Bulletin of the Iranian Mathematical Society Vol. 39 *No.* 2 (*2013*), *pp 289-305.*

QUASIRECOGNITION BY THE PRIME GRAPH OF $L_3(q)$ WHERE $3 < q < 100$

S. S. SALEHI AMIRI, A. R. KHALILI ASBOEI, A. IRANMANESH*
AND A. TEHRANIAN

Communicated by Ali Reza Ashrafi

Abstract. Let G be a finite group. We construct the prime graph of G, which is denoted by $\Gamma(G)$ as follows: the vertex set of this graph is the prime divisors of $|G|$ and two distinct vertices p and q are joined by an edge if and only if G contains an element of order pq. In this paper, we determine finite groups G with $\Gamma(G) = \Gamma(L_3(q))$, $2 \leq q < 100$ and prove that if $q \neq 2, 3$, then $L_3(q)$ is quasirecognizable by the prime graph, i.e., if G is a finite group with the same prime graph as the finite simple group $L_3(q)$, then G has a unique non-Abelian composition factor isomorphic to $L_3(q)$. As a consequence of our results we prove that the simple group $L_3(4)$ is recognizable and the simple groups $L_3(7)$ and $L_3(9)$ are $2-$recognizable by the prime graph.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the *prime graph* of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices p and p' are joined by an edge if and only if G has an element of order pp' (we write $p \sim p'$).

MSC(2010): Primary: 20D05; Secondary: 20D06, 20D20.
Keywords: Prime graph, element order, simple group, linear group.
Received: 20 September 2011, Accepted: 12 April 2012.
*Corresponding author

Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let π_1, π_2, ..., $\pi_{s(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.
The *spectrum* of a finite group G, which is denoted by $\pi_e(G)$ is the set of its element orders. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility, and hence it is uniquely determined by $\mu(G)$, the subset of its maximal elements.
A subset X of the vertices of a graph is called an independent set if the induced subgraph on X has no edge. Let G be a finite group and $r \in \pi(G)$. We denote by $\rho(G)$, some independent set of vertices in $\Gamma(G)$ with the maximal number of elements. Also some independent set of vertices in $\Gamma(G)$ containing r with the maximal number of elements is denoted by $\rho(r, G)$. Also let $t(G) = |\rho(G)|$ and $t(r, G) = |\rho(r, G)|$.
Let G be a non-Abelian finite simple group. The numbers $t(G)$ and $t(r, G)$ have been determined in [34] and [36]. Also for every finite non-Abelian simple group, we use these references for adjacency of vertices in a prime graph of the group.
A finite group G is called *recognizable by spectrum*, if every finite group H with $\pi_e(G) = \pi_e(H)$ is isomorphic to G. A finite simple non-Abelian group P is called *quasirecognizable by spectrum*, if each finite group H with $\pi_e(P) = \pi_e(H)$ has a unique non-Abelian composition factor isomorphic to P [2].
A finite group G is called *recognizable by prime graph*, if every finite group H with $\Gamma(G) = \Gamma(H)$ is isomorphic to G. A finite simple non-Abelian group P is called *quasirecognizable by prime graph*, if each finite group H with $\Gamma(P) = \Gamma(H)$ has a unique nonabelian composition factor isomorphic to P [13].
We denote by $k(\Gamma(G))$ the number of isomorphism classes of the finite groups H satisfying $\Gamma(G) = \Gamma(H)$. A finite group G is called $n-$*recognizable by prime graph* if $k(\Gamma(G)) = n$ [16].
We note that quasirecognition by prime graph implies quasirecognition by spectrum, but the converse is not true in general. Also quasirecognition by prime graph is in general harder to establish than quasirecognition by spectrum, since some methods fail in the former case.

The structure of finite groups G such that $\Gamma(G)$ is not connected has been determined by Gruenberg and Kegel (1975). Moreover it has been proved that $s(G) \leq 6$ and all the simple groups G such that $\Gamma(G)$ is not connected have been described in [9, 24, 37].

We denote by (a, b) the greatest common divisor of positive integers a and b. If G is a finite group, then we denote by P_q a Sylow $q-$subgroup of G.
Let π be a set of prime numbers and let G be a finite $\pi-$group. Then G has unique largest normal $\pi-$subgroup, which is denoted by $O_\pi(G)$ and is called the $\pi-$radical of G. In fact, $O_\pi(G)$ contains every normal $\pi-$subgroup of G. All further unexplained notations are standard and is referred to [4].

Finite groups G satisfying $\Gamma(G) = \Gamma(H)$ have been determined, in cases where H is one of the following groups: a sporadic simple group [7]; a CIT simple group [12]; $PSL(2, q)$, where $q = p^\alpha < 100$ [11]; $PSL(2, p)$, where $p > 3$ is a prime [14]; $G_2(7)$, ${}^2G_2(q)$, where $q = 3^{2n+1} > 3$, $(n > 0)$ [13, 38]; $PSL(2, q)$ [16, 17]; $L_{16}(2)$ [18]., $B_p(3)$, where p is an odd prime [30].

Also, the quasirecognizability of the following simple non-Abelian groups by their prime graphs have been obtained: Alternating group A_p where p and $p - 2$ are primes [15]., $L_9(2)$ [19]., $L_{10}(2)$ [20]., ${}^2F_4(q)$ where $q = 2^{2m+1}$ for some $m \geq 1$ [1]., ${}^2D_p(3)$ where $p = 2^n + 1 \geq 5$ is a prime [3]., $C_n(2)$, where $n \neq 3$ is odd [5], $L_n(2)$ and $U_n(2)$, where $n \geq 17$ [21].

In this paper we determine finite groups G such that $\Gamma(G) = \Gamma(L_3(q))$, where q is a prime power and $2 \leq q < 100$ and conclude that if $3 < q < 100$, then $L_3(q)$ is quasirecognizable by prime graph. As a consequence of our results we prove that the simple group $L_3(4)$ is recognizable by prime graph and the simple groups $L_3(7)$ and $L_3(9)$ are $2-$recognizable by prime graph. In fact the main theorem of our paper is as follow:

Main Theorem. Let $q = p^\alpha$ be a prime power, $L = L_3(q)$, where $2 \leq q < 100$ and G be a finite group satisfying $\Gamma(G) = \Gamma(L)$. Then G is one of the groups appeared in the second column of Table 1.

2. **Preliminaries**

We first quote some remarks and lemmas that are used in deducing the main theorem of this paper.

Remark 2.1. *Let G be a finite group and K be a normal subgroup of G. If $p \sim q$ in $\Gamma(G/K)$, then $p \sim q$ in $\Gamma(G)$. In fact if $xK \in G/K$ has order pq, then there is a power of x which has order pq.*

Remark 2.2. *We know that $\mu(L_2(q))$=$\{p, (q-1)/d, (q+1)/d\}$, where $q = p^\alpha$ and $d = (2, q-1)$,* [8] *(page. 213). By* [28], *$\Gamma(L_3(q))$ has two connected components: $\pi_1(G) = \pi(p(q^2-1))$ and $\pi_2(G) = \pi((q^3-1)/(q-1)(3, q-1))$. Also the set of element orders $L_3(q)$, can be found in* [26]*; we have:*

$$\mu(L_3(q)) = \begin{cases} \{q-1, \frac{1}{3}p(q-1), \frac{1}{3}(q^2-1), \frac{1}{3}(q^2+q+1)\} & if \quad d = 3, \\ \{p(q-1), (q^2-1), (q^2+q+1)\} & if \quad d = 1, \end{cases}$$

where $q = p^\alpha$ is odd number and $d = (3, q-1)$, and

$$\mu(L_3(2^n)) = \begin{cases} \{4, 2^n-1, \frac{2}{3}(2^n-1), \frac{1}{3}(2^{2n}-1), \frac{1}{3}(2^{2n}+2^n+1)\} & if \quad d = 3, \\ \{4, 2(2^n-1), 2^{2n}-1, 2^{2n}+2^n+1\} & if \quad d = 1, \end{cases}$$

where $d = (3, 2^n-1)$.

Remark 2.3. *Let p be a prime number and $(a, p) = 1$. Let $k \geq 1$ be the smallest positive integer such that $a^k \equiv 1 (mod\ p)$. Then k is called the order of a with respect to p and we denote k by $ord_p(a)$. Obviously by the Fermat little theorem $ord_p(a) \mid (p-1)$. Also, if $a^n \equiv 1$ (mod p), then $ord_p(a) \mid n$.*

Lemma 2.4. [31] *Let G be a nonsolvable complement of a Frobenius group. Then G has a normal subgroup G_0=$SL(2, 5) \times Z$, such that $|G : G_0| \leq 2$, $\pi(Z) \cap \{2, 3, 5\}$=1 and the Sylow subgroups of Z are cyclic.*

Lemma 2.5. [33] *Let G be a finite Frobenius group with kernel K and complement C. Then*

(1) *K is nilpotent,*
(2) *The Sylow $p-$subgroups of C are cyclic if $p > 2$ and cyclic or generalized quaternion if $p = 2$.*

Using [37, Theorem A], we can conclude the following lemma.

Lemma 2.6. *A finite group G with disconnected prime graph $\Gamma(G)$ satisfies one of the following conditions:*

(1) *$s(G) = 2$, $G = KC$ is a Frobenius group with kernel K and complement C, and connected components of $\Gamma(G)$ are $\pi(K)$ and*

$\pi(C)$. Moreover, K is nilpotent, hence $\pi(K)$ is a complete graph. If C is solvable, then $\Gamma(C)$ is complete; otherwise, $2, 3, 5 \in \pi(G)$ and $\pi(C)$ can be obtained from the complete graph with vertex set $\pi(C)$ by removing the edge $3 \sim 5$.

(2) *$s(G) = 2$ and G is a 2-Frobenius group, i.e., $G = ABC$, where A and AB are normal subgroups of G, B is a normal subgroup of BC, and AB and BC are Frobenius groups. The connected components of $\Gamma(G)$ are complete graphs $\Gamma(AC)$ and $\Gamma(B)$.*

(3) *There exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$, where K is a nilpotent normal subgroup of G; furthermore K and $\overline{G}/S$ are trivial or π_1-groups, $s(S) \geq s(G)$, and for every $2 \leq i \leq s(G)$, there exists $2 \leq j \leq s(S)$ such that $\pi_i(G) = \pi_j(S)$.*

Remark 2.7. *A 2-Frobenius group is solvable and the above lemma implies that $t(G) = 1$ or 2 for a solvable group.*

Lemma 2.8. [35] *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the followings hold:*

(1) *There exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for a maximal normal soluble subgroup K of G.*

(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K|.|\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*

(3) *One of the followings holds:*
   (a) *every prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide the product $|K|.|\overline{G}/S|$; in particular, $t(2, S) \geq t(2, G)$;*
   (b) *there exists a prime $r \in \pi(K)$ nonadjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong A_7$ or $A_1(q)$ for some odd q.*

Remark 2.9. *Note that the condition of Lemma 2.8, implies an insolubility of G, and so by the Feit-Thompson Theorem, it is not necessary to assume in the main theorem, that G is of even order.*

Lemma 2.10. [30] *Let G be a finite group such that $s(G) \geq 2$ and K be a normal π_1-subgroup of G. Let S be a finite simple group such that $S \leq G/K$ and S is not a π_1-group. If $K \neq 1$, and S contains a Frobenius subgroup with kernel F and a cyclic complement C such that $(|F|, |K|) = 1$, then $r|C| \in \pi_e(G)$, for every prime divisor r of $|K|$.*

Lemma 2.11. [22] *Let G be a group with disconnected prime graph, such that $t(G) \geq 3$. Also let K be the maximal normal solvable subgroup of G. Then K is a nilpotent π_1-group.*

Lemma 2.12. [39] *There are a total of 1972 isomorphism types of finite non-Abelian simple groups G such that all prime divisors of $|G|$ do not exceed 1000.*

The groups obtained by Lemma 2.12, are listed in Tables $1-4$ of [39]. Let $t > 1$ and n be natural numbers and let $\varepsilon \in \{+,-\}$. If there exists a prime that divides $t^n - (\varepsilon 1)^n$ and does not divide $t^i - (\varepsilon 1)^i$ for $1 \leq i < n$, then we denote this prime by $t_{[\varepsilon n]}$ and call it a *primitive divisor* of $t^n - (\varepsilon 1)^n$. A primitive divisor need not be exist, nor be unique. The following lemma generalizes Zsigmondy's theorem:

Lemma 2.13. [29] *Let $t, n > 1$ be natural numbers. Then, for all $\varepsilon \in \{+,-\}$, there exists a primitive divisor $t_{[\varepsilon n]}$ of $t^n - (\varepsilon 1)^n$ except in the following cases:*

(1) $\varepsilon = +$, $n = 6$, $t = 2$;
(2) $\varepsilon = +$, $n = 2$ *and* $t = 2^l - 1$ *for some* $l \geq 2$;
(3) $\varepsilon = -$, $n = 3$, $t = 2$;
(4) $\varepsilon = -$, $n = 2$ *and* $t = 2^l + 1$ *for some* $l \geq 0$.

In the next lemma we write $L_n^\varepsilon(q)$, where $\varepsilon \in \{+,-\}$, $L_n^+(q) = L_n(q)$ and $L_n^-(q) = U_n(q)$.

Lemma 2.14. [29] *Suppose that $L_n^\varepsilon(q)$ is a simple group for some natural numbers n and q and that the primitive prime divisor $r = q_{[\varepsilon n]}$ of $q^n - (\varepsilon 1)^n$ exists. Then $L_n^\varepsilon(q)$ contains a Frobenius subgroup whose kernel is of order r and cyclic complement is of order n. Moreover, if n is odd or q is even, then such a Frobenius subgroup exists in $SL_n^\varepsilon(q)$.*

Lemma 2.15. *Let q be a prime power and $L = L_3(q)$, where $2 < q < 100$. If S is a non-Abelian simple group such that $\Gamma(S)$ is a subgraph of $\Gamma(L)$ and there exists $i \geq 2$ such that $\pi_2(L) = \pi_i(S)$, then $L \cong S$.*

Proof. For every finite non-Abelian simple group, we use [34] and [36] for adjacency of vertices in a prime graph of the group. If $q \neq 41, 59, 71, 73, 89$ and 97, then $\pi(L_3(q)) \subseteq \{2, 3, ..., 997\}$. So the result follows from Lemma 2.12. Now suppose that $q = 89$ and S is a non-Abelian simple group such that $\{2, 8011\} \subseteq \pi(S) \subseteq \{2, 3, 5, 11, 89, 8011\}$. According to the classification of finite simple groups we consider the following cases:

case 1. S is isomorphic to an alternating group. Since $8011 \mid |S|$ and $7 \nmid |S|$, we get a contradiction.
case 2. S is isomorphic to a sporadic simple group. Since $8011 \nmid |S|$, we get a contradiction.
case 3. S is isomorphic to a simple group of Lie type. We use the list of orders of these groups given in [4]. If S is defined over the field F_q, where $q = p^m$, then since $p \mid |S|$ we must have $p = 2, 3, 5, 11, 89$ or 8011. We note that $\pi(S)$ contains all prime divisors of $q^2 - 1$, except ${}^2B_2(q)$, where $q = 2^{2m+1}$.

If $p = 8011$, then $2003 \in \pi(8011^2 - 1) \subseteq \pi(q^2 - 1)$, which is a contradiction. So $p \neq 8011$.

If $p = 2$, then $ord_3(2) = 2$, $ord_5(2) = 4$, $ord_{11}(2) = 10$, $ord_{89}(2) = 11$ and $ord_{8011}(2) = 2670$. Thus $\max \{ord_r(2) \mid r \in \{3, 5, 11, 89, 8011\}\} = 2670$ and we have $2m \leq 2670$, for otherwise $2^{2m} - 1 = q^2 - 1$ would be divisible by a prime not in $\pi(S)$ by Lemma 2.13. Hence $m \leq 1335$. Since $2^3 - 1 = 7$, then $3 \nmid m$. The relations $2^4 + 1 = 17$, $2^5 - 1 = 31$, $2^7 - 1 = 127$, $17 \mid 2^8 - 1$, $73 \mid 2^9 - 1$, $31 \mid 2^{10} - 1$, $23 \mid 2^{11} - 1$ and Lemma 2.13 imply that $m \in \{1, 2, 1335\}$. By using the GAP-Program $2^{2670} - 1$ is not prime and there exists a prime $k \notin \{2, 3, 5, 11, 89, 8011\}$ such that $k \mid 2^{2670} - 1$. So $m = 1$ or 2. It is easy to check there is no possibility in this case.

If $p = 3$, then $ord_2(3) = 1$, $ord_5(3) = 4$, $ord_{11}(3) = 5$, $ord_{89}(3) = 88$ and $ord_{8011}(3) = 2670$. Thus $\max \{ord_r(3) \mid r \in \{2, 5, 11, 89, 8011\}\} = 2670$. As above, m does not exceed 1335. Furthermore, $3 \nmid m$, since otherwise $13 \in \pi(3^3 - 1) \subseteq \pi(3^m - 1)$. The relations $3^4 + 1 = 2 \times 41$, $61 \mid 3^5 + 1$, $23 \mid 3^{88} - 1$ and Lemma 2.13 imply that $m \in \{1, 2, 1335\}$. By using the GAP-Program $3^{2670} - 1$ is not prime and there exists a prime $k \notin \{2, 3, 5, 11, 89, 8011\}$ such that $k \mid 3^{2670} - 1$. So $m = 1$ or 2. It is easy to check that there is no possibility in this case as well.

If $p = 5$, then $\max \{ord_r(5) \mid r \in \{2, 3, 11, 89, 8011\}\} = 445$. So similar to the previous case there is no possibility in this case as well.

If $p = 11$, then $\max \{ord_r(11) \mid r \in \{2, 3, 5, 89, 8011\}\} = 890$. So $m \leq 445$. It is easy to check that there is no possibility in this case, too.

If $p = 89$, then $\max \{ord_r(89) \mid r \in \{2, 3, 5, 11, 8011\}\} = 3$. So $m = 1$ and the only possibility is $S \cong L_3(89)$.

Similarly, we get our results for $q = 41, 59, 71, 73$ and 97. $\square$

Corollary 2.16. *Let $L = L_3(q)$, where $2 < q < 100$ and let S be a finite simple group such that $\Gamma(S) = \Gamma(L)$. Then S is isomorphic to L.*
Proof. *Straightforward from Lemma 2.15.*

Remark 2.17. *If S is a finite simple group such that $\Gamma(S) = \Gamma(L_3(2))$, then S isomorphic to $L_3(2)$ or $L_2(8)$, by Lemma 2.12.*

In the following we give the structure of the group of outer automorphisms of the group $L_n(q)$.

Lemma 2.18. [23] *Let $n \geq 3$, and $q = p^f$. Then $Out(L_n(q)) \cong \mathbb{Z}_{(n,\ q-1)} : \mathbb{Z}_f : \mathbb{Z}_2$.*

By the above Lemma, we conclude that $Out(L_3(q)) \cong \mathbb{Z}_{(3,\ q-1)} : \mathbb{Z}_f : \mathbb{Z}_2$.

Lemma 2.19. [25] *Let $L_3(q) < G \leq Aut(L_3(q))$, then $\Gamma(G)$ is not connected if and only if $G/L_3(q) \cong \langle\gamma\rangle \times \langle\beta\rangle$, where γ and β are field and graph automorphism, respectively.*

By the above lemma, if $(3, q-1) = 3$, then $L_3(q).3$ has a connected prime graph. So $\Gamma(L_3(q).3)$ is not subgraph of $\Gamma(L_3(q))$.

Lemma 2.20. [32] *Let G be a finite group and let K a nontrivial normal p-subgroup, for some prime p, and set $L = G/K$. Suppose that L contains an element x of order m coprime to p such that $\langle \varphi|_{\langle x\rangle}, 1|_{\langle x\rangle}\rangle > 0$ for every Brauer character φ of (an absolutely irreducible representation of) L in characteristic p. Then G contains elements of order pm.*

3. Proof of the Main Theorem

Let G be a finite group such that $\Gamma(G) = \Gamma(L_3(q))$, where $2 < q < 100$. By [34], we have $3 \leq t(G) \leq 4$ and $t(2, G) \geq 2$, except for $q = 3$. First, we consider this case separately.

The Case $L = L_3(3)$
By assumption $\Gamma(G) = \Gamma(L_3(3))$. So $s(G) = 2$ and we can apply Lemma 2.6. Suppose that G is a Frobenius or 2-Frobenius group. If G is nonsolvable, then G is a Frobenius group by Remark 2.7. Hence the Frobenius complement of G has a normal subgroup $SL_2(5) \times Z$ with index at most 2 by Lemma 2.4. Since $5 \notin \pi(G)$, we get a contradiction. Therefore G is solvable Frobenius or 2-Frobenius group. We note that there exists a finite Frobenius group G with $\Gamma(G) = \Gamma(L_3(3))$. Indeed, let H be an extension of the group of order 2 by S_4 such that the Sylow $2-$subgroup in H is a generalized quaternion group. Then $\mu(H) = \{6, 8\}$. By Lemma 8 in [27], there is a Frobenius group G which is an extension of an elementary Abelian $13-$group by H. Then $\mu(G) = \{6, 8, 13\} = \mu(L_3(3))$.

Therefore we conclude that $\Gamma(G) = \Gamma(L_3(3))$. Also, we note that there exists a finite $2-$Frobenius group G of order $3^6.13.2$ such that $\Gamma(G) = \Gamma(L_3(3))$. Now suppose that there exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$, where K is a π_1-group and nilpotent normal subgroup of G. It follows from Lemma 2.15, that $S \cong L_3(3)$. Since $Out(L_3(3)) \cong \mathbb{Z}_2$ by Lemma 2.18, then $\overline{G}/L_3(3) \leq \mathbb{Z}_2$. Therefore $G/O_\pi(G) \cong L_3(3)$ or $L_3(3).2$, where $\pi \subseteq \{2, 3\}$.

If $q \neq 3$, then $t(G) \geq 3$ and $t(2, G) \geq 2$. By Lemmas 2.8 and 2.11, there exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for the maximal normal solvable subgroup K of G, and K is nilpotent π_1-group.

Case $L = L_3(2)$

It follows from Remark 2.17, that $S \cong L_3(2)$ or $L_2(8)$. At first, suppose that $S \cong L_3(2)$. We know that $Out(L_3(2)) \cong \mathbb{Z}_2$, by Lemma 2.18. Then $\overline{G}/L_3(2) \leq \mathbb{Z}_2$. By [4], $L_3(2).2$ has an element of order 6, so $\Gamma(L_3(2).2)$ is not subgraph of $\Gamma(G)$. So we have $\overline{G} = G/K \cong L_3(2)$, where K is a nilpotent $2-$group. By [4], we know that $L_3(2)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3. If $2 \in \pi(K)$, then by Lemma 2.10, $2 \sim 3$ in $\Gamma(G)$, which is a contradiction. Therefore $K = 1$ and in this case we have $G \cong L_3(2)$. Now suppose that $S \cong L_2(8)$. We note that $Out(L_2(8)) \cong \mathbb{Z}_3$. Then $\overline{G}/L_2(8) \leq \mathbb{Z}_3$. Since $L_2(8).3$ has an element of order 6, then $\Gamma(L_2(8).3)$ is not subgraph of $\Gamma(G)$. So we have $G/K \cong L_2(8)$. Therefore, $G/O_2(G) \cong L_2(8)$.

Case $L = L_3(4)$

It follows from Lemma 2.15, that $S \cong L_3(4)$. We know that $Out(L_3(4)) \cong \mathbb{Z}_3 : \mathbb{Z}_2 : \mathbb{Z}_2 \cong \mathbb{Z}_2 \times S_3$, and so $\overline{G}/L_3(4) \leq Out(L_3(4)) \cong \mathbb{Z}_2 \times S_3$. By the notations of [4], the finite groups $L_3(4).2_1$, $L_3(4).3$, $L_3(4).6$ $L_3(4).2_2 \cong L_3(4).2'_2 \cong L_3(4).2''_2$ and $L_3(4).2_3 \cong L_3(4).2'_3 \cong L_3(4).2''_3$ have elements of order 6. Then the prime graphs of these groups are not subgraphs of $\Gamma(G)$. So $G/K \cong L_3(4)$, where K is a nilpotent $2-$group. We note that $L_3(2)$ is a maximal subgroup of $L_3(4)$ and contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3, by [4]. If $2 \in \pi(K)$, then by Lemma 2.10, $2 \sim 3$ in $\Gamma(G)$, which is a contradiction. Hence $K = 1$ and $G \cong L_3(4)$. Therefore $L_3(4)$ is recognizable by its prime graph.

Table 1.

L	G or $\overline{G} = G/O_\pi(G)$
$L_3(2)$	$G \cong L_3(2)$ $\overline{G} \cong L_2(8)$, where $\pi \subseteq \{2\}$
$L_3(3)$	G is solvable Frobenius or 2-Frobenius group or $\overline{G} \cong L_3(3)$, where $\pi \subseteq \{2,3\}$
$L_3(4)$	$G \cong L_3(4)$
$L_3(5)$	$\overline{G} \cong L_3(5)$ or $L_3(5).2$, where $\pi \subseteq \{2\}$
$L_3(7)$	$G \cong L_3(7)$ or $L_3(7).2$
$L_3(8)$	$\overline{G} \cong L_3(8)$, where $\pi \subseteq \{7\}$
$L_3(9)$	$G \cong L_3(9)$ or $L_3(9).2_1$
$L_3(11)$	$\overline{G} \cong L_3(11)$ or $L_3(11).2$, where $\pi \subseteq \{2,5\}$
$L_3(13)$	$\overline{G} \cong L_3(13)$ or $L_3(13).2$, where $\pi \subseteq \{2\}$
$L_3(16)$	$\overline{G} \leq L_3(16).(\mathbb{Z}_4 \times \mathbb{Z}_2)$, where $\pi \subseteq \{5\}$
$L_3(17)$	$\overline{G} \cong L_3(17)$ or $L_3(17).2$, where $\pi \subseteq \{2\}$
$L_3(19)$	$\overline{G} \cong L_3(19)$ or $L_3(19).2$, where $\pi \subseteq \{2,3,19\}$
$L_3(23)$	$\overline{G} \cong L_3(23)$ or $L_3(23).2$, where $\pi \subseteq \{2,11\}$
$L_3(25)$	$\overline{G} \leq L_3(25).(\mathbb{Z}_2 \times \mathbb{Z}_2)$, where $\pi \subseteq \{2\}$
$L_3(27)$	$\overline{G} \leq L_3(27).\mathbb{Z}_6$, where $\pi \subseteq \{2,13\}$
$L_3(29)$	$\overline{G} \cong L_3(29)$ or $L_3(29).2$, where $\pi \subseteq \{2,7\}$
$L_3(31)$	$\overline{G} \cong L_3(31)$ or $L_3(31).2$, where $\pi \subseteq \{2,5\}$
$L_3(32)$	$\overline{G} \leq L_3(32).\mathbb{Z}_{10}$, where $\pi \subseteq \{31\}$
$L_3(37)$	$\overline{G} \cong L_3(37)$ or $L_3(37).2$, where $\pi \subseteq \{2,3,37\}$
$L_3(41)$	$\overline{G} \cong L_3(41)$ or $L_3(41).2$, where $\pi \subseteq \{2,5\}$
$L_3(43)$	$\overline{G} \cong L_3(43)$ or $L_3(43).2$, where $\pi \subseteq \{2,7\}$
$L_3(47)$	$\overline{G} \cong L_3(47)$ or $L_3(47).2$, where $\pi \subseteq \{2,23\}$
$L_3(49)$	$\overline{G} \leq L_3(49).(\mathbb{Z}_2 \times \mathbb{Z}_2)$, where $\pi \subseteq \{2\}$
$L_3(53)$	$\overline{G} \cong L_3(53)$ or $L_3(53).2$, where $\pi \subseteq \{2,13\}$
$L_3(59)$	$\overline{G} \cong L_3(59)$ or $L_3(59).2$, where $\pi \subseteq \{2,29\}$
$L_3(61)$	$\overline{G} \cong L_3(61)$ or $L_3(61).2$, where $\pi \subseteq \{2,5\}$
$U_3(64)$	$\overline{G} \leq L_3(64).(\mathbb{Z}_6 \times \mathbb{Z}_2)$, where $\pi \subseteq \{2,3,7\}$
$L_3(67)$	$\overline{G} \cong L_3(67)$ or $L_3(67).2$, where $\pi \subseteq \{2,11\}$
$L_3(71)$	$\overline{G} \cong L_3(71)$ or $L_3(71).2$, where $\pi \subseteq \{2,5,7\}$
$L_3(73)$	$\overline{G} \cong L_3(73)$ or $L_3(73).2$, where $\pi \subseteq \{2,3\}$
$L_3(79)$	$\overline{G} \cong L_3(79)$ or $L_3(79).2$, where $\pi \subseteq \{2,13\}$
$L_3(81)$	$\overline{G} \leq L_3(81).(\mathbb{Z}_4 \times \mathbb{Z}_2)$, where $\pi \subseteq \{2,3,5\}$
$L_3(83)$	$\overline{G} \cong L_3(83)$ or $L_3(83).2$, where $\pi \subseteq \{2,41\}$
$L_3(89)$	$\overline{G} \cong L_3(89)$ or $L_3(89).2$, where $\pi \subseteq \{2,11\}$
$L_3(97)$	$\overline{G} \cong L_3(97)$ or $L_3(97).2$, where $\pi \subseteq \{2\}$

Case $L = L_3(5)$
It follows from Lemma 2.15, that $S \cong L_3(5)$. Then $\overline{G}/L_3(5) \leq Out(L_3(5)) \cong \mathbb{Z}_2$. So we have $G/K \cong L_3(7)$ or $L_3(7).2$, where K is a nilpotent $\{2,3,5\}-$group. We note that $L_3(5)$ contains a Frobenius subgroup with Frobenius kernel of order 31 and Frobenius complement of order 3 by Lemma 2.14. If $5 \in \pi(K)$, then by Lemma 2.10, $3 \sim 5$ in $\Gamma(G)$, which is a contradiction. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_5(G)$. Since K is a nilpotent group, then P *char* K. On the other hand, Since $K \trianglelefteq G$ we can conclude that $P \trianglelefteq G$. Since $3 \not\sim 5$ in $\Gamma(G)$, then Q acts fixed point free on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $L_3(5)$ has no element of order 5^3, by Remark 2.2. Therefore we have $G/O_2(G) \cong L_3(5)$ or $L_3(5).2$.

Case $L = L_3(7)$
It follows from Lemma 2.15, that $S \cong L_3(7)$. We know that $Out(L_3(7)) \cong \mathbb{Z}_3 : \mathbb{Z}_2 \cong S_3$. Then $\overline{G}/L_3(7) \leq S_3$. By Lemma 2.19, $L_3(7).3$ has a connected prime graph, and so $\Gamma(L_3(7).3)$ is not a subgraph of $\Gamma(G)$. Hence we have $G/K \cong L_3(7)$ or $L_3(7).2$, where K is a nilpotent $\{2,3,7\}-$group. We know that $L_3(7)$ contains a Frobenius subgroup with Frobenius kernel of order 19 and Frobenius complement of order 3, by Lemma 2.14. If $7 \in \pi(K)$, then by Lemma 2.10, $3 \sim 7$ in $\Gamma(G)$, which is a contradiction. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_7(G)$. Since $3 \not\sim 7$ in $\Gamma(G)$, then Q acts fixed point free on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $L_3(7)$ has no element of order 7^3, by Remark 2.2. If $2 \in \pi(K)$, then let $x \in G/K$, $X = \langle x \rangle$ and $o(x) = 19$ and $z = exp(2\pi i/19)$. Now by using [10] about the irreducible characters of $L_3(7)$ (mod 2), we can see that $\langle \varphi_1|_X, 1|_X \rangle = (1 + 1 \times 18)/19 = 1 > 0$, $\langle \varphi_2|_X, 1|_X \rangle = (56 + (-1) \times 18)/19 = 2 > 0$, $\langle \varphi_i|_X, 1|_X \rangle = (152 + 0 \times 18)/19 = 8 > 0$, for $i = 3,4,5$, $\langle \varphi_j|_X, 1|_X \rangle = (288 + ((z + z^7 + z^{11}) + (z^{-1} + z^{-7} + z^{-11}) + (z^2 + z^{14} + z^{22}) + (z^{-2} + z^{-14} + z^{-22}) + (z^4 + z^{28} + z^{44}) + (z^{-4} + z^{-28} + z^{-44})) \times 3)/19 = (288 + (\sum_{i=1}^{18} z^i) \times 3)/19 = (288 + (-1) \times 3)/19 = 15 > 0$ for $j = 6,7,...,11$ and $\langle \varphi_{12}|_X, 1|_X \rangle = (342 + 0 \times 18)/19 = 18 > 0$. Therefore for every irreducible character φ of $L_3(7)$ (mod 2) we show that $\langle \varphi|_X, 1|_X \rangle = \sum_{x \in X} \varphi(x)/|X| > 0$. Now by using Lemma 2.20, it follows that $38 \in \pi_e(G)$. Then $2 \sim 19$ in $\Gamma(G)$, which is a contradiction. Therefore $G \cong L_3(7)$ or $L_3(7).2$ and $k(\Gamma(L_3(7)) = 2$.

Case $L = L_3(8)$

It follows from Lemma 2.15, that $S \cong L_3(8)$. We note that $Out(L_3(8)) \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$. Since $L_3(8).2$, $L_3(8).3$ and $L_3(8).6$ have elements of order 6, then $\Gamma(L_3(8).2)$, $\Gamma(L_3(8).3)$ and $\Gamma(L_3(8).6)$ are not subgraphs of $\Gamma(G)$. So we have $G/K \cong L_3(8)$, where K is a nilpotent $\{2,3,7\}-$group. We know that $L_3(8)$ contains a Frobenius subgroup with Frobenius kernel of order 73 and Frobenius complement of order 3, by Lemma 2.14. If $2 \in \pi(K)$, then by Lemma 2.10, $2 \sim 3$ in $\Gamma(G)$, which is a contradiction. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_7(G)$. Since $3 \not\sim 7$ in $\Gamma(G)$, then Q acts fixed point free on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $L_3(8)$ has no element of order 7^3, by Remark 2.2. Therefore $G/O_7(G) \cong L_3(8)$.

Case $L = L_3(9)$

It follows from Lemma 2.15, that $S \cong L_3(9)$. We note that $Out(L_3(9)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $L_3(9).2_2$ and $L_3(9).2_3$, have elements of order 26 and 14 respectively, then $\Gamma(L_3(9).2_2)$ and $\Gamma(L_3(9).2_3)$ are not subgraphs of $\Gamma(G)$. So we have $G/K \cong L_3(9)$ or $L_3(9).2_1$, where K is a nilpotent $\{2,3,5\}-$group. We know that $L_3(9)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3, by Lemma 2.14. If $5 \in \pi(K)$, then by Lemma 2.10, $3 \sim 5$ in $\Gamma(G)$, which is a contradiction. We may assume that K is an elementary Abelian p-group for $p \in \{2,3\}$. If $2 \in \pi(K)$, then Let $x \in G/K$, $X = \langle x \rangle$, $o(x) = 7$. Now by using [10] about the irreducible characters of $L_3(9)$ (mod 2), we can see that $\langle \varphi_1|_X, 1|_X \rangle = (1 + 1 \times 6)/7 = 1 > 0$, $\langle \varphi_2|_X, 1|_X \rangle = (90 + (-1) \times 6)/7 = 12 > 0$, $\langle \varphi_3|_X, 1|_X \rangle = \langle \varphi_4|_X, 1|_X \rangle = (640 + 3(1/2(-1 + i\sqrt{7})) + 3(1/2(-1 - i\sqrt{7})))/7 = (640 - 3)/7 = 91 > 0$, $\langle \varphi_i|_X, 1|_X \rangle = (640 + 3 \times 6)/7 = 94 > 0$ for $i = 5, 6, 7, 8$, similarly $\langle \varphi_j|_X, 1|_X \rangle = (640 + 3(1/2(-1 + i\sqrt{7})) + 3(1/2(-1 - i\sqrt{7})))/7 = (640 - 3)/7 = 91 > 0$ for $j = 9, \ldots, 32$, $\langle \varphi_{33}|_X, 1|_X \rangle = \langle \varphi_{34}|_X, 1|_X \rangle = \langle \varphi_{35}|_X, 1|_X \rangle = (728 + 0 \times 6)/7 = 104 > 0$. Therefore for every irreducible character φ of $L_3(9)$ (mod 2) we show that $\langle \varphi|_X, 1|_X \rangle = \sum_{x \in X} \varphi(x)/|X| > 0$. Now by using Lemma 2.20, it follows that $14 \in \pi_e(G)$. Then $2 \sim 7$ in $\Gamma(G)$, which is a contradiction. If $3 \in \pi(K)$, then Let $x \in G/K$, $X = \langle x \rangle$, $o(x) = 5$. Now by [10], we can check easily $\langle \varphi|_X, 1|_X \rangle = \sum_{x \in X} \varphi(x)/|X| > 0$ for every irreducible character φ of $L_3(9)$ (mod 3). By Lemma 2.20, it follows that $15 \in \pi_e(G)$. Then

$3 \sim 5$ in $\Gamma(G)$, which is a contradiction. Hence $K = 1$ and $G \cong L_3(9)$ or $L_3(9).2_1$. Therefore $k(\Gamma(L_3(9)) = 2$.

The proof of the other cases are similar and for convenience, we give some of them, namely $L_3(11)$, $L_3(13)$, $L_3(16)$, $L_3(25)$ and $L_3(32)$.

Case $L = L_3(11)$
It follows from Lemma 2.15, that $S \cong L_3(11)$. We know that $Out(L_3(11)) \cong \mathbb{Z}_2$. Then $\overline{G}/L_3(11) \leq \mathbb{Z}_2$. So we have $G/K \cong L_3(11)$ or $L_3(11).2$, where K is a nilpotent $\{2, 3, 5, 11\}-$group. We know that $L_3(11)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3 by Lemma 2.14. If $11 \in \pi(K)$, then by Lemma 2.10, $3 \sim 11$ in $\Gamma(G)$, which is a contradiction. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_{11}(G)$. We know that $3 \not\sim 11$ in $\Gamma(G)$. So Q acts fixed point free on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $L_3(11)$ has no element of order 11^3. Therefore $G/O_\pi(G) \cong L_3(11)$ or $L_3(11).2$, where $\pi \subseteq \{2, 5\}$.

Case $L = L_3(13)$
It follows from Lemma 2.15, that $S \cong L_3(13)$. We know that $Out(L_3(13)) \cong \mathbb{Z}_3 : \mathbb{Z}_2$, by Lemma 2.18. By Lemma 2.19, $L_3(13).3$ has a connected prime graph, and so $\Gamma(L_3(13).3)$ is not subgraph of $\Gamma(G)$. Hence $\overline{G}/L_3(13) \leq \mathbb{Z}_2$, and so $G/K \leq L_3(13).\mathbb{Z}_2$, where K is a nilpotent $\{2, 3, 7, 13\}-$group. We know that $L_3(13)$ contains a Frobenius subgroup with Frobenius kernel of order 61 and Frobenius complement of order 3 by Lemma 2.14. If $13 \in \pi(K)$, then by Lemma 2.5, $3 \sim 13$ in $\Gamma(G)$, which is a contradiction. Similarly $7 \notin \pi(K)$. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_{13}(G)$. We know that $3 \not\sim 13$ in $\Gamma(G)$. So Q acts fixed point free on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $L_3(13)$ has no element of order 13^3. Therefore $G/O_2(G) \leq L_3(13).\mathbb{Z}_2$.

Case $L = L_3(16)$
It follows from Lemma 2.15, that $S \cong L_3(16)$. We know that $Out(L_3(16)) \cong \mathbb{Z}_3 : \mathbb{Z}_4 : \mathbb{Z}_2$, by Lemma 2.18. By Lemma 2.19, $L_3(16).3$ has a connected prime graph, and so $\Gamma(L_3(16).3)$ is not a subgraph of $\Gamma(G)$. Hence

$\overline{G}/L_3(16) \leq (\mathbb{Z}_4 \times \mathbb{Z}_2)$, and so $G/K \leq L_3(16).(\mathbb{Z}_4 \times \mathbb{Z}_2)$, where K is a nilpotent $\{2,3,5,17\}-$group. We know that $L_3(16)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3 by Lemma 2.14. If $2 \in \pi(K)$, then by Lemma 2.10, $2 \sim 3$ in $\Gamma(G)$, which is a contradiction. Since $S \ncong A_7$, $A_1(q)$ and $2 \not\sim 3$ in $\Gamma(G)$, then by Lemma 2.8, we have $3 \nmid |K||\overline{G}/L_3(16)|$. So $3 \notin \pi(K)$. Similarly $17 \notin \pi(K)$. Therefore $G/O_5(G) \leq L_3(16).(\mathbb{Z}_4 \times \mathbb{Z}_2)$.

Case $L = L_3(25)$
It follows from Lemma 2.15, that $S \cong L_3(25)$. We know that $Out(L_3(25)) \cong \mathbb{Z}_3 : \mathbb{Z}_2 : \mathbb{Z}_2$, by Lemma 2.18. By Lemma 2.19, $L_3(25).3$ has a connected prime graph, and so $\Gamma(L_3(25).3)$ is not a subgraph of $\Gamma(G)$. Hence $\overline{G}/L_3(25) \leq (\mathbb{Z}_2 \times \mathbb{Z}_2)$, where K is a nilpotent $\{2,3,5,13\}-$group. We know that $L_3(25)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3 by Lemma 2.14. If $5 \in \pi(K)$, then by Lemma 2.10, $3 \sim 5$ in $\Gamma(G)$, which is a contradiction. Similarly, $13 \in \pi(K)$. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_5(G)$. We know that $3 \not\sim 5$ in $\Gamma(G)$. So Q acts fixed point free on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $L_3(25)$ has no element of order 5^4. Therefore, $G/O_2(G) \leq L_3(25).(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Case $L = L_3(32)$
It follows from Lemma 2.15, that $S \cong L_3(32)$. We know that $Out(L_3(32)) \cong \mathbb{Z}_5 \times \mathbb{Z}_2 \cong \mathbb{Z}_{10}$. Hence $\overline{G}/L_3(32) \leq \mathbb{Z}_{10}$ and so $G/K \leq L_3(32).\mathbb{Z}_{10}$. We know that $L_3(32)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3 by Lemma 2.14. If $2 \in \pi(K)$, then by Lemma 2.10, $2 \sim 3$ in $\Gamma(G)$, which is a contradiction. Since $S \ncong A_7$, $A_1(q)$ and $2 \not\sim 3$ in $\Gamma(G)$, then by Lemma 2.8, we have $3 \nmid |K||\overline{G}/L_3(32)|$. So $3 \notin \pi(K)$. Similarly, $11 \notin \pi(K)$. Therefore $G/O_{31}(G) \leq L_3(32).\mathbb{Z}_{10}$.

By the main theorem and definitions of recognizability, quasirecognizability and $n-$recognizability of prime graph, we can conclude the following corollaries.

Corollary 3.1. *The finite simple group $L_3(q)$ for $3 < q < 100$ is quasirecognizable by its prime graph.*

Corollary 3.2. *The finite simple group $L_3(4)$ is recognizable by its prime graph and the finite simple groups $L_3(7)$ and $L_3(9)$ are $2-$recognizable by its prime graph.*

By [27], we know that the simple groups $L_3(3)$ is not recognizable by spectrum, and so it is not recognizable by its prime graph. It seems that if $p \geq 7$ is a prime number, then the simple group $L_3(p)$ is $2-$recognizable by its prime graph. Finally, we pose the following problem:

Problem: Is the simple group $L_3(p)$, $2-$recognizable by its prime graph, where $p \geq 7$ is a prime number?

Acknowledgments

The authors would like to thank from the referees for the valuable comments. Partial support by the Center of Excellence of Algebraic Hyperstructures and its Applications of Tarbiat Modares University (CEAHA) is gratefully acknowledged by the third author (AI).

References

[1] Z. Akhlaghi, M. Khatami and B. Khosravi, Quasirecognition by prime graph of the simple group ${}^2F_4(q)$, *Acta. Math. Hungar.* **122** (2009), no. 4, 387–397.

[2] O. A. Alekseeva and A. S. Kondrat′ev, Quasirecognition of one class of finite simple groups by the set of element orders, *Sib. Math. J.* **44** (2003), no. 2, 195–207.

[3] A. Babai , B. Khosravi and N. Hasani, Quasirecognition by prime graph of ${}^2D_p(3)$ where $p = 2^n + 1 \geq 5$ is a prime, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 3, 343–350.

[4] J. Conway, R. Curtis, S. Norton, R. Parker and R. Wilson, Atlas of Finite Groups, Oxford University Press, Eynsham, 1985.

[5] M. Foroudi Ghasemabadi and A. Iranmanesh, Quasirecognition by the prime graph of the group $C_n(2)$, where $n \neq 3$ is odd, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 3, 529–540.

[6] The GAP Group, GAP Groups, Algorithms, and Programming, Version 4.4.12; 2008, (http://www.gap-system.org).

[7] M. Hagie, The prime graph of a sporadic simple group, *Comm. Algebra* **31** (2003), no. 9, 4405–4424.

[8] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin-New York, 1967.

[9] N. Iiyori and H. Yamaki, Prime graph components of the simple groups of Lie type over the field of even characteristic, *J. Algebra* **155** (1993), no. 2, 335–343.

[10] C. Janson, K. Lux, R. A. Parker and R. A. Wilson, An Atlas of Brauer Characters, Oxford University Press, New York, 1995.

[11] B. Khosravi and S. S. Salehi Amiri, On the prime graph of $L_2(q)$, where $q = p^\alpha < 100$, *Quasigroups Related Systems* **14** (2006), no. 2, 179–190.

[12] B. Khosravi, B. Khosravi and B. Khosravi, Groups with the same prime graph as a CIT simple group, *Houston J. Math.* **33** (2007), no. 4, 967–977.

[13] A. Khosravi and B. Khosravi, Quasirecognition of the simple group ${}^2G_2(q)$ by a prime graph, *Sib. Math. J.* **48** (2007), no. 3, 570–577.

[14] B. Khosravi, B. Khosravi and B. Khosravi, On the prime graph of $PSL(2, p)$, where $p > 3$ is a prime number, *Acta. Math. Hungar.* **116** (2007), no. 4, 295–307.

[15] B. Khosravi and A. Zarea Moghanjoghi, Quasirecognition by prime graph of some alternating groups, *Int. J. Contemp. Math. Sci.* **2** (2007), no. 25-28, 1351–1358.

[16] A. Khosravi and B. Khosravi, 2-recognizability of $\mathrm{PSL}(2, p^2)$ by a prime graph, *Sib. Math. J.* **49** (2008), no. 4, 749–757.

[17] B. Khosravi, n-recognition by prime graph of the simple group $PSL(2, q)$, *J. Algebra Appl.* **7** (2008), no. 6, 735–748.

[18] B. Khosravi, B. Khosravi and B. Khosravi, A characterization of the finite simple group $L_{16}(2)$ by its prime graph, *Manuscripta Math.* **126** (2008), no. 1, 49–58.

[19] B. Khosravi, Some characterizations of $L_9(2)$ related to its prime graph, *Publ. Math. Debrecen* **75** (2009), no. 3-4, 375–385.

[20] B. Khosravi, Quasirecognizability of $L_{10}(2)$ by a prime graph, *Sib. Math. J.* **50** (2009), no. 2, 355–359.

[21] B. Khosravi and H. Moradi, Quasirecognition by prime graph of finite simple groups $L_n(2)$ and $U_n(2)$, *Acta Math. Hungar.* **132** (2011), no. 1-2, 140–153.

[22] B. Khosravi, Z. Akhlaghi and M. Khatami, Quasirecognition by prime graph of simple group $D_n(3)$, *Publ. Math. Debrecen* **78** (2011), no. 2, 469–484.

[23] P. Kleidman and M. Liebeck, The Subgroup Structure of Finite Classical Groups, London Math. Soc. Lecture Note Ser., 129, Cambridge University Press, Cambridge, 1990.

[24] A. S. Kondtrat′ev, Prime graph components of finite groups, *Math. USSR-Sb.* **67** (1990), no. 1, 235–247.

[25] M. S. Lucido, Prime graph components of finite almost simple groups, *Rend. Sem. Mat. Univ. Padova.* **102** (1999) 1–22.

[26] M. S. Luchido and A. R. Moghaddamfar, Groups with complete prime graph connected components, *J. Group Theory* **7** (2004), no. 3, 373–384.

[27] V. D. Mazurov, Recognition of finite simple groups $S_4(q)$ by their element orders, *Algebra Logic* **41** (2002), no. 2, 93–110.

[28] V. D. Mazurov, Characterizations of groups by arithmetic properties, *Algebra Colloq.* **11** (2004), no. 1, 129–140.

[29] V. D. Mazurov and A. V. Zavarnitsine, On element orders in coverings of the simple groups $L_n(q)$ and $U_n(q)$, *Proc. Steklov Inst. Math.* **257** (2007) 145–154.

[30] Z. Momen and B. Khosravi, On r-recognition by prime graph of $B_p(3)$ where p is an odd prime, *Monatsh. Math.* (2012), no. 2, 239–253.

[31] D. S. Passman, Permutation Groups, W. A. Benjamin Inc., New York-Amsterdam, 1968.

[32] C. E. Praeger and W. Shi, A characterization of some alternating and symmetric groups, *Comm. Algebra* **22** (1994), no. 5, 1507–1530.

[33] D. J. S. Robinson, A Course on the Theory of Groups, Springer-Verlag, New York, 1982.

[34] A. V. Vasil′ev and E. P. Vdovin, An adjacency criterion in the prime graph of a finite simple group, *Algebra Logic* **44** (2005), no. 6, 381–406.
[35] A. V. Vasil′ev and I. B. Gorshkov, On the recognition of finite simple groups with a connected prime graph, *Sib. Math. J.* **50** (2009), no. 2, 233–238.
[36] A. V. Vasil′ev and E. P. Vdovin, Cocliques of maximal size in the prime graph of a finite simple group, *Algebra Logic* **50** (2011), no. 4, 291–322.
[37] J. S. Williams, Prime graph components of finite groups, *J. Algebra* **69** (1981), no. 2, 487–513.
[38] A. V. Zavarnitsin, Recognition of finite groups by the prime graph, *Algebra Logic* **45** (2006), no. 4, 220–231.
[39] A. V. Zavarnitsin, Finite simple groups with narrow prime spectrum, *Sib. Élektron. Mat. Izv.* (2009) 1–12.
[40] A. V. Zavarnitsine, Uniqueness of the prime graph of $L_{16}(2)$, *Sib. Électron. Math. Izv.* **7** (2010) 119–121.

Seyed Sadegh Salehi Amiri
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
Email: salehissss@yahoo.com

Alireza Khalili Asboei
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
Email: khaliliasbo@yahoo.com

Ali Iranmanesh
Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University P.O. Box 14115-137, Tehran, Iran
Email: iranmanesh@modares.ac.ir

Abolfazl Tehranian
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
Email: tehranian1340@yahoo.com