_2/Z(G_2)$ such that its restriction to $H_1/H_1 \cap Z(G_1)$ is an isomorphism from $H_1/H_1 \cap Z(G_1)$ to $H_2/H_2 \cap Z(G_2)$ and β is an isomorphism from $[H_1, G_1]$ to $[H_2, G_2]$. Then the pair (α, β) is called a

relative isoclinism from (H_1, G_1) to (H_2, G_2) if the following diagram is commutative:

$$\frac{H_1}{H_1 \cap Z(G_1)} \times \frac{G_1}{Z(G_1)} \xrightarrow{\alpha^2} \frac{H_2}{H_2 \cap Z(G_2)} \times \frac{G_2}{Z(G_2)}$$

$$\gamma_1 \downarrow \qquad\qquad\qquad\qquad \downarrow \gamma_2$$

$$[H_1, G_1] \xrightarrow{\quad\beta\quad} [H_2, G_2]$$

where

$$\gamma_1(h_1(H_1 \cap Z(G_1)), g_1 Z(G_1)) = [h_1, g_1]$$

and

$$\gamma_2(h_2(H_2 \cap Z(G_2)), g_2 Z(G_2)) = [h_2, g_2]$$

for each $h_1 \in H_1$, $h_2 \in H_2$, $g_1 \in G_1$ and $g_2 \in G_2$. If $H_1 = G_1$ and $H_2 = G_2$, then we say that G_1 and G_2 are isoclinic.

As an immediate consequent of the above definition we have the following result.

Lemma 4.4. *If G_1 and G_2 are two isoclinic groups, then $\mathcal{D}(G_1) = \mathcal{D}(G_2)$.*

Using Lemma 4.4 and Theorem 4.1 we obtain the following results. Note that the generalized quarernion groups Q_{4n} and quasi-dihedral groups QD_{2^n} ($n \geq 3$) are isoclinic with the groups D_{4n} and D_{2^n}, respectively.

Corollary 4.5. *If $n = 2^k m$ (m odd), then $|\mathcal{D}(Q_{4n})| = 2(k+1)\tau(m) - 1$.*

Corollary 4.6. *If $n \geq 3$, then $|\mathcal{D}(QD_{2^n})| = 2n - 3$.*

Acknowledgments

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