

## CONTACT CR-WARPED PRODUCT SUBMANIFOLDS IN KENMOTSU SPACE FORMS

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**ABSTRACT.** In the present paper, we give a necessary and sufficient condition for contact CR-warped product to be contact CR-product in Kenmotsu space forms.

### 1. Introduction

The geometry of manifolds endowed with geometrical structures has been intensively studied and several important results have been published. An important class of such manifolds is formed Kenmotsu manifolds. In [5], K. Kenmotsu introduced and studied a new class of almost contact manifolds called Kenmotsu manifolds. Afterward, many authors studied the geometry of the submanifolds of a Kenmotsu manifold because the geometry of submanifolds of a Kenmotsu manifold is rich and interesting.

The notation CR-warped product submanifolds of a Kaehler manifolds were introduced by B.Y. Chen [3]. Then contact CR-submanifolds of Sasakian manifolds with definite metric were introduced and studied by K. Matsumoto [8] and I. Haseqawa [4] and further studied in [1,6,7,9].

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Afterward, the concept of contact CR-warped product was introduced by many geometers in various manifold types with differential geometric point of view [see references]. Many authors established general inequalities for contact CR-warped product submanifolds in Sasakian and Kenmotsu manifolds in terms of the warping function for a contact CR-warped submanifolds isometrically immersed in a contact metric manifold.

In [4], I. Hasegawa and I. Mihai obtained a sharp inequality for the squared norm of the second fundamental form in terms of the warping function for contact CR-warped products in Sasakian manifolds.

In [1], Arslan and the other authors studied contact CR-warped products in Kenmotsu manifolds. They provided some necessary and sufficient conditions for contact CR-warped product to be totally geodesic, totally umbilical and space form.

In [2], we studied warped product semi-invariant submanifolds in locally Riemannian product manifolds and we gave a necessary and sufficient condition for a warped product to be Riemannian product.

In the present paper, we obtain a sharp estimates for squared norm of the second fundamental form in terms of the warping function for contact CR-warped product submanifolds in Kenmotsu space forms. The equality case is considered and some new results are derived.

For the papers in this subject, we refer to the references.

## 2. Preliminaries

In this section, we give some notations used throughout this paper. We recall some necessary facts and formulas from the theory of Kenmotsu manifolds and their submanifolds.

A  $(2m + 1)$ -dimensional Riemannian manifold  $(\bar{M}, g)$  is said to be an almost contact metric manifold if it admits an endomorphism  $\phi$  of its tangent bundle  $T\bar{M}$ , a vector field  $\xi$  and a 1-form  $\eta$ , satisfying

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0$$

and

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any vector fields  $X, Y$  tangent to  $\bar{M}$ . Furthermore, an almost contact metric manifold is called a Kenmotsu manifold if  $\phi$  and  $\xi$  satisfy;

$$(2.3) \quad \begin{aligned} (\bar{\nabla}_X \phi)Y &= g(\phi X, Y)\xi - \eta(Y)\phi X \\ \bar{\nabla}_X \xi &= -\phi^2 X = X - \eta(X)\xi, \end{aligned}$$

where  $\bar{\nabla}$  denotes the Levi-Civita connection on  $\bar{M}$  [7].

Now, let  $\bar{M}$  be a  $(2n+1)$ -dimensional Kenmotsu manifold with structure tensors  $(\phi, \xi, \eta, g)$  and let  $M$  be an  $m$ -dimensional isometrically immersed submanifold in  $\bar{M}$ . Moreover, we denote the Levi-Civita connection on  $\bar{M}$  by  $\bar{\nabla}$  and let  $\nabla$  be the induced connection on  $M$  by  $\bar{\nabla}$ . Then the Gauss and Weingarten formula's for  $M$  in  $\bar{M}$  are, respectively, given by

$$(2.4) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

$$(2.5) \quad \bar{\nabla}_X V = -A_V X + \nabla_X^\perp V$$

for any vector fields  $X, Y$  tangent to  $M$  and vector  $V$  normal to  $M$ , where  $\nabla^\perp$  is the normal connection on  $T^\perp M$ ,  $h$  and  $A$  denote the second fundamental form and shape operator of  $M$  in  $\bar{M}$ , respectively. It is well known that  $A$  and  $h$  are related by

$$(2.6) \quad g(h(X, Y), V) = g(A_V X, Y).$$

We denote the Riemannian curvature tensors of  $\bar{\nabla}$  and the induced connection  $\nabla$  by  $\bar{R}$  and  $R$ , respectively. Then the Gauss and Codazzi equations are, respectively, given by

$$(2.7) \quad (\bar{R}(X, Y)Z)^\top = R(X, Y)Z + A_{h(X, Z)}Y - A_{h(Y, Z)}X$$

and

$$(2.8) \quad (\bar{R}(X, Y)Z)^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z)$$

for any vector fields  $X, Y, Z$  tangent to  $M$ , where the covariant derivative of  $h$  is defined by

$$(2.9) \quad (\bar{\nabla}_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields  $X, Y, Z$  tangent to  $M$ , and,  $(\bar{R}(X, Y)Z)^\perp$  and  $(\bar{R}(X, Y)Z)^\top$  denote the normal and tangent components of  $\bar{R}(X, Y)Z$ ,

respectively, with respect to the submanifold  $M$  [3].

For any vector field  $X$  tangent to  $M$ , we set

$$(2.10) \quad \phi X = fX + \omega X,$$

where  $fX$  and  $\omega X$  are the tangential and normal components of  $\phi X$ , respectively. Then  $f$  is an endomorphism of  $TM$  and  $\omega$  is a normal-bundle valued 1-form of  $TM$ . For the same reason, for any vector field  $V$  normal to  $M$ , we set

$$(2.11) \quad \phi V = BV + CV,$$

where  $BV$  and  $CV$  are the tangential and normal components of  $\phi V$ , respectively. Then  $B$  is an endomorphism of the normal bundle  $T^\perp M$  to  $TM$  and  $C$  is a normal-bundle valued 1-form of  $T^\perp M$ .

A Kenmotsu manifold with constant  $\phi$ -holomorphic sectional curvature  $c$  is called a Kenmotsu space form and is denoted by  $\bar{M}(c)$ . Then its curvature tensor  $\bar{R}$  is expressed by

$$(2.12) \quad \begin{aligned} \bar{R}(X, Y)Z &= \frac{c-3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ &\quad - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi \\ &\quad + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \end{aligned}$$

for any vector fields  $X, Y, Z$  tangent to  $\bar{M}$  [6].

### 3. Contact CR-Warped Product Submanifolds in Kenmotsu Manifolds

In this section, we define contact CR-submanifolds in a Kenmotsu manifold and study their fundamental properties from the view point of theory of submanifolds.

Let  $M_1$  and  $M_2$  be two Riemannian manifolds with Riemannian metrics  $g_1$  and  $g_2$ , respectively, and let  $f$  be a positive definite smooth function on  $M_1$ . We consider the product manifold  $M_1 \times M_2$  with its projections  $\pi : M_1 \times M_2 \rightarrow M_1$  and  $\eta : M_1 \times M_2 \rightarrow M_2$ . The warped product  $M = M_1 \times_f M_2$  is a manifold  $M_1 \times M_2$  equipped with the Riemannian metric such that

$$g(X, Y) = g_1(\pi_*X, \pi_*Y) + (f \circ \pi)^2 g_2(\eta_*X, \eta_*Y),$$

for any  $X, Y \in \Gamma(TM)$ , where  $*$  stands for differential of map and  $\Gamma(TM)$  denotes the set of the differentiable vector fields on  $M$ . Thus we have  $g = g_1 + f^2 g_2$ . The function  $f$  is called warping function of the warped product manifold  $M = M_1 \times_f M_2$ . If we denote the Levi Civita connection on  $M$  by  $\nabla$ , then we have the following Proposition for the warped product manifold [3].

**Proposition 3.1.** *Let  $M = M_1 \times_f M_2$  be a warped product manifold. For  $X, Y \in \Gamma(TM_1)$  and  $Z, V \in \Gamma(TM_2)$ , we have*

- (1)  $\nabla_X Y \in \Gamma(TM_1)$  is the lift of  $\nabla_X Y$  on  $M_1$
- (2)  $\nabla_X V = \nabla_V X = X(\ln f)V$
- (3)  $\text{nor}\nabla_Z V = -g(Z, V)\text{grad}\ln f$
- (4)  $\text{tan}\nabla_Z V = \nabla'_Z V \in \Gamma(TM_2)$  is the lift of  $\nabla'_Z V$  on  $M_2$ , where  $\nabla'$  denotes the Levi-Civita connection of  $g_2$ .

If the warping function  $f$  is constant, then the warped product is said to be a Riemannian product.

Let  $M$  be a  $m$ -dimensional Riemannian manifold with Riemannian metric  $g$  and let  $\{e_1, e_2, \dots, e_m\}$  be an orthonormal basis for  $\Gamma(TM)$ . For a smooth function  $f$  on  $M$ , the gradient and Hessian of  $f$  are, respectively, defined by

$$(3.1) \quad Xf = g(\text{grad}f, X)$$

and

$$(3.2) \quad H^f(X, Y) = XYf - (\nabla_X Y)f = g(\nabla_X \text{grad}f, Y)$$

for any  $X, Y \in \Gamma(TM)$ . The Laplacian of  $f$  is defined by

$$(3.3) \quad \Delta f = \sum_{i=1}^m \{(\nabla_{e_i} e_i)f - e_i e_i f\} = - \sum_{i=1}^m g(\nabla_{e_i} \text{grad}f, e_i).$$

From (3.2) and (3.3), it is easily seen that the Laplacian is the negative of the trace of the Hessian.

From the integration theory on manifolds, since  $M$  is a compact orientable Riemannian manifold without boundary, we have

$$(3.4) \quad \int_M \Delta f dV = 0,$$

where  $dV$  is the volume element of  $M$  [2].

**Definition 3.2.** Let  $M$  be an isometrically immersed submanifold of a Kenmotsu manifold  $\bar{M}$  such that  $M$  is tangent to  $\xi$ . Then  $M$  is said to be a contact CR-submanifold if there exists a pair of orthogonal differentiable distributions such as  $D$  and  $D^\perp$  on  $M$  such that

- 1.)  $TM = D \oplus D^\perp \oplus \{\xi\}$ , where  $\{\xi\}$  is the 1-dimensional distribution spanned by  $\xi$ .
- 2.)  $D$  is an invariant distribution with respect to  $\phi$ , that is,  $\phi D_p \subset D_p$ , for all  $p \in M$ ,
- 3.)  $D^\perp$  is an anti-invariant distribution with respect to  $\phi$ , that is,  $\phi D_p^\perp \subset T_p^\perp M$ , for all  $p \in M$  [1].

In this paper, we consider warped product manifolds which are of the form  $M = M_T \times_f M_\perp$  in a Kenmotsu manifold  $\bar{M}$  such that  $M$  is tangent to  $\xi$ , where  $M_T$  is an invariant submanifold and  $M_\perp$  is an anti-invariant submanifold of  $\bar{M}$  and we call it contact CR-warped product, where  $M_T$  and  $M_\perp$  denote the integral manifolds of the distributions of  $D$  and  $D^\perp$ , respectively, in Definition 3.2.

We give an example of contact CR-warped product submanifold of type  $M = M_T \times_f M_\perp$  in Kenmotsu manifold with  $\xi$  tangent to  $M_T$ .

**Example 3.3.** Let  $\mathbb{R}^9 = \mathbb{C}^4 \times \mathbb{R}$  be the 9-dimensional Euclidean space endowed with the almost contact metric structure  $(\phi, \xi, \eta, g)$  defined by

$$\phi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, t) = (-x_5, -x_6, -x_7, -x_8, x_1, x_2, x_3, x_4, 0),$$

$$\xi = e^t \frac{\partial}{\partial t}, \quad \eta = e^t dt, \quad g = e^{2t} \langle, \rangle,$$

where  $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, t)$  and  $\langle, \rangle$  denote the cartesian coordinates and the Euclidean metric tensor of  $\mathbb{R}^9$ , respectively. It is well known that  $\mathbb{R}^9$  is a Kenmotsu manifold.

Now, we define a submanifold by

$$M = \{(x_1, 0, x_3, 0, x_2, 0, 0, x_4, t) \in \mathbb{R}^9\}$$

and choose a frame  $\{e_1, e_2, e_3, e_4, e_5\}$  of orthogonal vector fields on  $M$  as

$$e_1 = \frac{\partial}{\partial x_1} + x_5 \frac{\partial}{\partial t}, e_2 = \frac{\partial}{\partial x_5}, e_3 = \frac{\partial}{\partial x_3} + x_7 \frac{\partial}{\partial t}, e_4 = \frac{\partial}{\partial x_8}, e_5 = \xi = \frac{\partial}{\partial t}.$$

Then it is easy to observe that  $D_T = \text{span}\{e_1, e_2, e_5\}$  and  $D^\perp = \text{span}\{e_3, e_4\}$  define the invariant and anti-invariant differentiable distributions on

Kenmotsu manifold  $\mathbb{R}^9$ . If we denote the integral manifolds of  $D_T$  and  $D^\perp$  by  $M_T$  and  $M_\perp$ , respectively, then it is easy to check that  $M = M_T \times_f M_\perp$  is a contact CR-warped product submanifold with warping function  $f(t) = e^t$ .

#### 4. Two Theorems For Contact CR Warped Product Submanifolds

In this section, we give the main results of this paper. Firstly, we give the following two lemmas and a theorem for later use.

**Lemma 4.1.** *Let  $M = M_T \times_f M_\perp$  be a contact CR-warped product submanifold of a Kenmotsu manifold  $\bar{M}$ . Then we have*

$$(4.1) \quad g(h(X, Y), \phi Y) = -\|Y\|^2 \phi X \ln f$$

and

$$(4.2) \quad g(h(\phi X, Y), \phi Y) = \|Y\|^2 X \ln f$$

for any  $X \in \Gamma(TM_T)$  and  $Y \in \Gamma(TM_\perp)$ .

*Proof.* For any  $X \in \Gamma(TM_T)$  and  $Y \in \Gamma(TM_\perp)$ , using (2.3), (2.4) and considering Proposition 3.1 (2), we have

$$\begin{aligned} g(h(X, Y), \phi Y) &= g(\bar{\nabla}_Y X, \phi Y) = -g(\phi \bar{\nabla}_Y X, Y) \\ &= -g(\bar{\nabla}_Y \phi X - (\bar{\nabla}_Y \phi)X, Y) \\ &= -g(\nabla_Y \phi X, Y) + g(g(\phi Y, X)\xi - \eta(X)\phi Y, Y) \\ &= -g(Y, Y)\phi X \ln f \end{aligned}$$

and

$$\begin{aligned} g(h(\phi X, Y), \phi Y) &= g(\bar{\nabla}_Y \phi X, \phi Y) = g((\bar{\nabla}_Y \phi)X + \phi \bar{\nabla}_Y X, \phi Y) \\ &= g(g(\phi Y, X)\xi - \eta(X)\phi Y, \phi Y) + g(\phi \bar{\nabla}_Y X, \phi Y) \\ &= g(\phi h(X, Y) + \phi \nabla_Y X, \phi Y) = g(\phi Y, \phi Y)X \ln f \\ &= g(Y, Y)X \ln f \end{aligned}$$

which prove our assertions. □

**Lemma 4.2.** *Let  $M = M_T \times_f M_\perp$  be a contact CR-warped product submanifold of a Kenmotsu manifold  $\bar{M}$ . Then we have*

$$(4.3) \quad \|h(X, Y)\|^2 = g(h(\phi X, Y), \phi h(X, Y)) + g(Y, Y)(\phi X \ln f)^2,$$

for any  $X \in \Gamma(TM_T)$  and  $Y \in \Gamma(TM_\perp)$ .

*Proof.* Making use of (2.3), (2.4) and considering Proposition 3.1 and Lemma 4.1 we have

$$\begin{aligned}
g(h(\phi X, Y), \phi h(X, Y)) &= g(\phi h(X, Y), \bar{\nabla}_Y \phi X - \nabla_Y \phi X) \\
&= g(\phi h(X, Y), (\bar{\nabla}_Y \phi)X + \phi \bar{\nabla}_Y X - (\phi X \ln f)Y) \\
&= g(\phi h(X, Y), g(\phi Y, X)\xi - \eta(X)\phi Y + \phi \bar{\nabla}_Y X) \\
&\quad + g(h(X, Y), \phi X \ln f \phi Y) \\
&= g(h(X, Y), \bar{\nabla}_Y X) + g(h(X, Y), \phi Y)\phi X \ln f \\
&= g(h(X, Y), h(X, Y)) + g(\bar{\nabla}_Y X, \phi Y)\phi X \ln f \\
&= \|h(X, Y)\|^2 - g(\phi \bar{\nabla}_Y X, Y)\phi X \ln f \\
&= \|h(X, Y)\|^2 - g(\bar{\nabla}_Y \phi X - (\bar{\nabla}_Y \phi)X, Y)\phi X \ln f \\
&= \|h(X, Y)\|^2 - g(\nabla_Y \phi X - g(\phi Y, X)\xi \\
&\quad + \eta(X)\phi Y, Y)\phi X \ln f \\
&= \|h(X, Y)\|^2 - g((\phi X \ln f)Y, Y)\phi X \ln f \\
&= \|h(X, Y)\|^2 - g(Y, Y)(\phi X \ln f)^2.
\end{aligned}$$

This completes the proof of the lemma.  $\square$

**Theorem 4.3.** *Let  $M = M_T \times_f M_\perp$  be a contact CR-warped product submanifold of a Kenmotsu space form  $\bar{M}(c)$ . Then we have*

$$\begin{aligned}
2\|h(X, Y)\|^2 &= \{H^{\ln f}(X, X) + H^{\ln f}(\phi X, \phi X) + (\frac{c-3}{2})g(X, X) \\
(4.4) \quad &\quad + 2(\phi X \ln f)^2\}g(Y, Y),
\end{aligned}$$

for any  $X \in \Gamma(TM_T)$  and  $Y \in \Gamma(TM_\perp)$ .

*Proof.* Using (2.8), (2.9) and considering  $\bar{\nabla}$ , the Levi-Civita connection, we have

$$\begin{aligned}
g(\bar{R}(X, \phi X)Y, \phi Y) &= g((\bar{\nabla}_X h)(\phi X, Y) - (\bar{\nabla}_{\phi X} h)(X, Y), \phi Y) \\
&= g(\bar{\nabla}_X h(\phi X, Y) - h(\nabla_X \phi X, Y) - h(\nabla_X Y, \phi X), \phi Y) \\
&\quad - g(\bar{\nabla}_{\phi X} h(X, Y) - h(\nabla_{\phi X} X, Y) - h(\nabla_{\phi X} Y, X), \phi Y) \\
&= Xg(h(\phi X, Y), \phi Y) - g(\bar{\nabla}_X \phi Y, h(\phi X, Y)) \\
&\quad - g(h(\nabla_X \phi X, Y), \phi Y) - g(h(\nabla_X Y, \phi X), \phi Y) \\
&\quad - \phi Xg(h(X, Y), \phi Y) + g(h(X, Y), \bar{\nabla}_{\phi X} \phi Y) \\
&\quad + g(h(\nabla_{\phi X} X, Y), \phi Y) + g(h(\nabla_{\phi X} Y, X), \phi Y).
\end{aligned}$$



On the other hand, considering Lemma 4.1 and Proposition 3.1, we have

$$\begin{aligned}
g(\bar{R}(X, \phi X)Y, \phi Y) &= X[(X \ln f)g(Y, Y)] - g(h(\phi X, Y), (\bar{\nabla}_X \phi)Y \\
&+ \phi(\bar{\nabla}_X Y)) - g(h(\nabla_X \phi X, Y), \phi Y) \\
&- X \ln f g(h(\phi X, Y), \phi Y) - \phi X[-\phi X \ln f g(Y, Y)] \\
&+ g(h(X, Y), (\bar{\nabla}_{\phi X} \phi)Y + \phi(\bar{\nabla}_{\phi X} Y)) \\
&+ g(h(\nabla_{\phi X} X, Y), \phi Y) + \phi X \ln f g(h(X, Y), \phi Y) \\
&= X(X \ln f)g(Y, Y) + 2g(\nabla_X Y, Y)X \ln f \\
&- g(h(\phi X, Y), \phi h(X, Y)) - g(h(\phi X, Y), \phi \nabla_X Y) \\
&+ \phi \nabla_X \phi X \ln f g(Y, Y) - X \ln f (X \ln f)g(Y, Y) \\
&+ \phi X(\phi X \ln f)g(Y, Y) + 2\phi X \ln f g(\nabla_{\phi X} Y, Y) \\
&+ g(h(X, Y), \phi h(\phi X, Y)) + g(h(X, Y), \phi X \ln f \phi Y) \\
&- \phi \nabla_{\phi X} X \ln f g(Y, Y) + \phi X \ln f(-\phi X \ln f)g(Y, Y) \\
&= X(X \ln f)g(Y, Y) + 2X \ln f (X \ln f)g(Y, Y) \\
&- 2g(h(\phi X, Y), \phi h(X, Y)) - X \ln f g(h(\phi X, Y), \phi Y) \\
&+ \phi \nabla_X \phi X \ln f g(Y, Y) - \phi \nabla_{\phi X} X \ln f g(Y, Y) \\
&+ \phi X(\phi X \ln f)g(Y, Y) + 2\phi X \ln f(\phi X \ln f)g(Y, Y) \\
&- (X \ln f)^2 g(Y, Y) - 2(\phi X \ln f)^2 g(Y, Y).
\end{aligned}$$

Summing up, we conclude

$$\begin{aligned}
g(\bar{R}(X, \phi X)Y, \phi Y) &= X(X \ln f)g(Y, Y) - 2g(h(\phi X, Y), \phi h(X, Y)) \\
&+ \phi \nabla_X \phi X \ln f g(Y, Y) - \phi \nabla_{\phi X} X \ln f g(Y, Y) \\
&+ \phi X(\phi X \ln f)g(Y, Y) \\
&= \{X(X \ln f) + \phi X(\phi X \ln f) + \phi \nabla_X \phi X \ln f \\
(4.5) \quad &- \phi \nabla_{\phi X} X \ln f\}g(Y, Y) - 2g(h(\phi X, Y), \phi h(X, Y)).
\end{aligned}$$

From (4.5) and Lemma 4.2, we arrive to

$$\begin{aligned}
g(\bar{R}(X, \phi X)Y, \phi Y) &= \{X(X \ln f) + \phi X(\phi X \ln f) + \phi \nabla_X \phi X \ln f \\
&- \phi \nabla_{\phi X} X \ln f + 2(\phi X \ln f)^2\}g(Y, Y) - 2\|h(X, Y)\|^2.
\end{aligned}$$

On the other hand, considering Proposition 3.1,  $M_T$  is totally geodesic in  $M$  and  $grad \ln f \in \Gamma(TM_T)$ . Thus by direct calculations, we have

$$\begin{aligned}
\phi \nabla_X \phi X \ln f &= g(\phi \nabla_X \phi X, grad \ln f) = -g(\nabla_X \phi X, \phi grad \ln f) \\
&= -g(\bar{\nabla}_X \phi X, \phi grad \ln f) = -g((\bar{\nabla}_X \phi)X, \phi grad \ln f) \\
&\quad - g(\phi \bar{\nabla}_X X, \phi grad \ln f) \\
&= -g(g(\phi X, X)\xi - \eta(X)\phi X, \phi grad \ln f) \\
&\quad - g(\phi \nabla_X X, \phi grad \ln f) \\
&= \eta(X)g(\phi X, \phi grad \ln f) + g(\nabla_X X, \phi^2 grad \ln f) \\
&= g(\nabla_X X, -grad \ln f + \eta(grad \ln f)\xi) = -\nabla_X X \ln f \\
&\quad - \xi \ln f g(\nabla_X \xi, X) \\
(4.6) \quad &= -\nabla_X X \ln f - g(X, X)\xi \ln f.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
\phi \nabla_{\phi X} X \ln f &= g(\phi \bar{\nabla}_{\phi X} X, grad \ln f) \\
&= g(\bar{\nabla}_{\phi X} \phi X - (\bar{\nabla}_{\phi X} \phi)X, grad \ln f) \\
&= g(\bar{\nabla}_{\phi X} \phi X, grad \ln f) \\
&\quad - g(g(\phi^2 X, X)\xi - \eta(X)\phi^2 X, grad \ln f) \\
&= \nabla_{\phi X} \phi X \ln f + g(\phi X, \phi X)g(\xi, grad \ln f) \\
&\quad + \eta(X)g(\phi^2 X, grad \ln f) \\
&= \nabla_{\phi X} \phi X \ln f + g(\phi X, \phi X)\xi \ln f \\
(4.7) \quad &= \nabla_{\phi X} \phi X \ln f + g(X, X)\xi \ln f.
\end{aligned}$$

In [1], it was proved  $\xi \ln f = 1$ . So by substituting (4.6) and (4.7) into (4.5), we find

$$\begin{aligned}
g(\bar{R}(X, \phi X)Y, \phi Y) &= \{X(X \ln f) + \phi X(\phi X \ln f) - \nabla_X X \ln f \\
&\quad - \nabla_{\phi X} \phi X - 2g(X, X) + 2(\phi X \ln f)^2\}g(Y, Y) \\
&\quad - 2\|h(X, Y)\|^2 \\
&= \{H^{\ln f}(X, X) + H^{\ln f}(\phi X, \phi X) \\
(4.8) \quad &\quad - 2g(X, X) + 2(\phi X \ln f)^2\}g(Y, Y) - 2\|h(X, Y)\|^2.
\end{aligned}$$

Moreover, using (2.12), we get

$$\begin{aligned}
g(\bar{R}(X, \phi X)Y, \phi Y) &= -\frac{c+1}{2}g(\phi X, \phi X)g(Y, Y) \\
(4.9) \quad &= -\frac{c+1}{2}g(X, X)g(Y, Y),
\end{aligned}$$

for any  $X \in \Gamma(TM_T)$  and  $Y \in \Gamma(TM_\perp)$ . Finally, from (4.8) and (4.9) we conclude that

$$\begin{aligned}
 -\left(\frac{c+1}{2}\right)g(X, X)g(Y, Y) &= \{H^{\ln f}(X, X) + H^{\ln f}(\phi X, \phi X) \\
 &- 2g(X, X) + 2(\phi X \ln f)^2\}g(Y, Y) \\
 (4.10) \qquad \qquad \qquad &- 2\|h(X, Y)\|^2,
 \end{aligned}$$

which is equivalent to (4.4). □

Now, let  $\{e_o = \xi, e_1, e_2, \dots, e_p, \phi e_1, \phi e_2, \dots, \phi e_p, e^1, e^2, \dots, e^q\}$  be an orthonormal basis of  $\Gamma(TM)$  such that  $e_o, e_1, e_2, \dots, e_p, \phi e_1, \phi e_2, \dots, \phi e_p$  are tangent to  $\Gamma(TM_T)$  and  $e^1, e^2, \dots, e^q$  are tangent to  $\Gamma(TM_\perp)$ . Moreover, let  $\{\phi e^1, \phi e^2, \dots, \phi e^q, N_1, N_2, \dots, N_{2r}\}$  be an orthonormal basis of  $\Gamma(TM^\perp)$  such that  $\{\phi e^1, \phi e^2, \dots, \phi e^q\}$  are tangent to  $\phi TM_\perp$  and  $\{N_1, N_2, \dots, N_{2r}\}$  are tangent to  $\Gamma(\nu)$ , where  $\nu$  denote the orthogonal distribution of  $\phi D^\perp$  in  $T^\perp M$ .

Now we can give the main theorem of this paper.

**Theorem 4.4.** *Let  $M = M_T \times_f M_\perp$  be a compact orientable contact CR-warped product submanifold of a Kenmotsu space form  $\bar{M}(c)$ . Then  $M$  is a contact CR-product if*

$$(4.11) \qquad \sum_{i=1}^p \sum_{j=1}^q \|h_2(e_i, e^j)\|^2 \geq \left(\frac{c-3}{4}\right)pq,$$

where  $h_2$  denotes the component of  $h$  in  $\Gamma(\nu)$ .

*Proof.* Using (3.3), the Laplacian of  $\ln f$  is given by

$$\begin{aligned}
 -\Delta \ln f &= \sum_{i=1}^p g(\nabla_{e_i} \text{grad} \ln f, e_i) + \sum_{i=1}^p g(\nabla_{\phi e_i} \text{grad} \ln f, \phi e_i) \\
 &+ \sum_{j=1}^q (\nabla_{e^j} \text{grad} \ln f, e^j) + g(\nabla_\xi \text{grad} \ln f, \xi).
 \end{aligned}$$

Here, considering  $\nabla$  to be the Levi-Civita connection,  $M_T$  is totally geodesic in  $M$ . Hence  $\text{grad} \ln f \in \Gamma(TM_T)$  and by Proposition 3.1, we

have

$$\begin{aligned}
 -\Delta \ln f &= \sum_{i=1}^p \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} \\
 &+ \sum_{j=1}^q \{(e^j g(\text{grad} \ln f, e^j) - g(\nabla_{e^j} e^j, \text{grad} \ln f))\} \\
 &+ g(\nabla_{\xi} \text{grad} \ln f, \xi) \\
 &= \sum_{i=1}^p \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} \\
 &- \sum_{j=1}^q \{-g(e^j, e^j)g(\text{grad} \ln f, \text{grad} \ln f)\} + \xi g(\text{grad} \ln f, \xi) \\
 &- g(\nabla_{\xi} \xi, \text{grad} \ln f) \\
 (4.12) \quad &= \sum_{i=1}^p \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} + q \|\text{grad} \ln f\|^2.
 \end{aligned}$$

In (4.10) let  $X = e_i$  and  $Y = e^j$ ,  $1 \leq i \leq p$  and  $1 \leq j \leq q$ . Then by direct calculations, we have

$$\begin{aligned}
 -\left(\frac{c+1}{2}\right)pq &= \left\{ \sum_{i=1}^p \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)\} - 2p \right. \\
 &\quad \left. + 2 \sum_{i=1}^p (\phi e_i \ln f)^2 \right\} q - 2 \sum_{i=1}^p \sum_{j=1}^q \|h(e_i, e^j)\|^2.
 \end{aligned}$$

From (4.12) and the last equation, we arrive at

$$\begin{aligned}
 -\left(\frac{c+1}{2}\right)pq &= \{-\Delta \ln f - q \|\text{grad} \ln f\|^2 - 2p + 2 \sum_{i=1}^p (\phi e_i \ln f)^2\} q \\
 &- 2 \sum_{i=1}^p \sum_{j=1}^q \|h(e_i, e^j)\|^2,
 \end{aligned}$$

that is,

$$(4.13) \quad \begin{aligned} \Delta \ln f &= 2 \sum_{i=1}^p (\phi e_i \ln f)^2 + \left(\frac{c-3}{2}\right) p - q \|\text{grad} \ln f\|^2 \\ &- \frac{2}{q} \sum_{i=1}^p \sum_{j=1}^q \|h(e_i, e^j)\|^2. \end{aligned}$$

Furthermore, from linear algebra rules, we know that  $h$  can be written as

$$h(e_i, e^j) = \sum_{k=1}^p g(h(e_i, e^j), \phi e^k) \phi e^k + \sum_{\ell=1}^{2r} g(h(e_i, e^j), N_\ell) N_\ell.$$

Also, making use of (4.1), we have

$$(4.14) \quad \begin{aligned} \sum_{i=1}^p \sum_{j=1}^q g(h(e_i, e^j), h(e_i, e^j)) &= \sum_{i=1}^p \sum_{k,j=1}^q g(h(e_i, e^j), \phi e^k)^2 \\ &+ \sum_{i=1}^p \sum_{j=1}^q \sum_{\ell=1}^{2r} g(h(e_i, e^j), N_\ell)^2 \\ &= q \sum_{i=1}^p (\phi e_i \ln f)^2 + \sum_{i=1}^p \sum_{j=1}^q \|h_2(e_i, e^j)\|^2. \end{aligned}$$

Finally, substituting (4.14) into (4.13), we get

$$-q \Delta \ln f = 2 \sum_{i=1}^p \sum_{j=1}^q \|h_2(e_i, e^j)\|^2 - \left(\frac{c-3}{2}\right) pq + q^2 \|\text{grad} \ln f\|^2.$$

From (3.4), we conclude that

$$(4.15) \quad \begin{aligned} \int_M \left\{ 2 \sum_{i=1}^p \sum_{j=1}^q \|h_2(e_i, e^j)\|^2 - \left(\frac{c-3}{2}\right) pq \right. \\ \left. + q^2 \|\text{grad} \ln f\|^2 \right\} dV = 0. \end{aligned}$$

Here, if

$$\sum_{i=1}^p \sum_{j=1}^q \|h_2(e_i, e^j)\|^2 \geq \left(\frac{c-3}{4}\right) pq,$$

which implies  $\|grad \ln f\| = 0$  because  $q \neq 0$ , that is, the warping function  $f$  is constant. So contact CR-warped product becomes a contact CR-product.  $\square$

From the integral formula (4.15) we derive the following corollaries.

**Corollary 4.5.** *Let  $M = M_T \times_f \bar{M}_\perp$  be a compact orientable contact CR-warped product submanifold of a Kenmotsu space form  $\bar{M}(c)$ . Then  $M$  is a contact CR-product if and only if*

$$(4.16) \quad \sum_{i=1}^p \sum_{j=1}^q \|h_2(e_i, e^j)\|^2 = \left(\frac{c-3}{4}\right)pq.$$

*Proof.* If (4.16) is satisfied, then (4.15) implies that  $f = constant$ , that is,  $M$  is a contact CR-product.

Conversely, if  $M$  is a contact CR-product, from Lemma 4.1 we know that  $h(X, Y) \in \Gamma(\nu)$ , for any  $X \in \Gamma(TM_T)$  and  $Y \in \Gamma(TM_\perp)$ . So the equality (4.16) is satisfied  $\square$

**Corollary 4.6.** *There exist no compact orientable contact CR-warped products in Kenmotsu space forms  $\bar{M}(c)$  such that  $c < 3$ .*

**Corollary 4.7.** *There exist no compact orientable contact CR-warped products in  $\mathbb{R}^{2n+1}$  with a usual almost contact metric structure  $(\phi, \xi, \eta, \langle, \rangle)$ .*

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