HYERS-ULAM-RASSIAS STABILITY OF A COMPOSITE FUNCTIONAL EQUATION IN VARIOUS NORMED SPACES

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Abstract. In this paper, we prove the generalized Hyers-Ulam (or Hyers-Ulam-Rassias) stability of the following composite functional equation

\[ f(f(x) - f(y)) + f(x) + f(y) = f(x + y) + f(x - y), \]

in various normed spaces.

1. Introduction and preliminaries

Let \( \Gamma^+ \) denote the set of all probability distribution functions \( F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1] \) such that \( F \) is left-continuous and nondecreasing on \( \mathbb{R} \) and \( F(0) = 0, F(+\infty) = 1 \). It is clear that the set \( \mathcal{D}^+ = \{ F \in \Gamma^+ : l^-F(-\infty) = 1 \} \), where \( l^-f(x) = \lim_{t \to x^-} f(t) \), is a subset of \( \Gamma^+ \). The set \( \Gamma^+ \) is partially ordered by the usual point-wise ordering of functions, that is, \( F \leq G \) if and only if \( F(t) \leq G(t) \) for all \( t \in \mathbb{R} \). For any \( a \geq 0 \), the element \( H_a(t) \) of \( \mathcal{D}^+ \) is defined by

\[
H_a(t) = \begin{cases} 
0, & \text{if } t \leq a, \\
1, & \text{if } t > a.
\end{cases}
\]

Definition 1.1. A function \( T : [0, 1]^2 \to [0, 1] \) is a continuous triangular norm (briefly, a \( t \)-norm) if \( T \) satisfies the following conditions:

(a) \( T \) is commutative and associative;


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(b) $T$ is continuous;
(c) $T(x, 1) = x$ for all $x \in [0, 1]$;
(d) $T(x, y) \leq T(z, w)$ whenever $x \leq z$ and $y \leq w$ for all $x, y, z, w \in [0, 1]$.

**Definition 1.2.** A random normed space (briefly, RN-space) is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:
(a) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;
(b) $\mu_{\alpha x}(t) = \mu_x \left( \frac{t}{|\alpha|} \right)$ for all $\alpha \neq 0$, $x \in X$ and $t \geq 0$;
(c) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

**Definition 1.3.** By a non-Archimedean field we mean a field $K$ equipped with a function (valuation) $|\cdot| : K \to [0, \infty)$ such that for all $r, s \in K$, the following conditions hold:
(i) $|r| = 0$ if and only if $r = 0$;
(ii) $|rs| = |r||s|$;
(iii) $|r + s| \leq \max\{|r|, |s|\}$.

**Remark 1.4.** Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

**Definition 1.5.** Let $X$ be a vector space over a scalar field $K$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot|| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:
(i) $||x|| = 0$ if and only if $x = 0$;
(ii) $||rx|| = |r||x||$ ($r \in K, x \in X$);
(iii) The strong triangle inequality (ultrametric); namely $||x + y|| \leq \max\{||x||, ||y||\}$, $x, y \in X$.

Then $(X, ||\cdot||)$ is called a non-Archimedean space.

**Definition 1.6.** A sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are $p$-adic numbers. A key property of $p$-adic numbers is that they do not satisfy the Archimedean axiom: “for $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$”.

**Example 1.7.** Fix a prime number $p$. For any nonzero rational number $x$, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b} p^{n_x}$, where $a$ and $b$ are integers not divisible by $p$. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric
$d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the $p$-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x} a_k p^k$ where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq n_x} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field.

Arriola and Beyer [1] investigated the Hyers-Ulam stability of approximate additive functions $f : \mathbb{Q}_p \rightarrow \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \rightarrow \mathbb{R}$ is a continuous function for which there exists a fixed $\epsilon$: 

$$|f(x + y) - f(x) - f(y)| \leq \epsilon$$

for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive function $T : \mathbb{Q}_p \rightarrow \mathbb{R}$ such that 

$$|f(x) - T(x)| \leq \epsilon$$

for all $x \in \mathbb{Q}_p$.

However, the following example shows that similar result is not true in non-Archimedean normed spaces.

**Example 1.8.** Let $p > 2$ and let $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ be defined by $f(x) = 2$. Then for $\epsilon = 1$,

$$|f(x + y) - f(x) - f(y)| = 1 \leq \epsilon$$

for all $x, y \in \mathbb{Q}_p$. However, the sequences $\left\{ \frac{f(2^n x)}{2^n} \right\}_{n=1}^{\infty}$ and $\left\{ 2^n f \left( \frac{x}{2^n} \right) \right\}_{n=1}^{\infty}$ are not Cauchy. In fact, by using the fact that $|2| = 1$, we have

$$\left| \frac{f(2^n x)}{2^n} - \frac{f(2^{n+1} x)}{2^{n+1}} \right| = |2^{-n} \cdot 2 - 2^{-(n+1)} \cdot 2| = |2^{-n}| = 1$$

and

$$\left| 2^n f \left( \frac{x}{2^n} \right) - 2^{n+1} f \left( \frac{x}{2^{n+1}} \right) \right| = |2^n \cdot 2^{-(n+1)} \cdot 2| = |2^{n+1}| = 1$$

for all $x, y \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence these sequences are not convergent in $\mathbb{Q}_p$.

**Definition 1.9.** Let $X$ be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. 


Theorem 1.10. Let \((X,d)\) be a complete generalized metric space and let \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then, for all \(x \in X\), either

\[(1.1)\quad d(J^n x, J^{n+1} x) = \infty\]

for all nonnegative integers \(n\), or there exists a positive integer \(n_0\) such that

(a) \(d(J^n x, J^{n+1} x) < \infty\) for all \(n_0 \geq n_0\);
(b) the sequence \(\{J^n x\}\) converges to a fixed point \(y^*\) of \(J\);
(c) \(y^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X : d(J^{n_0} x, y) < \infty\}\);
(d) \(d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)\) for all \(y \in Y\).

A classical question in the theory of functional equations is the following: “When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?” If the problem admits a solution, we say that the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [31] in 1940. In the following year, Hyers [10] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [19] proved a generalization of Hyers’ theorem for additive mappings. The result of Rassias has provided a significant influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Furthermore, in 1994, a generalization of Rassias’s theorem was obtained by Gavruta [8] by replacing the bound \(\epsilon(\|x\|^p + \|y\|^p)\) by a general control function \(\varphi(x, y)\). In 1897, Hensel [9] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [11,12].

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([2]-[8], [14]-[29]).

In Sections 2 and 3, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [30].
In this paper, we prove the Hyers-Ulam-Rassias stability of the functional equation
\[(1.2) \quad f(f(x) - f(y)) + f(x) + f(y) = f(x + y) + f(x - y)\]
in random and non-Archimedean normed spaces.

2. Random stability of the functional equation (1.2): a direct method

In this section, using a direct method, we prove the Hyers-Ulam-Rassias stability of the functional equation (1.2) in random normed spaces.

**Theorem 2.1.** Let $X$ be a real linear space, $(Z, \mu', \min)$ an RN-space and $\varphi : X^2 \to Z$ a function such that there exists $0 < \alpha < \frac{1}{2}$ with
\[(2.1) \quad \mu'_{\varphi(x, x)}(t) \geq \mu'_{\alpha \varphi(x, y)}(t)\]
for all $x \in X$ and $t > 0$ and
\[
\lim_{n \to \infty} \mu'_{\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\left(\frac{t}{2^n}\right) = 1
\]
for all $x, y \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ is a mapping such that
\[(2.2) \quad \mu_f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y) \geq \mu'_{\varphi(x, y)}(t)\]
for all $x, y \in X$ and $t > 0$. Then the limit
\[A(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)\]
exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that and
\[(2.3) \quad \mu_{f(x) - A(x)}(t) \geq \mu'_{\varphi(x, x)}\left(\frac{(1 - 2\alpha)t}{\alpha}\right)\]
for all $x \in X$ and $t > 0$.

**Proof.** Putting $y = x$ in (2.2), we see that
\[(2.4) \quad \mu_f(2x) - 2f(x) \geq \mu'_{\varphi(x, x)}(t)\]
Replacing $x$ by $\frac{x}{2}$ in (2.4), we obtain
\[(2.5) \quad \mu_{2f\left(\frac{x}{2}\right) - f(x)}(t) \geq \mu'_{\varphi\left(\frac{x}{2}, \frac{x}{2}\right)}(t)\]
for all $x \in X$. Replacing $x$ by $\frac{x}{2^n}$ in (2.5) and using (2.1), we obtain

$$\mu_{2^{n+1}}f\left(\frac{x}{2^{n+1}}\right)-2^n f\left(\frac{x}{2^n}\right)(t) \geq \mu'_{\varphi}\left(\frac{x}{2^{n+1}}, \frac{x}{2^n}\right) \geq \frac{t}{2^n} \mu'_{\varphi(x,x)}$$

and so

$$\mu_{2^n}f\left(\frac{x}{2^n}\right)-f(x)\left(\sum_{k=0}^{n-1} 2^k \alpha^{k+1} t\right) = \mu'_{\varphi} \left(\sum_{k=0}^{n-1} 2^k \alpha^{k+1} t\right).$$

This implies that

$$(2.6) \quad \mu_{2^n}f\left(\frac{x}{2^n}\right)-f(x)(t) \geq \mu'_{\varphi(x,x)} \left(\frac{t}{\sum_{k=0}^{n-1} 2^k \alpha^{k+1}}\right).$$

Replacing $x$ by $\frac{x}{2^n}$ in (2.6), we obtain

$$(2.7) \quad \mu_{2^{n+p}}f\left(\frac{x}{2^{n+p}}\right)-2^p f\left(\frac{x}{2^p}\right)(t) \geq \mu'_{\varphi(x,x)} \left(\frac{t}{\sum_{k=0}^{n+p-1} 2^k \alpha^{k+1}}\right) \to 1 \text{ when } n \to +\infty,$$

so $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete RN-space $(Y, \mu, \min)$ and so there exists a point $A(x) \in Y$ such that

$$\lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) = A(x).$$

Fix $x \in X$ and put $p = 0$ in (2.7). Then we obtain

$$\mu_{2^n}f\left(\frac{x}{2^n}\right)-f(x)(t) \geq \mu'_{\varphi(x,x)} \left(\frac{t}{\sum_{k=0}^{n-1} 2^k \alpha^{k+1}}\right)$$

and so, for any $\delta > 0$,

$$\mu_{A(x)-f(x)}(t+\delta) \geq T \left(\mu_{A(x)-2^n f\left(\frac{x}{2^n}\right)}(\delta), \mu_{2^n f\left(\frac{x}{2^n}\right)-f(x)}(t)\right) \geq T \left(\mu_{A(x)-2^n f\left(\frac{x}{2^n}\right)}(\delta), \mu'_{\varphi(x,x)} \left(\frac{t}{\sum_{k=0}^{n-1} 2^k \alpha^{k+1}}\right)\right).$$

$$(2.8)$$
Taking \( n \to \infty \) in (2.8), we get
\[
\mu_{A(x)-f(x)}(t + \delta) \geq \mu'_{\varphi(x,x)} \left( \frac{(1 - 2\alpha)t}{\alpha} \right).
\]
Since \( \delta \) is arbitrary, by taking \( \delta \to 0 \) in (2.9), we get
\[
\mu_{A(x)-f(x)}(t) \geq \mu'_{\varphi(x,x)} \left( \frac{(1 - 2\alpha)t}{\alpha} \right).
\]
Replacing \( x \) and \( y \) by \( \frac{2}{2n} \) and \( \frac{2r}{2n} \) in (2.2), respectively, we get
\[
\mu_{2^n[f\left(\frac{2}{2n}\right) - f\left(\frac{2r}{2n}\right) - f\left(\frac{2r}{2n}\right) + f\left(\frac{2}{2n}\right) + f\left(\frac{2}{2n}\right) + f\left(\frac{2r}{2n}\right)]}(t) \geq \mu'_{\varphi\left(\frac{2}{2n}, \frac{2r}{2n}\right)} \left( \frac{t}{2^n} \right)
\]
for all \( x, y \in X \) and \( t > 0 \). Since \( \lim_{n \to \infty} \mu'_{\varphi\left(\frac{2}{2n}, \frac{2r}{2n}\right)} \left( \frac{t}{2^n} \right) = 1 \), we conclude that \( A \) satisfies (1.2). On the other hand,
\[
2A\left(\frac{X}{2}\right) - A(x) = \lim_{n \to \infty} 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) = 0.
\]
This implies that \( A : X \to Y \) is an additive mapping. To prove the uniqueness of the additive mapping \( A \), assume that there exists another additive mapping \( L : X \to Y \) which satisfies (2.3). Then we have
\[
\mu_{A(x)-L(x)}(t) = \lim_{n \to \infty} \mu_{2^n A\left(\frac{2}{2n}\right) - 2^n L\left(\frac{2}{2n}\right)}(t)
\]
\[
\geq \lim_{n \to \infty} \min \left\{ \mu_{2^n A\left(\frac{2}{2n}\right) - 2^n f\left(\frac{2}{2n}\right)} \left( \frac{t}{2^n} \right), \mu_{2^n L\left(\frac{2}{2n}\right)} \left( \frac{t}{2^n} \right) \right\}
\]
\[
\geq \lim_{n \to \infty} \mu'_{\varphi\left(\frac{2}{2n}, \frac{2r}{2n}\right)} \left( \frac{(1 - 2\alpha)t}{2^{n+1} \alpha} \right) = \lim_{n \to \infty} \mu'_{\varphi(x,x)} \left( \frac{(1 - 2\alpha)t}{2^{n+1} \alpha n + 1} \right).
\]
Since \( \lim_{n \to \infty} \frac{(1 - 2\alpha)t}{2^{n+1} \alpha n + 1} = \infty \), we get \( \lim_{n \to \infty} \mu'_{\varphi(x,x)} \left( \frac{(1 - 2\alpha)t}{2^{n+1} \alpha n + 1} \right) = 1 \).
Therefore, it follows that \( \mu_{A(x)-L(x)}(t) = 1 \) for all \( t > 0 \) and so \( A(x) = L(x) \). This completes the proof.

**Corollary 2.2.** Let \( X \) be a real normed linear space, \((Z, \mu', \min)\) an RN-space and \((Y, \mu, \min)\) a complete RN-space. Let \( r \) be a positive real number with \( r > 1 \), \( z_0 \in Z \) and \( f : X \to Y \) a mapping satisfying
\[
\mu f(f(x) - f(y)) - f(x+y) - f(x-y) + f(x) + f(y)(t) \geq \mu'_{\|x\|', \|y\|'}(t)
\]
for all \( x, y \in X \) and \( t > 0 \). Then the limit \( A(x) = \lim_{n \to \infty} 2^n f\left(\frac{2}{2n}\right) \) exists for all \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that
\[
\mu f(x) - A(x)(t) \geq \mu'_{\|x\|'} \left( \frac{(2^r - 2)t}{2} \right).
\]
for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 2^{-r}$ and let $\varphi : X^2 \to Z$ be a mapping defined by $\varphi(x, y) = (\|x\|^r + \|y\|^r)z_0$. Then, from Theorem 2.1, the conclusion follows. \hfill \Box

**Theorem 2.3.** Let $X$ be a real linear space, $(Z, \mu', \min)$ an RN-space and $\varphi : X^2 \to Z$ a function such that there exists $0 < \alpha < 2$ such that $\mu'_{\varphi(x, 2y)}(t) \geq \mu'_{\varphi(x,y)}(t)$ for all $x \in X$ and $t > 0$ and $\lim_{n \to \infty} \mu'_{\varphi(2^n x, 2^n y)}(2^n t) = 1$ for all $x, y \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ is a mapping satisfying (2.2). Then the limit $A(x) = \lim_{n \to \infty} f(2^n x)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that and

\begin{equation}
(2.11) \quad \mu_{f(x) - A(x)}(t) \geq \mu'_{\varphi(x,x)}((2 - \alpha)t)
\end{equation}

for all $x \in X$ and $t > 0$.

Proof. Putting $y = x$ in (2.2), we see that

\begin{equation}
(2.12) \quad \mu_{f(2x) - f(x)}(t) \geq \mu'_{\varphi(x,x)}(2t).
\end{equation}

Replacing $x$ by $2^n x$ in (2.12), we obtain that

\begin{equation}
(2.13) \quad \mu_{f(2^{n+1} x) - f(2^n x)}(2^{-n} t) \geq \mu'_{\varphi(2^n x, 2^n x)}(2^{n+1} t) \geq \mu_{\varphi(x,x)} \left(\frac{2^{n+1} t}{\alpha^n}\right).
\end{equation}

The rest of the proof is similar to the proof of Theorem 2.1. \hfill \Box

**Corollary 2.4.** Let $X$ be a real normed linear space, $(Z, \mu', \min)$ an RN-space and $(Y, \mu, \min)$ a complete RN-space. Let $r$ be a positive real number with $0 < r < 1$, $z_0 \in Z$ and $f : X \to Y$ a mapping satisfying (2.10). Then the limit $A(x) = \lim_{n \to \infty} f(2^n x)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

\begin{equation}
\mu_{f(x) - A(x)}(t) \geq \mu'_{\|x\|^r z_0} \left(\frac{(2 - 2^r)t}{2}\right)
\end{equation}

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 2^r$ and let $\varphi : X^2 \to Z$ be a mapping defined by $\varphi(x, y) = (\|x\|^r + \|y\|^r)z_0$. Then, from Theorem 2.3, the conclusion follows. \hfill \Box
3. Random stability of the functional equation (1.2): a fixed point method

Throughout this section, using a fixed point method, we prove Hyers-Ulam-Rassias stability of functional equation (1.2) in RN-spaces.

**Theorem 3.1.** Let $X$ be a linear space, $(Y, \mu, T_M)$ a complete RN-space and $\Phi$ a mapping from $X^2$ to $D^+$ such that there exists $0 < \alpha < \frac{1}{2}$ such that
\begin{equation}
\Phi_{2x,2y}(t) \leq \Phi_{x,y}(\alpha t)
\end{equation}
for all $x, y \in X$ and $t > 0$ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$). Let $f : X \to Y$ be a mapping satisfying
\begin{equation}
\mu f(x) - f(y) - f(x+y) + f(x) + f(y) + f(x-y) \geq \Phi_{x,y}(t)
\end{equation}
for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$
\begin{equation}
A(x) := \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)
\end{equation}
exists and $A : X \to Y$ is a unique additive mapping such that
\begin{equation}
\mu f(x) - A(x) \geq \Phi_{x,x}\left(\frac{(1-2\alpha)t}{\alpha}\right)
\end{equation}
for all $x \in X$ and $t > 0$.

**Proof.** Putting $y = x$ in (3.2) and replacing $x$ by $\frac{x}{2}$, we have
\begin{equation}
\mu_{2f\left(\frac{x}{2}\right)} - f(x) \geq \Phi_{x,x}\left(\frac{t}{2}\right)
\end{equation}
for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y\}$ and the generalized metric $d$ in $S$ defined by
\begin{equation}
d(f, g) = \inf_{u \in (0, \infty)} \{\mu_{g(x) - h(x)}(ut) \geq \Phi_{x,x}(t), \forall x \in X, t > 0\},
\end{equation}
where $\inf \emptyset = +\infty$. It is easy to show that $(S, d)$ is complete (see [14], Lemma 2.1). Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that
\begin{equation}
Jh(x) := 2h\left(\frac{x}{2}\right)
\end{equation}
for all $x \in X$.

First, we prove that $J$ is a strictly contractive mapping with the Lipschitz constant $2\alpha$. In fact, let $g, h \in S$ be such that $d(g, h) < \epsilon$. Then we have
\begin{equation}
\mu_{g(x) - h(x)}(\epsilon t) \geq \Phi_{x,x}(t)
\end{equation}
for all \( x \in X \) and \( t > 0 \) and so
\[
\mu_{Jg(x)-Jh(x)}(2\alpha et) = \mu_{2g(\frac{x}{2})-2h(\frac{x}{2})}(2\alpha et) = \mu_{g(\frac{x}{2})-h(\frac{x}{2})} (\alpha et) \\
\geq \Phi_{\frac{x}{2},\frac{x}{2}}(\alpha t) \\
\geq \Phi_{x,x}(t)
\]
for all \( x \in X \) and \( t > 0 \). Thus \( d(g,h) < \epsilon \) implies that \( d(Jg,Jh) < 2\alpha \epsilon \).

This means that \( d(Jg,Jh) \leq 2\alpha d(g,h) \) for all \( g,h \in S \). It follows from (3.4) that
\[
d(f,Jf) \leq \alpha.
\]

By Theorem 1.10, there exists a mapping \( A : X \to Y \) satisfying the following:
(1) \( A \) is a fixed point of \( J \), that is,
\[
A\left(\frac{x}{2}\right) = \frac{1}{2} A(x)
\]
for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( J \) in the set \( \Omega = \{ h \in S : d(g,h) < \infty \} \). This implies that \( A \) is a unique mapping satisfying (3.7) such that there exists \( u \in (0,\infty) \) satisfying
\[
\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,x}(t) \text{ for all } x \in X \text{ and } t > 0.
\]
(2) \( d(J^n f, A) \to 0 \) as \( n \to \infty \). This implies the equality
\[
\lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)
\]
for all \( x \in X \).
(3) \( d(f,A) \leq \frac{d(f,Jf)}{1-2\alpha} \) with \( f \in \Omega \), which implies the inequality
\[
d(f,A) \leq \frac{\alpha}{1-2\alpha}
\]
and so
\[
\mu_{f(x)-A(x)} \left( \frac{\alpha t}{1-2\alpha} \right) \geq \Phi_{x,x}(t)
\]
for all \( x \in X \) and \( t > 0 \). This implies that the inequality (3.3) holds.

On the other hand, replacing \( x,y \) by \( \frac{x}{2^n} \) and \( \frac{y}{2^n} \), respectively, in (3.2), we have
\[
\mu_{2^n[f(f(\frac{x}{2^n}))-f(\frac{y}{2^n})]-f(\frac{x+y}{2^n})+f(\frac{y}{2^n})+f(\frac{y}{2^n})}(t) \geq \Phi_{\frac{x}{2^n},\frac{y}{2^n}} \left( \frac{t}{2^n} \right)
\]
for all \( x,y \in X, t > 0 \) and \( n \geq 1 \) and so, from (3.1), it follows that
\[
\Phi_{\frac{x}{2^n},\frac{y}{2^n}} \left( \frac{t}{2^n} \right) \geq \Phi_{x,y} \left( \frac{t}{2^n\alpha^n} \right) \to 1 \text{ as } n \to +\infty
\]
for all \( x, y \in X \) and \( t > 0 \). Therefore
\[
\mu_{A(A(x) - A(y)) - A(x + y) - A(x - y) + A(x) + A(y)}(t) = 1
\]
for all \( x, y \in X \) and \( t > 0 \). Thus the mapping \( A : X \to Y \) satisfies (1.2). Furthermore, since for all \( x, y \in X \), we have
\[
A(2x) - 2A(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) - 2 \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)
\]
\[
= 2 \left[ \lim_{n \to \infty} 2^{n-1} f\left(\frac{x}{2^n}\right) - \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) \right]
\]
\[
= 0,
\]
we conclude that \( A : X \to Y \) is additive. This completes the proof. □

**Corollary 3.2.** Let \( X \) be a real normed space, \( \theta \geq 0 \) and let \( r \) be a real number with \( r > 1 \). Let \( f : X \to Y \) be a mapping satisfying
\[
(3.8) \quad \mu_f(f(x) - f(y) - f(x + y) - f(x - y) + f(x) + f(y))(t) \geq \frac{t}{t + \theta(\|x\|^r + \|y\|^r)}
\]
for all \( x, y \in X \) and \( t > 0 \). Then \( A(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) \) exists for all \( x \in X \) and \( A : X \to Y \) is a unique additive mapping such that
\[
\mu_{f(x) - A(x)}(t) \geq \frac{(2^r - 2)t}{(2^r - 2)t + 2\theta\|x\|^r}
\]
for all \( x \in X \) and \( t > 0 \).

**Proof.** The proof follows from Theorem 3.1 if we take
\[
\Phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^r + \|y\|^r)}
\]
for all \( x, y \in X \) and \( t > 0 \). In fact, if we choose \( \alpha = 2^{-r} \), then we get the desired result. □

**Theorem 3.3.** Let \( X \) be a linear space, \((Y, \mu, T_M)\) a complete RN-space and \( \Phi \) a mapping from \( X^2 \) to \( D^+ \) such that for some \( 0 < \alpha < 2 \), \( \Phi \left( \frac{t}{2^n} \right) \leq \Phi_{x,y}(\alpha t) \) for all \( x, y \in X \) and \( t > 0 \). Let \( f : X \to Y \) be a mapping satisfying (3.2). Then the limit \( A(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \) exists for all \( x \in X \) and \( A : X \to Y \) is a unique additive mapping such that
\[
(3.9) \quad \mu_f(f(x) - A(x))(t) \geq \Phi_{x,x}(2 - \alpha)t
\]
for all \( x \in X \) and \( t > 0 \).
Proof. Putting $y = x$ in (3.2), we have

\begin{equation}
\mu_{f(2x) - f(x)}(t) \geq \Phi_{x,x}(2t)
\end{equation}

for all $x \in X$ and $t > 0$. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that $Jh(x) := \frac{1}{2}h(2x)$ for all $x \in X$. It follows from (3.10) that

$$d(f, Jf) \leq \frac{1}{2}.$$ 

By Theorem 1.10, there exists a mapping $A : X \to Y$ satisfying the following:

1. $A$ is a fixed point of $J$, that is,

\begin{equation}
A(2x) = 2A(x)
\end{equation}

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that $A$ is a unique mapping satisfying (3.11) such that there exists $u \in (0, \infty)$ satisfying $\mu_{f(x) - A(x)}(ut) \geq \Phi_{x,x}(t)$ for all $x \in X$ and $t > 0$.

2. $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} \frac{f(2^n x)}{2^n} = A(x)$$

for all $x \in X$.

3. $d(f, A) \leq \frac{d(f, Jf)}{1 - \frac{r}{2}}$ with $f \in \Omega$, which implies the inequality

$$\mu_{f(x) - A(x)} \left( \frac{t}{2 - \alpha} \right) \geq \Phi_{x,x}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.9) holds. The rest of the proof is similar to the proof of Theorem 3.1. \qed

**Corollary 3.4.** Let $X$ be a real normed space, $\theta \geq 0$ and let $r$ be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping satisfying (3.8). Then the limit $A(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x) - A(x)}(t) \geq \frac{(2 - 2^r)t}{(2 - 2^r)t + 2\theta \|x\|^r}$$

for all $x \in X$ and $t > 0$. 

Proof. The proof follows from Theorem 3.3 if we take
\[ \Phi_{x,y}(t) = \frac{t}{t + \theta(||x||^r + ||y||^r)} \]
for all \( x, y \in X \) and \( t > 0 \). In fact, if we choose \( \alpha = 2^r \), then we get the desired result. \( \square \)

4. Non-Archimedean stability of functional equation (1.2): a fixed point method

In this section, using a fixed point approach, we prove the Hyers-Ulam-Rassias stability of functional equation (1.2) in non-Archimedean normed spaces. Throughout this section, \( X \) is a non-Archimedean normed spaces and that \( Y \) is a complete non-Archimedean normed spaces. Also we assume that \( |2| \neq 1 \).

**Theorem 4.1.** Let \( \zeta : X^2 \to [0, \infty) \) be a function such that there exists \( L < 1 \) with
\[ |2|\zeta \left( \frac{x}{2}, \frac{y}{2} \right) \leq L \zeta(x,y) \]
for all \( x, y \in X \). If \( f : X \to Y \) is a mapping satisfying
\[ \|f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\| \leq \zeta(x,y) \]
for all \( x, y \in X \), then there is a unique additive mapping \( A : X \to Y \) such that
\[ \|f(x) - A(x)\| \leq \frac{L \zeta(x,x)}{|2| - |2|L}. \]

**Proof.** Putting \( y = x \) in (4.2), we have
\[ \|f(2x) - 2f(x)\| \leq \zeta(x,x) \]
for all \( x \in X \). Replacing \( x \) by \( \frac{x}{2} \) in (4.4), we obtain
\[ \left\| 2f\left( \frac{x}{2} \right) - f(x) \right\| \leq \zeta \left( \frac{x}{2}, \frac{x}{2} \right) \]
for all \( x \in X \). Consider the set \( S^* := \{g : X \to Y\} \) and the generalized metric \( d^* \) in \( S^* \) defined by
\[ d^*(f,g) = \inf \left\{ \mu \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq \mu \zeta(x,x), \forall x \in X \right\}, \]

\[ \|f(x) - A(x)\| \leq \frac{L \zeta(x,x)}{|2| - |2|L}. \]
where \( \inf \emptyset = +\infty \). It is easy to show that \((S^*, d^*)\) is complete (see [14], Lemma 2.1). Now, we consider a linear mapping \( J^* : S^* \rightarrow S^* \) such that

\[
J^* h(x) := 2h\left(\frac{x}{2}\right)
\]

for all \( x \in X \). Let \( g, h \in S^* \) be such that \( d^*(g, h) = \epsilon \). Then we have \( \|g(x) - h(x)\| \leq \epsilon \zeta(x, x) \) for all \( x \in X \) and so

\[
\|J^*g(x) - J^*h(x)\| = \left\| 2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right) \right\| \leq |2\epsilon \zeta\left(\frac{x}{2}, \frac{x}{2}\right) | \leq |2\epsilon L | \zeta(x, x)
\]

for all \( x \in X \). Thus \( d^*(g, h) = \epsilon \) implies that \( d^*(J^*g, J^*h) \leq L \epsilon \). This means that \( d^*(J^*g, J^*h) \leq Ld^*(g, h) \) for all \( g, h \in S^* \). It follows from (4.5) that

\[
d^*(f, J^*f) \leq L |2|.
\]

By Theorem 1.10, there exists a mapping \( A : X \rightarrow Y \) satisfying the following:

1. \( A \) is a fixed point of \( J^* \), that is,
   \[
   A\left(\frac{x}{2}\right) = \frac{1}{2} A(x)
   \]
   for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( J^* \) in the set \( \Omega = \{ h \in S^* : d^*(g, h) < \infty \} \). This implies that \( A \) is a unique mapping satisfying (4.9) such that there exists \( \mu \in (0, \infty) \) satisfying \( \|f(x) - A(x)\| \leq \mu \zeta(x, x) \) for all \( x \in X \).
2. \( d^*(J^{*n}f, A) \rightarrow 0 \) as \( n \rightarrow \infty \). This implies the equality
   \[
   \lim_{n \rightarrow \infty} 2^nf\left(\frac{x}{2^n}\right) = A(x)
   \]
   for all \( x \in X \).
3. \( d^*(f, A) \leq \frac{d^*(f, J^*f)}{1 - L} \) with \( f \in \Omega \), which implies the inequality
   \[
   d^*(f, A) \leq \frac{L}{|2| - |2|L}.
   \]

This implies that the inequality (4.3) holds. By (4.2), we have

\[
\left\| 2^n \left[ f \left( f \left( \frac{x}{2^n} \right) - f \left( \frac{y}{2^n} \right) \right) - f \left( \frac{x+y}{2^n} \right) - f \left( \frac{x-y}{2^n} \right) + f \left( \frac{x}{2^n} \right) + f \left( \frac{y}{2^n} \right) \right] \right\| \leq |2|^n \zeta\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \leq |2|^n \cdot \frac{L}{|2|} \zeta(x, y)
\]
for all $x, y \in X$ and $n \geq 1$ and so $\|f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\| = 0$ for all $x, y \in X$. On the other hand

$$2A \left( \frac{x}{2} \right) - A(x) = \lim_{n \to \infty} 2^{n+1} f \left( \frac{x}{2^{n+1}} \right) - \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) = 0.$$ 

Therefore, the mapping $A : X \to Y$ is additive. This completes the proof. \[ \square \]

**Corollary 4.2.** Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying

$$\|f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then the limit $A(x) = \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\|f(x) - A(x)\| \leq \frac{2|2|\theta\|x\|^p}{|2|^{p+1} - |2|^2}$$

for all $x \in X$.

**Proof.** The proof follows from Theorem 4.1 if we take $\zeta(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. In fact, if we choose $L = |2|^{-1-p}$, then we get the desired result. \[ \square \]

Similarly, we have the following results for which we sketch the proofs.

**Theorem 4.3.** Let $\zeta : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\zeta(2x, 2y) \leq |2|L\zeta(x, y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (4.2). Then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{\zeta(x, x)}{|2| - |2|L}.$$ 

**Proof.** It follows from (4.4) that

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \leq \frac{\zeta(x, x)}{|2|}$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 4.1. \[ \square \]

**Corollary 4.4.** Let $\theta \geq 0$ and let $p$ be a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying (4.11). Then the limit $A(x) =$
\[
\lim_{n \to \infty} f(2^nx) = \frac{f(2^nx)}{2^n}
\]
exists for all \( x \in X \) and \( A : X \to Y \) is a unique additive mapping such that
\[
\| f(x) - A(x) \| \leq \frac{2\theta \| x \|^p}{|2| - |2|^p}
\]
for all \( x \in X \).

**Proof.** The proof follows from Theorem 4.3 if we take \( \zeta(x, y) = \theta(\| x \|^p + \| y \|^p) \) for all \( x, y \in X \). In fact, if we choose \( L = |2|^{p-1} \), then we get the desired result. \( \square \)

5. **Non-Archimedean stability of functional equation (1.2): a direct method**

In this section, using a direct method, we prove the Hyers-Ulam-Rassias stability of the functional equation (1.2) in non-Archimedean space. Throughout this section, \( G \) is an additive semigroup and \( X \) is a non-Archimedean Banach space.

**Theorem 5.1.** Let \( \zeta : G \times G \to [0, +\infty) \) be a function such that
\[
\lim_{n \to \infty} 2^n \zeta\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\]
for all \( x, y \in G \). Suppose that, for any \( x \in G \), the limit
\[
\Psi(x) = \lim_{n \to \infty} \max\left\{ 2^{k+1} \zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) : 0 \leq k < n \right\}
\]
exists and \( f : G \to X \) is a mapping satisfying
\[
\| f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y) \| \leq \zeta(x, y).
\]
Then, for all \( x \in G \), \( T(x) := \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) \) exists and satisfies the inequality
\[
\| f(x) - T(x) \| \leq \frac{1}{|2|} \Psi(x).
\]
Moreover, if
\[
\lim_{j \to \infty} \lim_{n \to \infty} \max\left\{ 2^{k+1} \zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) : j \leq k < n + j \right\} = 0,
\]
then \( T \) is the unique additive mapping satisfying (5.4).
Proof. By (4.5), we get
\[ \| 2f \left( \frac{x}{2} \right) - f(x) \| \leq \zeta \left( \frac{x}{2}, \frac{x}{2} \right) \]
for all \( x \in G \). Replacing \( x \) by \( \frac{x}{2^n} \) in (5.6), we obtain
\[ \| 2^{n+1} f \left( \frac{x}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) \| \leq |2|^n \zeta \left( \frac{x}{2^n+1}, \frac{x}{2^n+1} \right). \]
Thus, it follows from (5.1) and (5.7) that the sequence \( \left\{ 2^n f \left( \frac{x}{2^n} \right) \right\} \) is a Cauchy sequence. Since \( X \) is complete, it follows that \( \left\{ 2^n f \left( \frac{x}{2^n} \right) \right\} \) is convergent. Set \( T(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) \). By induction, one can show that
\[ \| 2^n f \left( \frac{x}{2^n} \right) - f(x) \| \leq \max \left\{ |2|^{k+1} \zeta \left( \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}} \right) : 0 \leq k < n \right\} \]
for all \( n \geq 1 \) and \( x \in G \). By taking \( n \to \infty \) in (5.8) and using (5.2), one obtains (5.4). By (5.1) and (5.3), we get
\[ \| T(T(x) - T(y)) - T(x + y) - T(x - y) + T(x) + T(y) \| \]
\[ = \lim_{n \to \infty} \| 2^n \left[ f \left( f \left( \frac{x}{2^n} \right) - f \left( \frac{y}{2^n} \right) \right) - f \left( \frac{x + y}{2^n} \right) - f \left( \frac{x - y}{2^n} \right) \right] \]
\[ + f \left( \frac{x}{2^n} \right) + f \left( \frac{y}{2^n} \right) \]
\[ \leq \lim_{n \to \infty} |2|^n \zeta \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = 0 \]
for all \( x, y \in G \). Therefore, the mapping \( T : G \to X \) satisfies (1.2). To prove the uniqueness property of \( T \), let \( S \) be another mapping satisfying (5.4). Then we have
\[ \| T(x) - S(x) \| = \lim_{j \to \infty} |2|^j \left\| T \left( \frac{x}{2^j} \right) - S \left( \frac{x}{2^j} \right) \right\| \]
\[ \leq \lim_{j \to \infty} |2|^j \max \left\{ \| T \left( \frac{x}{2^j} \right) - f \left( \frac{x}{2^j} \right) \|, \| f \left( \frac{x}{2^j} \right) - S \left( \frac{x}{2^j} \right) \| \right\} \]
\[ \leq \lim_{j \to \infty} \lim_{n \to \infty} \frac{1}{|2|} \max \left\{ |2|^{k+1} \zeta \left( \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}} \right) : j \leq k < n + j \right\} \]
\[ = 0 \]
for all \( x \in G \). Therefore, \( T = S \). This completes the proof. \( \square \)
Corollary 5.2. Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying

$$\xi \left( \frac{t}{|2|} \right) \leq \xi \left( \frac{1}{|2|} \right) \xi(t), \quad \xi \left( \frac{1}{|2|} \right) < \frac{1}{|2|}$$

for all $t \geq 0$. Let $\kappa > 0$ and $f : G \to X$ be a mapping such that (5.9)

$$\|f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\| \leq \kappa(\xi(|x|) + \xi(|y|))$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \leq 2\kappa \xi(|x|)$$

Proof. If we define $\zeta : G \times G \to [0, \infty)$ by $\zeta(x, y) := \kappa(\xi(|x|) + \xi(|y|))$, then we have

$$\lim_{n \to \infty} |2^n \zeta\left( \frac{x}{2^n}, \frac{y}{2^n} \right) \leq \lim_{n \to \infty} \left( 2|\xi\left( \frac{1}{|2|} \right) \right)^n \left[ \kappa(\xi(|x|) + \xi(|y|)) \right] = 0$$

for all $x, y \in G$. On the other hand, for all $x \in G$,

$$\Psi(x) = \lim_{n \to \infty} \max \{ |2|^{k+1} \zeta\left( \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}} \right) : 0 \leq k < n \}$$

exists. Also, we have

$$\lim_{j \to \infty} \lim_{n \to \infty} \max \left\{ |2|^{k+1} \zeta\left( \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}} \right) : j \leq k < n + j \right\} = \lim_{j \to \infty} |2|^{j+1} \zeta\left( \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) = 0.$$

Thus, applying Theorem 5.1, we have the conclusion. This completes the proof. □

Theorem 5.3. Let $\zeta : G \times G \to [0, +\infty)$ be a function such that

(5.10) $$\lim_{n \to \infty} \frac{\zeta(2^n x, 2^n y)}{|2|^n} = 0$$

for all $x, y \in G$. Suppose that, for every $x \in G$, the limit

(5.11) $$\Psi(x) = \lim_{n \to \infty} \max \left\{ \frac{\zeta(2^k x, 2^k x)}{|2|^k} : 0 \leq k < n \right\}$$
exists and let $f : G \to X$ be a mapping satisfying (5.3), then, the limit $T(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G$ and satisfies the inequality

\[(5.12) \quad \|f(x) - T(x)\| \leq \frac{1}{|2|} \Psi(x).\]

Moreover, if

\[(5.13) \quad \lim_{j \to \infty} \lim_{n \to \infty} \max \left\{ \frac{\zeta(2^k x, 2^k x)}{|2|^k} ; j \leq k < n + j \right\} = 0,

then $T$ is the unique mapping satisfying (5.12).

Proof. By (4.4), we have

\[(5.14) \quad \left\| f(x) - \frac{f(2x)}{2} \right\| \leq \frac{\zeta(x, x)}{|2|} \]

for all $x \in G$. Replacing $x$ by $2^n x$ in (5.14), we obtain

\[(5.15) \quad \left\| \frac{f(2^n x)}{2^n} - \frac{f(2^{n+1} x)}{2^{n+1}} \right\| \leq \frac{\zeta(2^n x, 2^n x)}{|2|^{n+1}}.

Thus it follows from (5.10) and (5.15) that the sequence $\left\{ \frac{f(2^n x)}{2^n} \right\}_{n \geq 1}$ is convergent. Set $T(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$. On the other hand, it follows from (5.15) that

\[
\left\| \frac{f(2^p x)}{2^p} - \frac{f(2^q x)}{2^q} \right\| = \left\| \sum_{k=p}^{q-1} \frac{f(2^k x)}{2^k} - \frac{f(2^{k+1} x)}{2^{k+1}} \right\|
\leq \max \left\{ \left\| \frac{f(2^k x)}{2^k} - \frac{f(2^{k+1} x)}{2^{k+1}} \right\| : p \leq k < q \right\}
\leq \frac{1}{|2|} \max \left\{ \frac{\zeta(2^k x, 2^k x)}{|2|^k} : p \leq k < q \right\}
\]

for all $x \in G$ and all integers $p, q \geq 0$ with $q > p \geq 0$. Letting $p = 0$, taking $q \to \infty$ in the last inequality and using (5.11), we obtain (5.12).

The rest of the proof is similar to the proof of Theorem 5.1. This completes the proof. \qed

**Corollary 5.4.** Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying

\[\xi(|2| t) \leq \xi(|2|) \xi(t), \quad \xi(|2|) < |2|

for all $t \geq 0$. Then for all $x, y \in G$, the function $\xi \circ f : G \to X$ is a mapping satisfying (5.3) and (5.12).
for all \( t \geq 0 \). Let \( \kappa > 0 \) and let \( f : G \to X \) be a mapping satisfying (5.9). Then there exists a unique additive mapping \( T : G \to X \) such that

\[
\| f(x) - T(x) \| \leq \frac{2\kappa \xi(|x|)}{2}.
\]

Proof. If we define \( \zeta : G \times G \to [0, \infty) \) by \( \zeta(x, y) := \kappa(\xi(|x|) + \xi(|y|)) \) and apply Theorem 5.3, then we get the conclusion. \( \square \)

References


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