Bulletin of the Iranian Mathematical Society Vol. 39 *No. 3 (2013), pp 415-429.*

CONTACT CR-WARPED PRODUCT SUBMANIFOLDS IN KENMOTSU SPACE FORMS

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Communicated by Mohammad Bagher Kashani

ABSTRACT. In the present paper, we give a necessary and sufficient condition for contact CR-warped product to be contact CR-product in Kenmotsu space forms.

1. **Introduction**

The geometry of manifolds endowed with geometrical structures has been intensively studied and several important results have been published. An important class of such manifolds is formed Kenmotsu manifolds. In [5], K. Kenmotsu introduced and studied a new class of almost contact manifolds called Kenmotsu manifolds. Afterward, many authors studied the geometry of the submanifolds of a Kenmotsu manifold because the geometry of submanifolds of a Kenmotsu manifold is rich and interesting.

The notation CR-warped product submanifolds of a Kaehler manifolds were introduced by B.Y. Chen [3]. Then contact CR-submanifolds of Sasakian manifolds with definite metric were introduced and studied by K. Matsumoto [8] and I. Haseqawa [4] and further studied in [1,6,7,9].

MSC(2010): Primary: 53C15; Secondary: 53C42.
Keywords: Kenmotsu space form, Contact CR-warped product, Contact CR-product.
Received: 29 December 2011, Accepted: 2 April 2012.
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Afterward, the concept of contact CR-warped product was introduced by many geometers in various manifold types with differential geometric point of view [see references]. Many authors established general inequalities for contact CR-warped product submanifolds in Sasakian and Kenmotsu manifolds in terms of the warping function for a contact CR-warped submanifolds isometrically immersed in a contact metric manifold.

In [4] , I. Haseqawa and I. Mihai obtained a sharp inequality for the squared norm of the second fundamental form in terms of the warped function for contact CR-warped products in Sasakian manifolds.

In [1] , Arslan and the other authors studied contact CR-warped products in Kenmotsu manifolds. They provided some necessary and sufficient conditions for contact CR-warped product to be totally geodesic, totally umbilical and space form.

In [2], we studied warped product semi-invariant submanifolds in locally Riemannian product manifolds and we gave a necessary and sufficient condition for a warped product to be Riemannian product.

In the present paper, we obtain a sharp estimates for squared norm of the second fundamental form in terms of the warping function for contact CR-warped product submanifolds in Kenmotsu space forms. The equality case is considered and some new results are derived.

For the papers in this subject, we refer to the references.

2. **Preliminaries**

In this section, we give some notations used throughout this paper. We recall some necessary facts and formulas from the theory of Kenmotsu manifolds and their submanifolds.

A $(2m+1)$-dimensional Riemannian manifold $(\bar{M}, g)$ is said to be an almost contact metric manifold if it admits an endomorphism ϕ of its tangent bundle $T\bar{M}$, a vector field ξ and a 1-form η, satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi) \tag{2.2}$$

for any vector fields X, Y tangent to $\bar{M}$. Furthermore, an almost contact metric manifold is called a Kenmotsu manifold if ϕ and ξ satisfy;

$$\begin{aligned} (\bar{\nabla}_X \phi)Y &= g(\phi X, Y)\xi - \eta(Y)\phi X \\ \bar{\nabla}_X \xi &= -\phi^2 X = X - \eta(X)\xi, \end{aligned} \tag{2.3}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$ [7].

Now, let $\bar{M}$ be a $(2n+1)$-dimensional Kenmotsu manifold with structure tensors (ϕ, ξ, η, g) and let M be an m-dimensional isometrically immersed submanifold in $\bar{M}$. Moreover, we denote the Levi-Civita connection on $\bar{M}$ by $\bar{\nabla}$ and let ∇ be the induced connection on M by $\bar{\nabla}$. Then the Gauss and Weingarten formula's for M in $\bar{M}$ are, respectively, given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.4}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.5}$$

for any vector fields X, Y tangent to M and vector V normal to M, where $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$, h and A denote the second fundamental form and shape operator of M in $\bar{M}$, respectively. It is well known that A and h are related by

$$g(h(X,Y),V) = g(A_V X, Y). \tag{2.6}$$

We denote the Riemannian curvature tensors of $\bar{\nabla}$ and the induced connection ∇ by $\bar{R}$ and R, respectively. Then the Gauss and Codazzi equations are, respectively, given by

$$(\bar{R}(X,Y)Z)^{\top} = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \tag{2.7}$$

and

$$(\bar{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \tag{2.8}$$

for any vector fields X, Y, Z tangent to M, where the covariant derivative of h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.9}$$

for any vector fields X, Y, Z tangent to M, and, $(\bar{R}(X,Y)Z)^{\perp}$ and $(\bar{R}(X,Y)Z)^{\top}$ denote the normal and tangent components of $\bar{R}(X,Y)Z$, respectively, with respect to the submanifold M [3].

For any vector field X tangent to M, we set

$$\phi X = fX + \omega X, \tag{2.10}$$

where fX and ωX are the tangential and normal components of ϕX, respectively. Then f is an endomorphism of TM and ω is a normal-bundle valued 1-form of TM. For the same reason, for any vector field V normal to M, we set

$$\phi V = BV + CV, \tag{2.11}$$

where BV and CV are the tangential and normal components of ϕV, respectively. Then B is an endomorphism of the normal bundle $T^{\perp}M$ to TM and C is a normal-bundle valued 1-form of $T^{\perp}M$.

A Kenmotsu manifold with constant ϕ-holomorphic sectional curvature c is called a Kenmotsu space form and is denoted by $\bar{M}(c)$. Then its curvature tensor $\bar{R}$ is expressed by

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ &+ g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \end{aligned} \tag{2.12}$$

for any vector fields X, Y, Z tangent to $\bar{M}$ [6].

3. Contact CR-Warped Product Submanifolds in Kenmotsu Manifolds

In this section, we define contact CR-submanifolds in a Kenmotsu manifold and study their fundamental properties from the view point of theory of submanifolds.

Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and let f be a positive definite smooth function on M_1. We consider the product manifold $M_1 \times M_2$ with its projections $\pi : M_1 \times M_2 \longrightarrow M_1$ and $\eta : M_1 \times M_2 \longrightarrow M_2$. The warped product $M = M_1 \times_f M_2$ is a manifold $M_1 \times M_2$ equipped with the Riemannian metric such that

$$g(X,Y) = g_1(\pi_* X, \pi_* Y) + (fo\pi)^2 g_2(\eta_* X, \eta_* Y),$$

for any $X, Y \in \Gamma(TM)$, where $*$ stands for differential of map and $\Gamma(TM)$ denotes the set of the differentiable vector fields on M. Thus we have $g = g_1 + f^2 g_2$. The function f is called warping function of the warped product manifold $M = M_1 \times_f M_2$. If we denote the Levi Civita connection on M by ∇, then we have the following Proposition for the warped product manifold [3].

Proposition 3.1. *Let $M = M_1 \times_f M_2$ be a warped product manifold. For $X, Y \in \Gamma(TM_1)$ and $Z, V \in \Gamma(TM_2)$, we have*
(1) $\nabla_X Y \in \Gamma(TM_1)$ is the lift of $\nabla_X Y$ on M_1
(2) $\nabla_X V = \nabla_V X = X(\ln f)V$
(3) $nor\nabla_Z V = -g(Z, V) grad \ln f$
(4) $tan\nabla_Z V = \nabla'_Z V \in \Gamma(TM_2)$ is the lift of $\nabla'_Z V$ on M_2, where ∇' denotes the Levi-Civita connection of g_2.

If the warping function f is constant, then the warped product is said to be a Riemannian product.

Let M be a m-dimensional Riemannian manifold with Riemannian metric g and let $\{e_1, e_2, ..., e_m\}$ be an orthonormal basis for $\Gamma(TM)$. For a smooth function f on M, the gradient and Hessian of f are, respectively, defined by

$$Xf = g(gradf, X) \tag{3.1}$$

and

$$H^f(X, Y) = XYf - (\nabla_X Y)f = g(\nabla_X gradf, Y) \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$. The Laplacian of f is defined by

$$\triangle f = \sum_{i=1}^{m} \{(\nabla_{e_i} e_i)f - e_i e_i f\} = -\sum_{i=1}^{m} g(\nabla e_i gradf, e_i). \tag{3.3}$$

From (3.2) and (3.3), it is easily seen that the Laplacian is the negative of the trace of the Hessian.

From the integration theory on manifolds, since M is a compact orientable Riemannian manifold without boundary, we have

$$\int_M \Delta f dV = 0, \tag{3.4}$$

where dV is the volume element of M [2].

Definition 3.2. *Let M be an isometrically immersed submanifold of a Kenmotsu manifold $\bar{M}$ such that M is tangent to ξ. Then M is said to be a contact CR-submanifold if there exists a pair of orthogonal differentiable distributions such as D and $D^{\perp}$ on M such that*
1.) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where $\{\xi\}$ is the 1-dimesional distribution spanned by ξ.
2.) D is an invariant distribution with respect to ϕ, that is, $\phi D_p \subset D_p$, for all $p \in M$,
3.) $D^{\perp}$ is an anti-invariant distribution with respect to ϕ, that is, $\phi D_p^{\perp} \subset T_p^{\perp}M$, for all $p \in M$ [1].

In this paper, we consider warped product manifolds which are of the form $M = M_T \times_f M_{\perp}$ in a Kenmotsu manifold $\bar{M}$ such that M is tangent to ξ, where M_T is an invariant submanifold and $M_{\perp}$ is an anti-invariant submanifold of $\bar{M}$ and we call it contact CR-warped product, where M_T and $M_{\perp}$ denote the integral manifolds of the distributions of D and $D^{\perp}$, respectively, in Definition 3.2.

We give an example of contact CR-warped product submanifold of type $M = M_T \times_f M_{\perp}$ in Kenmotsu manifold with ξ tangent to M_T.

Example 3.3. *Let $\mathbb{R}^9 = \mathbb{C}^4 \times \mathbb{R}$ be the 9-dimensional Euclidean space endowed with the almost contact metric structure (ϕ, ξ, η, g) defined by*

$$\phi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, t) = (-x_5, -x_6, -x_7, -x_8, x_1, x_2, x_3, x_4, 0),$$
$$\xi = e^t \frac{\partial}{\partial t}, \quad \eta = e^t dt, \quad g = e^{2t} <, >,$$

where $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, t)$ and $<, >$ denote the cartesian coordinates and the Euclidean metric tensor of $\mathbb{R}^9$, respectively. It is well known that $\mathbb{R}^9$ is a Kenmotsu manifold.

Now, we define a submanifold by

$$M = \{(x_1, 0, x_3, 0, x_2, 0, 0, x_4, t) \in \mathbb{R}^9\}$$

and choose a frame $\{e_1, e_2, e_3, e_4, e_5\}$ of orthogonal vector fields on M as

$$e_1 = \frac{\partial}{\partial x_1} + x_5 \frac{\partial}{\partial t}, e_2 = \frac{\partial}{\partial x_5}, e_3 = \frac{\partial}{\partial x_3} + x_7 \frac{\partial}{\partial t}, e_4 = \frac{\partial}{\partial x_8}, e_5 = \xi = \frac{\partial}{\partial t}.$$

Then it is easy to observe that $D_T = span\{e_1, e_2, e_5\}$ and $D^{\perp} = span\{e_3, e_4\}$ define the invariant and anti-invariant differentiable distributions on

Kenmotsu manifold $\mathbb{R}^9$. *If we denote the integral manifolds of* D_T *and* $D^\perp$ *by* M_T *and* $M_\perp$, *respectively, then it is easy to check that* $M = M_T \times_f M_\perp$ *is a contact CR-warped product submanifold with warping function* $f(t) = e^t$.

4. Two Theorems For Contact CR Warped Product Submanifolds

In this section, we give the main results of this paper. Firstly, we give the following two lemmas and a theorem for later use.

Lemma 4.1. *Let* $M = M_T \times_f M_\perp$ *be a contact CR-warped product submanifold of a Kenmotsu manifold* $\bar{M}$. *Then we have*

$$g(h(X,Y),\phi Y) = -\|Y\|^2 \phi X \ln f \tag{4.1}$$

and

$$g(h(\phi X,Y),\phi Y) = \|Y\|^2 X \ln f \tag{4.2}$$

for any $X \in \Gamma(TM_T)$ *and* $Y \in \Gamma(TM_\perp)$.

Proof. For any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$, using (2.3), (2.4) and considering Proposition 3.1 (2), we have

$$\begin{aligned} g(h(X,Y),\phi Y) &= g(\bar{\nabla}_Y X, \phi Y) = -g(\phi \bar{\nabla}_Y X, Y) \\ &= -g(\bar{\nabla}_Y \phi X - (\bar{\nabla}_Y \phi)X, Y) \\ &= -g(\nabla_Y \phi X, Y) + g(g(\phi Y, X)\xi - \eta(X)\phi Y, Y) \\ &= -g(Y,Y)\phi X \ln f \end{aligned}$$

and

$$\begin{aligned} g(h(\phi X,Y),\phi Y) &= g(\bar{\nabla}_Y \phi X, \phi Y) = g((\bar{\nabla}_Y \phi)X + \phi \bar{\nabla}_Y X, \phi Y) \\ &= g(g(\phi Y, X)\xi - \eta(X)\phi Y, \phi Y) + g(\phi \bar{\nabla}_Y X, \phi Y) \\ &= g(\phi h(X,Y) + \phi \nabla_Y X, \phi Y) = g(\phi Y, \phi Y) X \ln f \\ &= g(Y,Y) X \ln f \end{aligned}$$

which prove our assertions. □

Lemma 4.2. *Let* $M = M_T \times_f M_\perp$ *be a contact CR-warped product submanifold of a Kenmotsu manifold* $\bar{M}$. *Then we have*

$$\|h(X,Y)\|^2 = g(h(\phi X,Y), \phi h(X,Y)) + g(Y,Y)(\phi X \ln f)^2, \tag{4.3}$$

for any $X \in \Gamma(TM_T)$ *and* $Y \in \Gamma(TM_\perp)$.

Proof. Making use of (2.3), (2.4) and considering Proposition 3.1 and Lemma 4.1 we have

$$\begin{aligned}
g(h(\phi X,Y),\phi h(X,Y)) &= g(\phi h(X,Y),\bar{\nabla}_Y\phi X-\nabla_Y\phi X)\\
&= g(\phi h(X,Y),(\bar{\nabla}_Y\phi)X+\phi\bar{\nabla}_Y X-(\phi X\ln f)Y)\\
&= g(\phi h(X,Y),g(\phi Y,X)\xi-\eta(X)\phi Y+\phi\bar{\nabla}_Y X)\\
&+ g(h(X,Y),\phi X\ln f\phi Y)\\
&= g(h(X,Y),\bar{\nabla}_Y X)+g(h(X,Y),\phi Y)\phi X\ln f\\
&= g(h(X,Y),h(X,Y))+g(\bar{\nabla}_Y X,\phi Y)\phi X\ln f\\
&= \|h(X,Y)\|^2-g(\phi\bar{\nabla}_Y X,Y)\phi X\ln f\\
&= \|h(X,Y)\|^2-g(\bar{\nabla}_Y\phi X-(\bar{\nabla}_Y\phi)X,Y)\phi X\ln f\\
&= \|h(X,Y)\|^2-g(\nabla_Y\phi X-g(\phi Y,X)\xi\\
&+ \eta(X)\phi Y,Y)\phi X\ln f\\
&= \|h(X,Y)\|^2-g((\phi X\ln f)Y,Y)\phi X\ln f\\
&= \|h(X,Y)\|^2-g(Y,Y)(\phi X\ln f)^2.
\end{aligned}$$

This completes the proof of the lemma. □

Theorem 4.3. *Let $M=M_T\times_f M_\perp$ be a contact CR-warped product submanifold of a Kenmotsu space form $\bar{M}(c)$. Then we have*

$$\begin{aligned}
2\|h(X,Y)\|^2 &= \{H^{\ln f}(X,X)+H^{\ln f}(\phi X,\phi X)+(\frac{c-3}{2})g(X,X)\\
&+ 2(\phi X\ln f)^2\}g(Y,Y),
\end{aligned}\tag{4.4}$$

for any $X\in\Gamma(TM_T)$ and $Y\in\Gamma(TM_\perp)$.

Proof. Using (2.8), (2.9) and considering $\bar{\nabla}$, the Levi-Civita connection, we have

$$\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) &= g((\bar{\nabla}_X h)(\phi X,Y)-(\bar{\nabla}_{\phi X}h)(X,Y),\phi Y)\\
&= g(\bar{\nabla}_X h(\phi X,Y)-h(\nabla_X\phi X,Y)-h(\nabla_X Y,\phi X),\phi Y)\\
&- g(\bar{\nabla}_{\phi X}h(X,Y)-h(\nabla_{\phi X}X,Y)-h(\nabla_{\phi X}Y,X),\phi Y)\\
&= Xg(h(\phi X,Y),\phi Y)-g(\bar{\nabla}_X\phi Y,h(\phi X,Y))\\
&- g(h(\nabla_X\phi X,Y),\phi Y)-g(h(\nabla_X Y,\phi X),\phi Y)\\
&- \phi Xg(h(X,Y),\phi Y)+g(h(X,Y),\bar{\nabla}_{\phi X}\phi Y)\\
&+ g(h(\nabla_{\phi X}X,Y),\phi Y)+g(h(\nabla_{\phi X}Y,X),\phi Y).
\end{aligned}$$

On the other hand, considering Lemma 4.1 and Proposition 3.1, we have

$$\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) &= X[(X\ln f g(Y,Y)] - g(h(\phi X,Y),(\bar{\nabla}_X\phi)Y \\
&+ \phi(\bar{\nabla}_X Y)) - g(h(\nabla_X\phi X,Y),\phi Y) \\
&- X\ln f g(h(\phi X,Y),\phi Y) - \phi X[-\phi X\ln f g(Y,Y)] \\
&+ g(h(X,Y),(\bar{\nabla}_{\phi X}\phi)Y + \phi(\bar{\nabla}_{\phi X}Y)) \\
&+ g(h(\nabla_{\phi X}X,Y),\phi Y) + \phi X\ln f g(h(X,Y),\phi Y) \\
&= X(X\ln f)g(Y,Y) + 2g(\nabla_X Y,Y)X\ln f \\
&- g(h(\phi X,Y),\phi h(X,Y)) - g(h(\phi X,Y),\phi\nabla_X Y) \\
&+ \phi\nabla_X\phi X\ln f g(Y,Y) - X\ln f(X\ln f)g(Y,Y) \\
&+ \phi X(\phi X\ln f)g(Y,Y) + 2\phi X\ln f g(\nabla_{\phi X}Y,Y) \\
&+ g(h(X,Y),\phi h(\phi X,Y)) + g(h(X,Y),\phi X\ln f\phi Y) \\
&- \phi\nabla_{\phi X}X\ln f g(Y,Y) + \phi X\ln f(-\phi X\ln f)g(Y,Y) \\
&= X(X\ln f)g(Y,Y) + 2X\ln f(X\ln f)g(Y,Y) \\
&- 2g(h(\phi X,Y),\phi h(X,Y)) - X\ln f g(h(\phi X,Y),\phi Y) \\
&+ \phi\nabla_X\phi X\ln f g(Y,Y) - \phi\nabla_{\phi X}X\ln f g(Y,Y) \\
&+ \phi X(\phi X\ln f)g(Y,Y) + 2\phi X\ln f(\phi X\ln f)g(Y,Y) \\
&- (X\ln f)^2 g(Y,Y) - 2(\phi X\ln f)^2 g(Y,Y).
\end{aligned}$$

Summing up, we conclude

$$\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) &= X(X\ln f)g(Y,Y) - 2g(h(\phi X,Y),\phi h(X,Y)) \\
&+ \phi\nabla_X\phi X\ln f g(Y,Y) - \phi\nabla_{\phi X}X\ln f g(Y,Y) \\
&+ \phi X(\phi X\ln f)g(Y,Y) \\
&= \{X(X\ln f) + \phi X(\phi X\ln f) + \phi\nabla_X\phi X\ln f \\
&- \phi\nabla_{\phi X}X\ln f\}g(Y,Y) - 2g(h(\phi X,Y),\phi h(X,Y)). \qquad (4.5)
\end{aligned}$$

From (4.5) and Lemma 4.2, we arrive to

$$\begin{aligned}
g(\bar{R}(X,\phi X)Y,\phi Y) = \{&X(X\ln f) + \phi X(\phi X\ln f) + \phi\nabla_X\phi X\ln f \\
&- \phi\nabla_{\phi X}X\ln f + 2(\phi X\ln f)^2\}g(Y,Y) - 2\|h(X,Y)\|^2.
\end{aligned}$$

On the other hand, considering Proposition 3.1, M_T is totally geodesic in M and $grad \ln f \in \Gamma(TM_T)$. Thus by direct calculations, we have

$$\begin{aligned}
\phi\nabla_X\phi X \ln f &= g(\phi\nabla_X\phi X, grad \ln f) = -g(\nabla_X\phi X, \phi grad \ln f) \\
&= -g(\bar{\nabla}_X\phi X, \phi grad \ln f) = -g((\bar{\nabla}_X\phi)X, \phi grad \ln f) \\
&- g(\phi\bar{\nabla}_X X, \phi grad \ln f) \\
&= -g(g(\phi X, X)\xi - \eta(X)\phi X, \phi grad \ln f) \\
&- g(\phi\nabla_X X, \phi grad \ln f) \\
&= \eta(X)g(\phi X, \phi grad \ln f) + g(\nabla_X X, \phi^2 grad \ln f) \\
&= g(\nabla_X X, -grad \ln f + \eta(grad \ln f)\xi) = -\nabla_X X \ln f \\
&- \xi \ln f g(\nabla_X\xi, X) \\
&= -\nabla_X X \ln f - g(X,X)\xi \ln f. \qquad (4.6)
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
\phi\nabla_{\phi X} X \ln f &= g(\phi\bar{\nabla}_{\phi X} X, grad \ln f) \\
&= g(\bar{\nabla}_{\phi X}\phi X - (\bar{\nabla}_{\phi X}\phi)X, grad \ln f) \\
&= g(\bar{\nabla}_{\phi X}\phi X, grad \ln f) \\
&- g(g(\phi^2 X, X)\xi - \eta(X)\phi^2 X, grad \ln f) \\
&= \nabla_{\phi X}\phi X \ln f + g(\phi X, \phi X)g(\xi, grad \ln f) \\
&+ \eta(X)g(\phi^2 X, grad \ln f) \\
&= \nabla_{\phi X}\phi X \ln f + g(\phi X, \phi X)\xi \ln f \\
&= \nabla_{\phi X}\phi X \ln f + g(X,X)\xi \ln f. \qquad (4.7)
\end{aligned}$$

In [1] , it was proved $\xi \ln f = 1$. So by substituting (4.6) and (4.7) into (4.5), we find

$$\begin{aligned}
g(\bar{R}(X,\phi X)Y, \phi Y) &= \{X(X \ln f) + \phi X(\phi X \ln f) - \nabla_X X \ln f \\
&- \nabla_{\phi X}\phi X - 2g(X,X) + 2(\phi X \ln f)^2\}g(Y,Y) \\
&- 2\|h(X,Y)\|^2 \\
&= \{H^{\ln f}(X,X) + H^{\ln f}(\phi X, \phi X) \\
&- 2g(X,X) + 2(\phi X \ln f)^2\}g(Y,Y) - 2\|h(X,Y)\|^2. \qquad (4.8)
\end{aligned}$$

Moreover, using (2.12), we get

$$\begin{aligned}
g(\bar{R}(X,\phi X)Y, \phi Y) &= -\frac{c+1}{2}g(\phi X, \phi X)g(Y,Y) \\
&= -\frac{c+1}{2}g(X,X)g(Y,Y), \qquad (4.9)
\end{aligned}$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$. Finally, from (4.8) and (4.9) we conclude that

$$
\begin{aligned}
-(\frac{c+1}{2})g(X,X)g(Y,Y) &= \{H^{\ln f}(X,X) + H^{\ln f}(\phi X, \phi X) \\
&- 2g(X,X) + 2(\phi X \ln f)^2\}g(Y,Y) \\
&- 2\|h(X,Y)\|^2, \qquad (4.10)
\end{aligned}
$$

which is equivalent to (4.4). □

Now, let $\{e_o = \xi, e_1, e_2, ..., e_p, \phi e_1, \phi e_2, ..., \phi e_p, e^1, e^2, ..., e^q\}$ be an orthonormal basis of $\Gamma(TM)$ such that $e_o, e_1, e_2, ..., e_p, \phi e_1, \phi e_2, ..., \phi e_p$ are tangent to $\Gamma(TM_T)$ and $e^1, e^2, ..., e^q$ are tangent to $\Gamma(TM_\perp)$. Moreover, let
$\{\phi e^1, \phi e^2, ..., \phi e^q, N_1, N_2, ..., N_{2r}\}$ be an orthonormal basis of $\Gamma(TM^\perp)$ such that $\{\phi e^1, \phi e^2, ..., \phi e^q\}$ are tangent to $\phi TM_\perp$ and $\{N_1, N_2, ..., N_{2r}\}$ are tangent to $\Gamma(\nu)$, where ν denote the orthogonal distribution of $\phi D^\perp$ in $T^\perp M$.

Now we can give the main theorem of this paper.

Theorem 4.4. *Let $M = M_T \times_f M_\perp$ be a compact orientable contact CR-warped product submanifold of a Kenmotsu space form $\bar{M}(c)$. Then M is a contact CR-product if*

$$
\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_2(e_i, e^j)\|^2 \geq (\frac{c-3}{4})pq, \qquad (4.11)
$$

where h_2 denotes the component of h in $\Gamma(\nu)$.

Proof. Using (3.3), the Laplacian of $\ln f$ is given by

$$
\begin{aligned}
-\Delta \ln f &= \sum_{i=1}^{p} g(\nabla_{e_i} grad \ln f, e_i) + \sum_{i=1}^{p} g(\nabla_{\phi e_i} grad \ln f, \phi e_i) \\
&+ \sum_{j=1}^{q} (\nabla_{e^j} grad \ln f, e^j) + g(\nabla_\xi grad \ln f, \xi).
\end{aligned}
$$

Here, considering ∇ to be the Levi-Civita connection, M_T is totally geodesic in M. Hence $grad \ln f \in \Gamma(TM_T)$ and by Proposition 3.1, we have

$$\begin{aligned}
-\Delta\ln f &= \sum_{i=1}^{p}\{H^{\ln f}(e_i,e_i)+H^{\ln f}(\phi e_i,\phi e_i)\} \\
&+ \sum_{j=1}^{q}\{(e^j g(grad\ln f,e^j)-g(\nabla_{e^j}e^j,grad\ln f)\}) \\
&+ g(\nabla_\xi grad\ln f,\xi) \\
&= \sum_{i=1}^{p}\{H^{\ln f}(e_i,e_i)+H^{\ln f}(\phi e_i,\phi e_i)\} \\
&- \sum_{j=1}^{q}\{-g(e^j,e^j)g(grad\ln f,grad\ln f)\}+\xi g(grad\ln f,\xi) \\
&- g(\nabla_\xi\xi,grad\ln f) \\
&= \sum_{i=1}^{p}\{H^{\ln f}(e_i,e_i)+H^{\ln f}(\phi e_i,\phi e_i)\}+q\|grad\ln f\|^2. \qquad (4.12)
\end{aligned}$$

In (4.10) let $X=e_i$ and $Y=e^j$, $1\le i\le p$ and $1\le j\le q$. Then by direct calculations, we have

$$\begin{aligned}
-(\frac{c+1}{2})pq &= \{\sum_{i=1}^{p}\{H^{\ln f}(e_i,e_i)+H^{\ln f}(\phi e_i,\phi e_i)\}-2p \\
&+ 2\sum_{i=1}^{p}(\phi e_i\ln f)^2\}q-2\sum_{i=1}^{p}\sum_{j=1}^{q}\|h(e_i,e^j)\|^2.
\end{aligned}$$

From (4.12) and the last equation, we arrive at

$$\begin{aligned}
-(\frac{c+1}{2})pq &= \{-\Delta\ln f-q\|grad\ln f\|^2-2p+2\sum_{i=1}^{p}(\phi e_i\ln f)^2\}q \\
&- 2\sum_{i=1}^{p}\sum_{j=1}^{q}\|h(e_i,e^j)\|^2,
\end{aligned}$$

that is,

$$\begin{aligned} \Delta \ln f &= 2\sum_{i=1}^{p}(\phi e_i \ln f)^2 + (\frac{c-3}{2})p - q\|grad \ln f\|^2 \\ &- \frac{2}{q}\sum_{i=1}^{p}\sum_{j=1}^{q}\|h(e_i, e^j)\|^2. \end{aligned} \tag{4.13}$$

Furthermore, from linear algebra rules, we know that h can be written as

$$h(e_i, e^j) = \sum_{k=1}^{p} g(h(e_i, e^j), \phi e^k)\phi e^k + \sum_{\ell=1}^{2r} g(h(e_i, e^j), N_\ell)N_\ell.$$

Also, making use of (4.1), we have

$$\begin{aligned} \sum_{i=1}^{p}\sum_{j=1}^{q} g(h(e_i, e^j), h(e_i, e^j)) &= \sum_{i=1}^{p}\sum_{k,j=1}^{q} g(h(e_i, e^j), \phi e^k)^2 \\ &+ \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{\ell=1}^{2r} g(h(e_i, e^j), N_\ell)^2 \\ &= q\sum_{i=1}^{p}(\phi e_i \ln f)^2 + \sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i, e^j)\|^2. \end{aligned} \tag{4.14}$$

Finally, substituting (4.14) into (4.13), we get

$$-q\Delta \ln f = 2\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i, e^j)\|^2 - (\frac{c-3}{2})pq + q^2\|grad \ln f\|^2.$$

From (3.4), we conclude that

$$\begin{aligned} \int_M \{2\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i, e^j)\|^2 &- (\frac{c-3}{2})pq \\ &+ q^2\|grad \ln f\|^2\} dV = 0. \end{aligned} \tag{4.15}$$

Here, if

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i, e^j)\|^2 \geq (\frac{c-3}{4})pq,$$

which implies $\|grad \ln f\| = 0$ because $q \neq 0$, that is, the warping function f is constant. So contact CR-warped product becomes a contact CR-product. □

From the integral formula (4.15) we derive the following corollaries.

Corollary 4.5. *Let $M = M_T \times_f M_\perp$ be a compact orientable contact CR-warped product submanifold of a Kenmotsu space form $\bar{M}(c)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\|h_2(e_i, e^j)\|^2 = (\frac{c-3}{4})pq. \tag{4.16}$$

Proof. If (4.16) is satisfied, then (4.15) implies that $f = constant$, that is, M is a contact CR-product.

Conversely, if M is a contact CR-product, from Lemma 4.1 we know that $h(X, Y) \in \Gamma(\nu)$, for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$. So the equality (4.16) is satisfied □

Corollary 4.6. *There exist no compact orientable contact CR-warped products in Kenmotsu space forms $\bar{M}(c)$ such that $c < 3$.*

Corollary 4.7. *There exist no compact orientable contact CR-warped products in $\mathbb{R}^{2n+1}$ with a usual almost contact metric structure $(\phi, \xi, \eta, <,>)$.*

Acknowledgments

I would like to express my gratitude to the referees for valuable comments and suggestions.

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