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EXISTENCE OF POSITIVE SOLUTIONS FOR A BOUNDARY VALUE PROBLEM OF A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. This paper presents conditions for the existence and multiplicity of positive solutions for a boundary value problem of a nonlinear fractional differential equation. We show that it has at least one or two positive solutions. The main tool is Krasnosel'skii fixed point theorem on cone and fixed point index theory.

1. Introduction

In this paper, we discuss the existence and multiplicity of positive solutions to boundary value problem of nonlinear fractional differential equation

$$D^{\alpha}_{0+}u(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3 \tag{1.1}$$

$$u(0) = u''(0) = 0, \quad u'(1) = \gamma u'(\eta) \tag{1.2}$$

where D^{α}_{0+} is the Caputo's differentiation and $\eta, \gamma \in (0,1)$.
Throughout the paper, we assume that f and a satisfy the following conditions.

(H1) $f : [0,\infty) \to [0,\infty)$ is continuous.

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(H2) $a \in L^{\infty}[0,1]$ and there exists $m > 0$ such that $a(t) \geq m$ a.e. $t \in [0,1]$.

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry and engineering. El-Shahed [4] used the Krasnosel'skii fixed point theorem on cone to show the existence and non-existence of positive solutions, and Kaufmann and Mboumi [6] studied the existence and multiplicity of positive solutions of nonlinear fractional boundary value problem, where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative. Qiu and Bai [8] studied the existence by using Krasnosel'skii fixed point theorem and nonlinear alternative of Leray-Schauder type in cones and Benchohra et al [1] studied the existence and uniqueness by using the non-linear alternative of Leray-Schauder type and Banach's, Schaefer's and Burton and Kirk fixed point theorem, where D^{α}_{0+} is the Caputo fractional derivative. In [2], the authors obtained existence and multiplicity of positive solutions for using the fixed point theorem due to Avery and Peterson. Motivated by the above works, we obtain some sufficient conditions for the existence of at least one and two positive solutions for (1.1) and (1.2).
The structure of the paper is as follows. In section 2, we present some necessary definitions and preliminary results that will be used later. In section 3, we discuss the existence of at least one positive solution for (1.1) and (1.2). In section 4, we analyse the existence of multiple positive solutions for (1.1) and (1.2). Finally, we give some examples to illustrate our results in section 5.

2. **Preliminaries**

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory [7].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbf{R}$ is given by*

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. *The Caputo's fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbf{R}$ is given by*

$$D^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds$$

where $n - 1 < \alpha \leq n$, provided that the right side is pointwise defined on $(0, \infty)$.

Remark 2.3. *If α is an integer, the derivative for order α is understood in the sense of usual differentiation.*

Lemma 2.4. *([8]) Let $n - 1 < \alpha \leq n$, $u \in C^n[0, 1]$. Then*

$$I^{\alpha}_{0^+} D^{\alpha}_{0^+} u(t) = u(t) - C_1 - C_2 t - .... - C_n t^{n-1}$$

where $C_i \in \mathbf{R}$, $i = 1, 2, ..., n$.

Lemma 2.5. *([8]) The relation*

$$I^{\alpha}_{0^+} I^{\beta}_{0^+} u(t) = I^{\alpha+\beta}_{0^+} u(t)$$

is valid in the following case

$$Re\beta > 0, \quad Re(\alpha + \beta) > 0, \quad u(t) \in L^1(a, b)$$

Definition 2.6. *Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called cone if*

(1) $x \in K, \lambda > 0$ *then* $\lambda x \in K$
(2) $x \in K$, $-x \in K$ *then* $x = 0$

We shall consider the Banach space $E = C[0, 1]$ equipped with standard norm

$$\|u\| = \max_{0 \leq t \leq 1} u(t)$$

The proof of existence of positive solution is based upon an applications of the following theorems.

Theorem 2.7. *([3][5]) Let E be a Banach space and let $K \subseteq E$ be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$ and let*

$$T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$$

be a completely continuous such that

(i) $\|Tu\| \leq \|u\|$ *if* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *if* $u \in K \cap \partial\Omega_2$
or
(ii) $\|Tu\| \geq \|u\|$ *if* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *if* $u \in K \cap \partial\Omega_2$
Then T has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Theorem 2.8. *([3]) Let E be a Banach space and K be a cone of E. For $r > 0$, define $K_r = \{u \in K : \|u\| \le r\}$ and assume that $T : K_r \to K$ is a completely continuous operator such that $Tu \neq u$ for $u \in \partial K_r$*

(1) *If $\|Tu\| \le \|u\|$ for all $u \in \partial K_r$ then $i(T, K_r, K) = 1$, where i is the fixed point index on K.*
(2) *If $\|Tu\| \ge \|u\|$ for all $u \in \partial K_r$ then $i(T, K_r, K) = 0$*

Consider the boundary value problem

$$D^{\alpha}_{0^+} u(t) + g(t) = 0, \quad 0 < t < 1 \tag{2.1}$$

$$u(0) = u''(0) = 0, \quad u'(1) = \gamma u'(\eta) \tag{2.2}$$

where $\eta, \gamma \in (0, 1)$.

Lemma 2.9. *Let $\gamma \neq 1$, $g \in L^1[0, 1]$. Then the boundary value problem (2.1) and (2.2) has a unique solution*

$$u(t) = \int_0^1 G_1(t, s)g(s)ds + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s)g(s)ds \tag{2.3}$$

where

$$G_1(t, s) = \begin{cases} \frac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \le s \le t \le 1 \\ \frac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)} & 0 \le t \le s \le 1 \end{cases}$$

$$G_2(\eta, s) = \begin{cases} \frac{(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(\eta-s)^{\alpha-2}}{\Gamma(\alpha)} & 0 \le s \le \eta \le 1 \\ \frac{(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)} & 0 \le \eta \le s \le 1 \end{cases}$$

Proof. From Lemmas 2.4 and 2.5

$$u(t) = -I^{\alpha}_{0^+} g(t) + C_1 + C_2 t + C_3 t^2$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, 3$.

$$\begin{aligned} D^1 u(t) &= -D^1 I^{\alpha}_{0^+} g(t) + C_2 + 2C_3 t \\ &= -D^1 I^1_{0^+} I^{\alpha-1}_{0^+} g(t) + C_2 + 2C_3 t \end{aligned}$$

thus,

$$u'(t) = -I^{\alpha-1}_{0^+} g(t) + C_2 + 2C_3 t \tag{2.4}$$

and

$$u''(t) = -I^{\alpha-2}_{0^+} g(t) + 2C_3$$

from (2.2), $C_1 = 0$ and $C_3 = 0$
From (2.4), one has

$$u'(1) = -I_{0^+}^{\alpha-1} g(1) + C_2 \tag{2.5}$$

and

$$\gamma u'(\eta) = -\gamma I_{0^+}^{\alpha-1} g(\eta) + \gamma C_2 \tag{2.6}$$

combining (2.5) and (2.6), we have

$$-I_{0^+}^{\alpha-1} g(1) + C_2 = -\gamma I_{0^+}^{\alpha-1} g(\eta) + \gamma C_2$$

therefore

$$\begin{aligned} C_2 &= \frac{1}{(1-\gamma)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} g(s)ds - \frac{\gamma}{(1-\gamma)\Gamma(\alpha-1)} \times \\ & \int_0^\eta (\eta - s)^{\alpha-2} g(s)ds \end{aligned}$$

so

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + \frac{t}{(1-\gamma)\Gamma(\alpha-1)} \times \\ & \int_0^1 (1-s)^{\alpha-2} g(s)ds - \frac{t\gamma}{(1-\gamma)\Gamma(\alpha-1)} \int_0^\eta (\eta-s)^{\alpha-2} g(s)ds \end{aligned} \tag{2.7}$$

splitting the second integral in two parts of the form

$$\frac{t}{\Gamma(\alpha-1)} + \frac{k}{(1-\gamma)\Gamma(\alpha-1)} = \frac{t}{(1-\gamma)\Gamma(\alpha-1)}$$

we have $k = \gamma t$.
replacing in (2.7)

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} g(s)ds \\ & + \frac{\gamma t}{(1-\gamma)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} g(s)ds - \frac{t\gamma}{(1-\gamma)\Gamma(\alpha-1)} \times \\ & \int_0^\eta (\eta-s)^{\alpha-2} g(s)ds \end{aligned}$$

Now, let

$$A = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} g(s)ds$$

and

$$B = \frac{\gamma t}{(1-\gamma)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}g(s)ds - \frac{t\gamma}{(1-\gamma)\Gamma(\alpha-1)} \times \int_0^{\eta} (\eta-s)^{\alpha-2}g(s)ds$$

by the above

$$A = \int_0^t \left[\frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right] g(s)ds + \int_t^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s)ds$$

and

$$B = \frac{\gamma t}{1-\gamma}\left[\int_0^{\eta}\left[\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right] g(s)ds + \int_{\eta}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s)ds\right]$$

Finally $\Gamma(p+1) = p!$, with $p > -1$ and $p \in \mathbf{R}$. This completes the proof. □

Lemma 2.10. *Let $\beta \in (0,1)$ be fixed. The kernel, $G_1(t,s)$, satisfies the following properties.*

(1) $0 \le G_1(t,s) \le G_1(1,s)$ *for all* $s \in (0,1)$.
(2) $\min_{\beta \le t \le 1} G_1(t,s) \ge \beta G_1(1,s)$ *for all* $s \in [0,1]$

Proof. (1) As $2 < \alpha \le 3$ and $0 \le s \le t \le 1$, we have

$$(\alpha-1)t(1-s)^{\alpha-2} > t(1-s)^{\alpha-2} \ge (t-s)(t-s)^{\alpha-2} = (t-s)^{\alpha-1}$$

thus, $G_1(t,s) > 0$. Note $\frac{\partial G_1(t,s)}{\partial t} \ge 0$ then $G_1(t,s)$ is increasing as a function of t, therefore

$$G_1(t,s) \le G_1(1,s) \quad \forall s \in [0,1]$$

(2) For $\beta \le t \le 1$, we have

$$\min_{\beta \le t \le 1} G_1(t,s) = G_1(\beta,s)$$

where

$$G_1(\beta,s) = \begin{cases} \frac{(\alpha-1)\beta(1-s)^{\alpha-2}-(\beta-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \le s \le \beta \\ \frac{(\alpha-1)\beta(1-s)^{\alpha-2}}{\Gamma(\alpha)} & \beta \le s \le 1 \end{cases}$$

(a) If $0 < s \leq \beta$

$$\min_{\beta \leq t \leq 1} G_1(t,s) = \frac{\beta(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)} - \frac{(\beta-s)^{\alpha-1}}{\Gamma(\alpha)} \tag{2.8}$$

On the other hand

$$\beta G_1(1,s) = \frac{\beta(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)} - \frac{\beta(1-s)^{\alpha-1}}{\Gamma(\alpha)} \tag{2.9}$$

Since $2 < \alpha \leq 3$ and

i) $\alpha - 1 > 1$, $\beta \in (0,1) \Rightarrow \beta^{\alpha-1} < \beta$

ii) $s \leq \beta \Rightarrow \frac{s}{\beta} \leq 1 \Rightarrow 1 - \frac{s}{\beta} \geq 0$

iii) $\beta < 1 \Rightarrow 1 < \frac{1}{\beta} \Rightarrow -s\frac{1}{\beta} < -s \Rightarrow 1 - s\frac{1}{\beta} < 1 - s$

thus, we have

$$(1 - \frac{s}{\beta})^{\alpha-1} < (1-s)^{\alpha-1}$$

from (2.8), we obtain

$$\begin{aligned} (\beta - s)^{\alpha-1} &= (\beta(1 - \frac{s}{\beta}))^{\alpha-1} \\ &= \beta^{\alpha-1}(1 - \frac{s}{\beta})^{\alpha-1} \\ &\leq \beta(1 - \frac{s}{\beta})^{\alpha-1} \\ &< \beta(1-s)^{\alpha-1} \end{aligned} \tag{2.10}$$

It follows from (2.8), (2.9) and (2.10), that (2) hold.

(b) If $\beta \leq s < 1$

$$\min_{\beta \leq t \leq 1} G_1(t,s) = \frac{\beta(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)} \tag{2.11}$$

$$\beta G_1(1,s) = \frac{\beta(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)} - \frac{\beta(1-s)^{\alpha-1}}{\Gamma(\alpha)} \tag{2.12}$$

It follows from (2.11) and (2.12) that (2) hold.

□

Lemma 2.11. *Let $g(t) \in C[0,1]$ and $g \geq 0$, then the unique solution of problem (2.1), (2.2) is nonnegative and satisfies*

$$\min_{\beta \leq t \leq 1} u(t) \geq \beta \|u\|$$

Proof. From the definition, $u(t)$ is nonnegative. From (2.3) and Lemma 2.10 we have

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)g(s)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)g(s)ds \\ &\leq \int_0^1 G_1(1,s)g(s)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)g(s)ds \end{aligned}$$

then

$$\|u\| \leq \int_0^1 G_1(1,s)g(s)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)g(s)ds$$

on the other hand,

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)g(s)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)g(s)ds \\ &\geq \int_0^1 \beta G_1(1,s)g(s)ds + \frac{\gamma\beta}{1-\gamma}\int_0^1 G_2(\eta,s)g(s)ds \\ &\geq \beta\left[\int_0^1 G_1(1,s)g(s)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)g(s)ds\right] \\ &\geq \beta\|u\| \end{aligned}$$

therefore

$$\min_{\beta\leq t\leq 1} u(t) \geq \beta\|u\|$$

□

Define the cone K by

$$K = \{u \in E : u(t) \geq 0 \ \ \textit{and} \ \ \min_{\beta\leq t\leq 1} u(t) \geq \beta\|u\|\}$$

and the map $T : K \to E$ by

$$Tu(t) = \int_0^1 G_1(t,s)a(s)f(u(s))ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u(s))ds$$

Remark 2.12. *By Lemma 2.9, the problem (1.1), (1.2) has a positive solution $u(t)$ if and only if $u(t)$ is a fixed point of T.*

Lemma 2.13. *T is completely continuous and $T(K) \subseteq K$.*

Proof. By Lemma 2.11, $T(K) \subseteq K$. In view of nonnegativeness and continuity of functions $G_i(x,y)$ with $i = 1,2$ and $a(t)f(u(t))$, we conclude that $T : K \to K$ is continuous.
Let $\Omega \subseteq K$ be bounded, that is, there exists $M > 0$ such that $\|u\| \leq M$

for all $u \in \Omega$.
Let

$$L = \max_{0 \le u \le M} |f(u)|$$

then using $u \in \Omega$, and by Lemmas 2.9, 2.10 and definition of $a(t)$, we have

$$\begin{aligned}
|Tu(t)| &= |\int_0^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds| \\
&\le \int_0^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\
&\le \int_0^1 G_1(1,s)a(s)f(u)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\
&\le \frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}a(s)f(u)ds + \frac{\gamma(\alpha-1)}{(1-\gamma)\Gamma(\alpha)} \times \\
&\quad \int_0^1 (1-s)^{\alpha-2}a(s)f(u)ds \\
&\le \left[\frac{(\alpha-1)L\|a\|_\infty}{\Gamma(\alpha)} + \frac{\gamma(\alpha-1)L\|a\|_\infty}{(1-\gamma)\Gamma(\alpha)}\right]\int_0^1 (1-s)^{\alpha-2}ds \\
&\le \left[\frac{(\alpha-1)L\|a\|_\infty}{\Gamma(\alpha)}\frac{1}{(1-\gamma)}\right]\int_0^1 (1-s)^{\alpha-2}ds \\
&= \left[\frac{(\alpha-1)L\|a\|_\infty}{\Gamma(\alpha)}\frac{1}{(1-\gamma)}\right]\frac{1}{\alpha-1} \\
&= \frac{L\|a\|_\infty}{(1-\gamma)\Gamma(\alpha)} \doteq l
\end{aligned}$$

Hence, $T(\Omega)$ is bounded.
On the other hand, let $u \in \Omega$, $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, then

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| \le\ & L\|a\|_\infty \Bigg[\int_0^1 [G_1(t_2,s) - G_1(t_1,s)]ds + \frac{|t_2-t_1|\gamma}{1-\gamma} \times \\
& \int_0^1 G_2(\eta,s)ds\Bigg]
\end{aligned}$$

The continuity of G_1 implies that the right-side of the above inequality tends to zero if $t_2 \to t_1$. Therefore, T is completely continuous by Arzela-Ascoli Theorem. □

We introduce the notation

$$f_a := \liminf_{u\to a} \tfrac{f(u)}{u} \qquad\qquad f^b := \limsup_{u\to b} \tfrac{f(u)}{u}$$

where $a, b = 0^+, \infty$
Let

$$N = \beta^2 m \left(\int_\beta^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma} \int_\beta^1 G_2(\eta,s)ds \right)$$

and

$$M = \|a\|_\infty \left(\int_0^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma} \int_0^1 G_2(\eta,s)ds \right)$$

In what follows, we will impose the following conditions
a. $f^0 = 0$ and $f_\infty = \infty$
b. $f_0 = \infty$ and $f^\infty = 0$
c. $f_0 = \infty$ and $f_\infty = \infty$
d. $f^0 = 0$ and $f^\infty = 0$
e. $0 \le f^0 < R$ and $r < f_\infty \le \infty$
f. $r < f_0 \le \infty$ and $0 \le f^\infty < R$
g. $\exists \rho > 0$ such that $f(u) < R\rho$, $0 < u \le \rho$
h. $\exists \rho > 0$ such that $f(u) > r\rho$, $\rho < u \le \frac{\rho}{\gamma}$

Remark 2.14. *We Note that **(a)** corresponds to the superlinear case and **(b)** corresponds to the sublinear case.*

Remark 2.15. *In condition **(e)** and **(f)**, $r = N^{-1}$ and $R = M^{-1}$. It is obvious that $r > R > 0$.*

3. **Existence of positive solutions**

Theorem 3.1. *Assume that $(H1 - H2)$ hold.*
*If **(a)**, **(b)**, **(e)** or **(f)** holds, then (1.1), (1.2) has at least one positive solution.*

Proof. (a)
(a1). Since $f^0 = 0$, $\exists H_1 > 0$ such that $f(u) \le \varepsilon u$ where $0 < u \le H_1$ and

$\varepsilon > 0$. Then for $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \{u \in X : \|u\| < H_1\}$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\ &\leq \int_0^1 G_1(1,s)a(s)f(u)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\ &\leq \int_0^1 G_1(1,s)\|a\|_\infty \varepsilon u ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\|a\|_\infty \varepsilon u ds \\ &\leq \varepsilon\left[\|a\|_\infty\left(\int_0^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)ds\right)\right]\|u\| \\ &= \varepsilon M\|u\| \end{aligned}$$

if $\varepsilon M \leq 1$ and taking the maximum in $0 \leq t \leq 1$, we have

$$\|Tu\| \leq \|u\|$$

(a2). since $f^\infty = \infty$, $\exists \bar{H}_2 > 0$ such that $f(u) \geq \delta u$ with $\bar{H}_2 \leq u$, $t \in [\beta, 1]$ and $\delta > 0$. For $u \in K \cap \partial\Omega_2$, where $\Omega_2 = \{u \in X : \|u\| < H_2\}$ with $H_2 = \max\{2H_1, \frac{\bar{H}_2}{\beta}\}$. Then $u \in K \cap \partial\Omega_2$ implies that $\min\limits_{\beta \leq t \leq 1} u(t) \geq \beta\|u\| = \beta H_2 > \bar{H}_2$. we have

$$\begin{aligned} Tu(t) &= \int_0^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\ &\geq \int_\beta^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma t}{1-\gamma}\int_\beta^1 G_2(\eta,s)a(s)f(u)ds \\ &\geq \int_\beta^1 \beta G_1(1,s)m\delta u ds + \frac{\gamma\beta}{1-\gamma}\int_\beta^1 G_2(\eta,s)m\delta u ds \\ &\geq \int_\beta^1 \beta^2 G_1(1,s)m\delta\|u\|ds + \frac{\gamma\beta^2}{1-\gamma}\int_\beta^1 G_2(\eta,s)m\delta\|u\|ds \\ &= \delta\left[\beta^2 m\left(\int_\beta^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma}\int_\beta^1 G_2(\eta,s)ds\right)\right]\|u\| \\ &= \delta N\|u\| \end{aligned}$$

if $\delta N \geq 1$ and taking the maximum of $Tu(t)$ with respect to t $0 \leq t \leq 1$, we have

$$\|Tu\| \geq \|u\|$$

Therefore, by Theorem 2.7, T has at least one fixed point, which is a positive solution of (1.1), (1.2). □

Proof. (b)
(b1). The proof is similar to that of (a2), so we omit it.
(b2). Since $f^\infty = 0$, $\exists \bar{H}_2 > 0$ such that $f(u) \leq \lambda u$ where $u \geq \bar{H}_2$ and $\lambda > 0$ satisfies that

$$\lambda M \leq 1$$

we consider two cases

(a) Suppose that f is bounded, $\exists L > 0$ such that $f(u) < L$ and $\Omega_2 = \{u \in X : \|u\| < H_2\}$ where $H_2 = \max\{2H_1, LM\}$. If $u \in K \cap \partial\Omega_1$, then by Lemma 2.10, we have

$$\begin{aligned} Tu(t) &\leq \int_0^1 G_1(1,s)a(s)f(u)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\ &\leq \int_0^1 G_1(1,s)\|a\|_\infty Luds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\|a\|_\infty Luds \\ &\leq L\left[\|a\|_\infty\left(\int_0^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)ds\right)\right] \\ &\leq H_2 = \|u\| \end{aligned}$$

therefore

$$\|Tu\| \leq \|u\|$$

(b) Suppose f is unbounded , by $(H1)$, $\exists H_2 > 0$ such that $H_2 > \max\{2H_1, \frac{\bar{H}_2}{\beta}\}$ and $f(u) \leq f(H_2)$ for $0 < u \leq H_2$ and let $\Omega_2 = \{u \in X : \|u\| < H_2\}$. If $u \in K \cap \partial\Omega_2$, then, by Lemma 2.10 we have

$$\begin{aligned} Tu(t) &\leq \int_0^1 G_1(1,s)a(s)f(u)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\ &\leq \int_0^1 G_1(1,s)\|a\|_\infty f(H_2)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\|a\|_\infty f(H_2)ds \\ &\leq \lambda\left[\|a\|_\infty\left(\int_0^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)ds\right)\right] H_2 \\ &\leq H_2 = \|u\| \end{aligned}$$

therefore

$$\|Tu\| \leq \|u\|$$

by Theorem 2.7, T has at least one fixed point, which is a positive solution of (1.1), (1.2). $\square$

Proof. (e)
(e1). Since $0 \le f^0 < R$, $\exists H_1 > 0$, $0 < \varepsilon_1 < R$ such that $f(u) \le (R-\varepsilon_1)u$, if $0 < u \le H_1$ and $t \in [0,1]$. Let $\Omega_1 = \{u \in X : \|u\| < H_1\}$. So for any $u \in K \cap \partial\Omega_1$, by Lemma 2.10, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\
&\le \int_0^1 G_1(1,s)a(s)f(u)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\
&\le \int_0^1 G_1(1,s)\|a\|_\infty(R-\varepsilon_1)uds + \frac{\gamma}{1-\gamma}\times \\
&\quad \int_0^1 G_2(\eta,s)\|a\|_\infty(R-\varepsilon_1)uds \\
&\le (R-\varepsilon_1)\left[\|a\|_\infty\left(\int_0^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)ds\right)\right]\|u\| \\
&= (R-\varepsilon_1)M\|u\| \\
&< \|u\|
\end{aligned}$$

Thus,

$$\|Tu\| < \|u\|$$

(e2). Since $r < f_\infty \le \infty$, $\exists \bar{H}_2 > 0$, $\varepsilon_2 > 0$ such that $f(u) \ge (r+\varepsilon_2)u$, for $u \ge \bar{H}_2$ and $\beta \le t \le 1$. Let $H_2 > \max\{2H_1, \frac{\bar{H}_2}{\beta}\}$ and $\Omega_2 = \{u \in X : \|u\| < H_2\}$. Then for $u \in K \cap \Omega_2$ implies $\min\limits_{\beta \le t \le 1} u(t) \ge \beta\|u\| = \beta H_2 > \bar{H}_2$. By Lemma 2.10 , we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\
&\ge \int_\beta^1 \beta G_1(1,s)a(s)f(u)ds + \frac{\gamma\beta}{1-\gamma}\int_\beta^1 G_2(\eta,s)a(s)f(u)ds \\
&\ge \int_\beta^1 \beta G_1(1,s)m(r+\varepsilon_2)uds + \frac{\gamma\beta}{1-\gamma}\int_\beta^1 G_2(\eta,s)m(r+\varepsilon_2)uds
\end{aligned}$$

$$\begin{aligned}
&\geq \int_\beta^1 \beta G_1(1,s)m(r+\varepsilon_2)\beta\|u\|ds + \frac{\gamma\beta}{1-\gamma} \times \\
&\quad \int_\beta^1 G_2(\eta,s)m(r+\varepsilon_2)\beta\|u\|ds \\
&\geq (r+\varepsilon_2)\left[\beta^2 m\left(\int_\beta^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma}\int_\beta^1 G_2(\eta,s)ds\right)\right]\|u\| \\
&= (r+\varepsilon_2)N\|u\| \\
&> \|u\|
\end{aligned}$$

Thus,

$$\|Tu\| > \|u\|$$

by Theorem 2.7, T has at least one fixed point, which is a positive solution of (1.1), (1.2). □

Proof. (f)
The proof is similar to that of (e), so we skip it. □

4. **Multiplicity results**

Theorem 4.1. *Assume that* $(H1-H2)$, ***(c)*** *and* ***(g)*** *hold, then* (1.1), (1.2) *has at least two positive solutions.*

Proof. Since $f_0 = \infty$, $\exists H_1 > 0$, $0 < H_1 < \rho$ such that $f(u) > ru$ with $0 < u \leq H_1$ and $t \in [\beta, 1]$. For $u \in K \cap \partial\Omega_1$ where $\Omega_1 = \{u \in X : \|u\| < H_1\}$ by Lemma 2.10 , we have that

$$\begin{aligned}
Tu(t) &= \int_0^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\
&> \int_\beta^1 \beta G_1(t,s)mruds + \frac{\gamma\beta}{1-\gamma}\int_\beta^1 G_2(\eta,s)mruds \\
&\geq \int_\beta^1 \beta^2 G_1(1,s)mr\|u\|ds + \frac{\gamma\beta^2}{1-\gamma}\int_\beta^1 G_2(\eta,s)mr\|u\|ds \\
&= r\left[\beta^2 m\left(\int_\beta^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma}\int_\beta^1 G_2(\eta,s)ds\right)\right]\|u\| \\
&= rN\|u\| = \|u\|
\end{aligned}$$

then

$$\|Tu\| > \|u\|$$

By Theorem 2.8
$$i(T, K_{H_1}, K) = 0$$

Since $f_\infty = \infty$, $\exists \bar{H}_2 > \rho$ such that $f(u) > ru$, $u \geq \bar{H}_2 > 0$ and $t \in [\beta, 1]$. Let $H_2 = \frac{\bar{H}_2}{\beta}$ and $\Omega_2 = \{u \in X : \|u\| < H_2\}$. For $u \in K \cap \partial\Omega_2$, $\min\limits_{\beta \leq t \leq 1} u(t) \geq \beta\|u\| = \beta H_2 = \bar{H}_2$. By Lemma 2.10, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\
&> \int_\beta^1 \beta G_1(t,s)mruds + \frac{\gamma\beta}{1-\gamma}\int_\beta^1 G_2(\eta,s)mruds \\
&\geq \int_\beta^1 \beta^2 G_1(1,s)mr\|u\|ds + \frac{\gamma\beta^2}{1-\gamma}\int_\beta^1 G_2(\eta,s)mr\|u\|ds \\
&= r\left[\beta^2 m\left(\int_\beta^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma}\int_\beta^1 G_2(\eta,s)ds\right)\right]\|u\| \\
&= rN\|u\| = \|u\|
\end{aligned}$$

so
$$\|Tu\| > \|u\|$$

By Theorem 2.8
$$i(T, K_{H_2}, K) = 0$$

Now, let $\Omega_3 = \{u \in X : \|u\| < \rho\}$, thus, for $u \in K \cap \partial\Omega_3$, we get from **(g)** that $f(u) < R\rho$ for $t \in [0,1]$, then

$$\begin{aligned}
Tu(t) &= \int_0^1 G_1(t,s)a(s)f(u)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\
&\leq \int_0^1 G_1(1,s)a(s)f(u)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)a(s)f(u)ds \\
&< \int_0^1 G_1(1,s)\|a\|_\infty R\rho ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\|a\|_\infty R\rho ds \\
&\leq R\left[\|a\|_\infty\left(\int_0^1 G_1(1,s)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)ds\right)\right]\rho \\
&= RM\rho = \|u\|
\end{aligned}$$

then
$$\|Tu\| < \|u\|$$

By Theorem 2.8
$$i(T, K_\rho, K) = 1$$

Therefore

$$i(T, K_{H_2} \setminus \overline{K_\rho}, K) = i(T, K_{H_2}, K) - i(T, K_\rho, K) = 0 - 1 = -1$$

$$i(T, K_\rho \setminus \overline{K_{H_1}}, K) = i(T, K_\rho, K) - i(T, K_{H_1}, K) = 1 - 0 = 1$$

Then there exist at least two positive solutions $u_1 \in K \cap (\bar{\Omega_3} \backslash \Omega_1)$ and $u_2 \in K \cap (\bar{\Omega_2} \backslash \Omega_3)$ of (1.1),(1.2) in K, such that

$$0 < \|u_1\| < \rho < \|u_2\|$$

□

Theorem 4.2. *Assume that* $(H1 - H2)$*,* ***(d)*** *and* ***(h)*** *hold, then* (1.1)*,* (1.2) *has at least two positive solutions.*

Proof. The proof of Theorem 4.2 is similar to that of Theorem 4.1, so we skip it. □

5. Examples

Example 1. *Superlinear and Sublinear Case*

a) *If* $f(u) = u^\alpha$*,* $\alpha > 1$*, the conclusions of Theorem 3.1(a), hold.*
b) *If* $f(u) = 1 + u^\alpha$*,* $\alpha \in (0, 1)$ *the conclusions of Theorem 3.1(b), hold.*

Example 2. *Let* $f(u) = \lambda \ln(1 + u) + u^2$*, fix* $\lambda > 0$*, sufficiently small. Clearly* $f^0 = \lambda$ *and* $f_\infty = \infty$*. By Theorem 3.1(e),* (1.1) *and 1.2 have at least one positive solution.*

Example 3. *Let* $f(u) = u^2e^{-u} + \mu \sin u$*, fix* $\mu > 0$ *sufficiently large. Then* $f_0 = \mu$ *and* $f^\infty = 0$*. By Theorem 3.1(f),* (1.1) *and* (1.2) *have at least one positive solution.*

Example 4. *Let* $f(u) = u^b + u^c - 1$*,* $a(t) = 1$*,* $\alpha = \frac{5}{2}$*,* $b \in (0, 1)$*,* $c > 1$*,* $\gamma = \frac{1}{4}$ *and* $\eta = \frac{1}{2}$ *then* $f_0 = \infty$ *and* $f_\infty = \infty$*. By a simple calculation,* $M = \int_0^1 G_1(1, s)ds + \frac{1}{3}\int_0^1 G_2(\frac{1}{2}, s)ds = \frac{4}{3\sqrt{\pi}}\left[\frac{14}{15} - \frac{\sqrt{2}}{12}\right]$ *then* $R \approx 1.633012$*.*
On the other hand, we could choose $\rho = 1$*, then* $f(u) \leq 1 < R\rho$ *for* $u \in [0, 1]$*. By Theorem 4.1,* (1.1) *and* (1.2) *have at least two positive solutions* u_1 *,* u_2 *and* $0 < \|u_1\| < 1 < \|u_2\|$*.*

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