GORENSTEIN FLAT AND GORENSTEIN INJECTIVE
DIMENSIONS OF SIMPLE MODULES

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Abstract. Let $R$ be a right $GF$-closed ring with finite left and right Gorenstein global dimension. We prove that if $I$ is an ideal of $R$ such that $R/I$ is a semi-simple ring, then the Gorenstein flat dimension of $R/I$ as a right $R$-module and the Gorenstein injective dimension of $R/I$ as a left $R$-module are identical. In particular, we show that for a simple module $S$ over a commutative Gorenstein ring $R$, the Gorenstein flat dimension of $S$ equals to the Gorenstein injective dimension of $S$.

1. Introduction

In classical homological algebra, the projective, injective and flat dimensions of modules are important and fundamental research objects. As a generalization of the notion of projective dimension of modules, Auslander and Bridger [1] introduced the $G$-dimension, denoted by $G\dim_R(M)$, for every finitely generated $R$-module $M$ over a two-sided Noetherian ring $R$. They proved the inequality $G\dim_R(M) \leq pd_R(M)$ with equality $G\dim_R(M) = pd_R(M)$ when $pd_R(M)$ is finite.

Several decades later, Enochs and Jenda [8, 9] defined the notion of Gorenstein projective dimension, as an extension of $G$-dimension of modules that are not necessarily finitely generated, and the Gorenstein...
injection dimension as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [10] introduced the Gorenstein flat dimension. Some references are [4, 5, 8, 9, 10].

Since then, the so-called Gorenstein homological algebra has been so interesting to many authors. A central topic in this study is to generalize the classical results in homological algebra to their Gorenstein counterpart. For example, the well known Regularity Theorem and Auslander-Buchsbaum Formula have already been proven to be true in the Gorenstein case, see [4]. Another interesting example of this kind, proved by Bennis and Mahdou in [3], is that the left Gorenstein global projective dimension of a ring $R$ equals to the left Gorenstein global injective dimension of $R$, and the common value of these two invariants is then called left Gorenstein global dimension.

The aim of this paper is to generalize the following results to the Gorenstein case: (1) Over a commutative ring $R$, a simple $R$-module is flat if and only if it is injective (see [14, Theorem 1.1]). (2) Over a commutative coherent ring $R$, the flat dimension of a simple module $S$ and the injective dimension of $S$ are identical (see [6, Lemma 3.1]).

In order to show that the Gorenstein counterpart of the above results also hold true in some cases, we need to assume that the ground ring is right $GF$-closed and with finite left and right Gorenstein global dimension. Recall that a ring $R$ is called left (resp. right) $GF$-closed [2] if the class of Gorenstein flat left (resp. right) $R$-modules is closed under extensions. The class of right $GF$-closed rings includes strictly the left coherent rings and rings with finite weak dimension by [2, Proposition 2.2]. In fact, with the above assumptions, we can give a more general result:

**Theorem 1.1.** If $R$ is a right $GF$-closed and with finite left and right Gorenstein global dimension, $I$ is an ideal of $R$ such that $R/I$ is a semisimple ring, then $Gfd_{R^{op}}(R/I) = Gid_R(R/I)$.

Note that the rings satisfying the conditions in Theorem 1.1 also have finite left and right weak Gorenstein global dimensions. Some familiar examples of rings with these properties are:

1. Any ring of finite left and right global dimension and any quasi-Frobenius ring.
2. Any Gorenstein ring $R$, which is a two-sided Noetherian ring with finite left and right self-injective dimension by [7, Theorem 12.3.1].
(3) Any Noetherian PI Hopf algebra over a field by [13].

As an immediate consequence of Theorem 1.1, we have that if $R$ is a commutative Gorenstein ring, then the Gorenstein flat dimension and Gorenstein injective dimension of a simple $R$-module $S$ are identical. In particular, $S$ is Gorenstein flat if and only if it is Gorenstein injective. This result generalizes the corresponding results of [14] and [6] in some sense.

Moreover, we prove that if $R \to \Lambda$ is a homomorphism of rings, where $R$ is right GF-closed and with finite left and right Gorenstein global dimension, and $\Lambda E$ is an injective cogenerator for the category of left $\Lambda$-modules, then the Gorenstein flat dimension of $\Lambda$ as a right $R$-module and the Gorenstein injective dimension of $E$ as a left $R$-module are identical. As an application, we have that if $\Lambda$ is an Artinian algebra with the center $R$ whose Gorenstein global dimension is finite, then, as $R$-modules, the Gorenstein flat dimension of $\Lambda$ and the Gorenstein injective dimension of $\mathbb{D}(\Lambda)$ are identical, where $\mathbb{D}$ is the usual duality of $\Lambda$.

2. Proof of main results and applications

We use $R$-Mod to denote the category of left $R$-modules and the category of right $R$-modules is denoted by $R^{op}$-Mod. Let $\mathcal{GP}_n$ denotes the class of left $R$-modules whose Gorenstein projective dimensions are at most $n$. For any $M \in R$-Mod (or $R^{op}$-Mod), $\text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})$ is denoted by $M^+$, where $Z$ is the additive group of integers and $Q$ is the additive group of rational numbers. For a left $R$-module, we denote the Gorenstein injective dimension of $M$ by $\text{Gid}_R M$, and for a right $R$ module $N$, we denote the Gorenstein flat dimension of $N$ by $\text{Gfd}_{R^{op}} N$.

Now we prove Theorem 1.1.

Proof. We first show that for any left $R$-module $M$ and a positive integer $n$, $\text{Ext}_R^n(M, R/I) = 0$ if and only if $\text{Tor}_R^n(R/I, M) = 0$. Assume that $\text{Ext}_R^n(M, R/I) = 0$. We have an isomorphism $\text{Ext}_R^n(M, (R/I)^+) \cong (\text{Tor}_R^n(R/I, M))^+$ by [7, Theorem 3.2.1]. Since $I(R/I)^+ = 0$, $(R/I)^+$ is a semi-simple left $R$-module and $(R/I)^+ \cong \bigoplus_{i \in \Gamma} S_i$, where each $S_i$ is a simple left $R$-module and $\Gamma$ is an index set. Since $R/I$ is semi-simple, each $S_i$ is a direct summand of $R/I$. So $\text{Ext}_R^n(M, \prod_{i \in \Gamma} S_i) \cong$
\[ \prod_{i \in \Gamma} \text{Ext}^n_R(M, S_i) = 0. \] Note that \( \prod_{i \in \Gamma} S_i \) is also a semi-simple module, so \( (R/I)^+ \cong \bigoplus_{i \in \Gamma} S_i \) is isomorphic to a direct summand of \( \prod_{i \in \Gamma} S_i \). Thus \( \text{Ext}^n_R(M, (R/I)^+) = 0 \) and therefore \( \text{Tor}^n_R(R/I, M) = 0 \). Conversely, assume that \( \text{Tor}^n_R(R/I, M) = 0 \). We have an isomorphism \( \text{Ext}^n_R(M, (R/I)^+) \cong (\text{Tor}^n_R((R/I)^+, M))^+ \). By a similar argument as above, we have \( \text{Tor}^n_R((R/I)^+, M) = 0 \). So \( (\text{Tor}^n_R((R/I)^+, M))^+ = 0 \).

On the other hand, the canonical evaluation homomorphism \( R/I \to (R/I)^+ \) is a monomorphism. Since \( I(R/I)^+ = 0 \), \( (R/I)^+ \) is a semi-simple left \( R \)-module and \( R/I \) is isomorphic to a direct summand of \( (R/I)^+ \). Thus \( \text{Ext}^n_R(M, R/I) = 0 \).

We now prove that \( Gfd_{R^{op}}(R/I) \geq Gid_R(R/I) \). Without loss of generality, suppose that \( Gfd_{R^{op}}(R/I) = m < \infty \). Then, for any \( i \geq m + 1 \) and injective left \( R \)-module \( E \), we have that \( \text{Tor}^i_R(R/I, E) = 0 \) by [2, Theorem 2.8]. Then we obtain that \( \text{Ext}^i_R(E, R/I) = 0 \) as above. It follows from [11, Theorem 2.22] that \( Gid_R(R/I) \leq m \). This proves \( Gfd_{R^{op}}(R/I) \geq Gid_R(R/I) \).

We next prove the converse inequality. Without loss of generality, suppose that \( Gid_R(R/I) = n < \infty \). Then, for any \( i \geq n + 1 \) and any injective left \( R \)-module \( E \), we have that \( \text{Ext}^i_R(E, R/I) = 0 \) by [11, Theorem 2.22]. We obtain that \( \text{Tor}^i_R(R/I, E) = 0 \) as above. It follows from [2, Theorem 2.8] that \( Gfd_{R^{op}}(R/I) \leq n \). This proves \( Gfd_{R^{op}}(R/I) \leq Gid_R(R/I) \). \( \Box \)

**Corollary 2.1.** If \( R \) is of finite left and right global dimension or a quasi-Frobenius ring or a Gorenstein ring or a Noetherian PI Hopf algebra over a field, then for any ideal \( I \) such that \( R/I \) is semi-simple, one has \( Gfd_{R^{op}}(R/I) = Gid_R(R/I) \). \( \Box \)

Recall that \( R \) is called a semi-local ring if \( R/J(R) \) is a semi-simple ring, where \( J(R) \) is the Jacobson radical of \( R \). It is well known that a semiperfect ring (more specially, a left (or right) Artinian ring or a semi-primary ring) is semilocal. By Theorem 1.1, we have the following:

**Corollary 2.2.** If \( R \) is a semi-local ring which is right GF-closed and of finite left and right Gorenstein global dimension, then \( Gfd_{R^{op}}(R/J(R)) = Gid_R(R/J(R)) \). Moreover, if \( R \) is a semi-primary ring, which is right GF-closed and with finite left and right Gorenstein global dimension, then \( Gfd_{R^{op}}(R/J(R)) = Gid_R(R/J(R)) = l.G.gldim(R) = Gpd_R(R/J(R)) \), where \( l.G.gldim(R) \) is the left Gorenstein global dimension of \( R \).
Proof. By [12, p23], we know that every left \( R \)-module \( M \) over a semi-primary ring has a finite filtration \( \{M_i\}_{i=1}^n \) for some fixed positive number \( n \) such that \( M_{i+1}/M_i \) is semisimple for every \( 1 \leq i \leq n \). Thus \( M \in GP_n \) if every \( M_{i+1}/M_i \in GP_n \) by [11, Theorem 2.5, Proposition 2.19]. So we have \( l.G.gldim(R) = Gpd_R(R/J(R)) \) since every simple module is a direct summand of \( R/J(R) \). Dually, \( Gid_R(R/J(R)) = l.G.gldim(R) \). \( \Box \)

**Corollary 2.3.** If \( R \) is a commutative ring, which is GF-closed and of finite Gorenstein global dimension, then for any simple \( R \)-module \( S \), \( Gfd_R(S) = Gid_R(S) \); therefore, \( S \) is Gorenstein flat if and only if it is Gorenstein injective. In particular, the assertions hold for a commutative Gorenstein ring.

Proof. For any simple \( R \)-module \( S \), we have \( S \cong R/m \) for some maximal ideal \( m \) of \( R \). So \( S \) is a simple ring, hence the assertion follows from Theorem 1.1. \( \Box \)

As an application of the techniques developed in the proof of Theorem 1.1, we give the following result:

**Theorem 2.4.** Let \( R \to \Lambda \) be a homomorphism of rings and \( R \) is right GF-closed and of finite left and right Gorenstein global dimension. If \( \Lambda E \) is an injective cogenerator for \( \Lambda \)-Mod, then \( Gfd_{R^{op}}(\Lambda) = Gid_R(E) \).

Proof. Let \( Q \) be any injective left \( R \)-module. Then for any \( n \geq 1 \), we have that \( Ext^n_R(Q,E) \cong Ext^n_R(Q,\text{Hom}_\Lambda(\Lambda, E)) \cong \text{Hom}_\Lambda(Tor^n_R(\Lambda, Q), E) \) by [7, Theorem 3.2.1]. Since \( \Lambda E \) is an injective cogenerator for \( \Lambda \)-Mod, \( Ext^n_R(Q,E) = 0 \) if and only if \( Tor^n_R(\Lambda, Q) = 0 \). It follows from [2, Theorem 2.8] and [11, Theorem 2.22] that \( Gfd_{R^{op}}(\Lambda) = Gid_R(E) \). \( \Box \)

Finally, we give two applications of Theorem 2.4.

**Corollary 2.5.** Let \( R \to \Lambda \) be a homomorphism of rings with \( R \) right GF-closed and of finite left and right Gorenstein global dimension. Then we have

1. \( Gfd_{R^{op}}(\Lambda) = Gid_R(\Lambda^+) \).
2. If \( \Lambda \) is a quasi-Frobenius ring, then \( Gfd_{R^{op}}(\Lambda) = Gid_R(\Lambda) \).

Proof. Note that \( \Lambda^+ \) is an injective cogenerator for \( \Lambda \)-Mod. On the other hand, it is well known that \( \Lambda \Lambda \) is an injective cogenerator for \( \Lambda \)-Mod if
Λ is a quasi-Frobenius ring. Thus both assertions follow immediately from Theorem 2.4.

Recall that an algebra Λ is called an Artinian algebra if Λ is a two-sided Artinian ring and Λ is finitely generated as an $R$-module, where $R$ is the center of Λ. It is well known that if Λ is an Artinian algebra, then its center $R$ is a commutative Artinian ring. We use $\mathcal{D}$ to denote the usual duality of Λ, that is, $\mathcal{D} = \text{Hom}_{\Lambda}(-, \Lambda/R/J(R))$. Note $\mathcal{D}(\Lambda)$ is an injective cogenerator for $\Lambda$-Mod if Λ is an Artinian algebra, so we have the following

**Corollary 2.6.** Let Λ be an Artinian algebra with center $R$. If $R$ is of finite Gorenstein global dimension, then $Gfd_R(\Lambda) = Gid_R(\mathcal{D}(\Lambda))$.

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