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ON NILPOTENT AND SOLVABLE POLYGROUPS

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Abstract. Applications of hypergroups have mainly appeared in special subclasses. One of the important subclasses is the class of polygroups. In this paper, we study the notions of nilpotent and solvable polygroups by using the notion of heart of a polygroup. In particular, we give a necessary and sufficient condition between nilpotency (solvability) of polygroups and fundamental groups.

1. **Introduction**

The concept of a hypergroup which is based on the notion of hyperoperation was introduced by Marty in [21] and studied extensively by many mathematicians. Hypergroup theory extends some well-known results in group theory and introduces new topics leading to a wide variety of applications, as well as to broadening of the fields of investigation. Surveys of the theory can be found in the books of Corsini [5], Davvaz and Leoreanu-Fotea [12], Corsini and Leoreanu [6] and Vougiouklis [24]. Since the beginning, several relations have been considered in groupoids and hyperstructures such as $\mathcal{A}$, $\mathcal{A}_n$, β, β_n, γ and γ_n, see Koskas [18], Corsini [5, 7], Davvaz [9–11, 17], Freni [14–16], Leoreanu [19], Migliorato [22], Vougiouklis [24, 25] and others. We recall here some basic notions of hypergroup theory. Applications of hypergroups have mainly

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appeared in special subclasses. For example, polygroups which form an important subclass of hypergroups were studied by Comer [3,4]. Quasi-canonical hypergroups (called "polygroups" by Comer) were introduced for the first time in [2], as a generalization of canonical hypergroups, introduced in [23].

Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. Let $\cdot$ be a *hyperoperation* (or *join operation*) on H, that is, $\cdot$ is a function from $H \times H$ into $\mathcal{P}^*(H)$. If $(a, b) \in H \times H$, its image under $\cdot$ in $\mathcal{P}^*(H)$ is denoted by $a \cdot b$. The join operation is extended to subsets of H in a natural way, that is, for non-empty subsets A, B of H, $A \cdot B = \cup\{a \cdot b \mid a \in A, b \in B\}$. The notation $a \cdot A$ is used for $\{a\} \cdot A$ and $A \cdot a$ for $A \cdot \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \cdot)$ is called a *semi-hypergroup* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in H$. Let $(H, \cdot)$ be a semi-hypergroup and let A be a nonempty subset of H. We say that A is a *complete part* of H if for any nonzero natural number n and for all $a_1, \ldots, a_n$ of H, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \Rightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

Let A be a nonempty part of H. The intersection of the parts of H which are complete and contain A is called the *complete closure* of A in H, it will be denoted by $C(A)$. A semi-hypergroup is a *hypergroup* if $a \cdot H = H \cdot a = H$ for all $a \in H$. A nonempty subset K of a hypergroup $(H, \cdot)$ is called a subhypergroup if it is a hypergroup. An element e of H is called an *identity element* if, for all $x \in H$, $x \in x \cdot e \cap e \cdot x$ and $a' \in H$ is called an *inverse* of a in H, with respect to e, if $e \in a \cdot a' \cap a' \cdot a$. Suppose that $(H, \cdot)$ and $(H', \circ)$ are two semi-hypergroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \circ f(b)$ for all a and b in H. We say that f is a *good homomorphism* if for all a and b in H, $f(a \cdot b) = f(a) \circ f(b)$.

If $(H, \cdot)$ is a hypergroup and $\rho \subseteq H \times H$ is an equivalence relation, we set

$$A \overline{\overline{\rho}} B \Leftrightarrow a \, \rho \, b, \quad \forall a \in A, \forall b \in B,$$

for all pairs (A, B) of non-empty subsets of H^2.

The relation ρ is called *strongly regular on the left* (*on the right*) if $x \, \rho \, y \Rightarrow a{\cdot}x \, \overline{\overline{\rho}} \, a{\cdot}y$ ($x \, \rho \, y \Rightarrow x{\cdot}a \, \overline{\overline{\rho}} \, y{\cdot}a$, respectively), for all $(x, y, a) \in H^3$. Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 1.1. *(Theorem 31, [5]). If $(H,\cdot)$ is a semi-hypergroup (hypergroup) and ρ is a strongly regular relation on H, then the quotient H/ρ is a semigroup (group) under the operation $\rho(x)\circ\rho(y)=\rho(z)$, for all $z\in x\cdot y$.*

We denote $\rho(x)$ by $\bar{x}$ and instead of $\bar{x}\circ\bar{y}$ we write $\bar{x}\bar{y}$.

For all $n>1$, we define the relation β_n on a semi-hypergroup H, as follows:

$$a\,\beta_n\,b \Leftrightarrow \exists(x_1,\ldots,x_n)\in H^n\ :\ \{a,b\}\subseteq \prod_{i=1}^{n} x_i,$$

and we set $\beta=\bigcup_{i=1}^{n}\beta_n$, where $\beta_1=\{(x,x)\mid x\in H\}$ is the diagonal relation on H. This relation was introduced by Koskas [18] and studied mainly by Corsini [5]. Suppose that β^* is the transitive closure of β. The relation β^* is a strongly regular relation [5]. The relation β^* is the smallest equivalence relation on a hypergroup H, such that the quotient H/β^* is a group. The *heart* ω_H of a hypergroup H is the set of all elements x of H, for which the equivalence class $\beta^*(x)$ is the identity of the group H/β^*, i.e., if $\varphi: H\longrightarrow H/\beta^*$ is the canonical map, then $\omega_H=\{x\in H\mid \varphi(x)=1_{H/\beta^*}\}$.

2. **Nilpotent Polygroups**

In this section, we introduce and analyze the definition of nilpotent polygroup P and we present some results about this new concept.

A *polygroup* is a system $\wp=<P,\cdot,e,^{-1}>$, where $e\in P$, $^{-1}$ is a unitary operation on P, $\cdot$ maps $P\times P$ into the non-empty subsets of P, and the following axioms hold for all x,y,z in P:

(P_1) $(x\cdot y)\cdot z=x\cdot(y\cdot z)$,
(P_2) $e\cdot x=x\cdot e=x$,
(P_3) $x\in y\cdot z$ implies $y\in x\cdot z^{-1}$ and $z\in y^{-1}\cdot x$.

The following elementary facts about polygroups follow easily from the axioms: $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$, $e^{-1}=e$, $(x^{-1})^{-1}=x$, and $(x\cdot y)^{-1}=y^{-1}\cdot x^{-1}$. Denote $A^{-1}=\{a^{-1}|\ a\in A\}$. A nonempty subset K of a polygroup $\langle P,\cdot,e,^{-1}\rangle$ is a subpolygroup of P if (1) $x,y\in K$ implies $x\cdot y\in K$; (2) $x\in K$ implies $x^{-1}\in K$. A subpolygroup N of a polygroup $\langle P,\cdot,e,^{-1}\rangle$ is normal in P if $x^{-1}\cdot N\cdot x\subseteq N$, for all $x\in P$. In the following we introduce a new construction of polygroups from groups.

Let $(G,\cdot)$ be a group and $P_G=G\cup\{a\}$, where $a\notin G$. We define on P_G the hyperoperations $\circ$ as follows:

(1) $a \circ a = e$;
(2) $e \circ x = x \circ e = x$, for every $x \in P_G$;
(3) $a \circ x = x \circ a = x$, for every $x \in P_G - \{e, a\}$;
(4) $x \circ y = x \cdot y$, for every $(x, y) \in G^2$ such that $y \neq x^{-1}$;
(5) $x \circ x^{-1} = \{e, a\}$, for every $x \in P_G - \{e, a\}$.

Proposition 2.1. *If G is a group, then $\langle P_G, \circ, e, ^{-1} \rangle$ is a polygroup.*

Proof. First of all, we prove the associativity of $\circ$. Suppose that $(x, y, z) \in P_G^3$.

(i) If $\{x, y, z\} \cap \{e, a\} = \emptyset$, then we have the following two cases.
Case 1. $x \neq y^{-1} \neq z$ and $x \neq z^{-1}$. In this case $(x \circ y) \circ z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \circ (y \circ z)$.
Case 2. There exists $\{u, v\} \subseteq \{x, y, z\}$ such that $u = v^{-1}$. Without loss of generality suppose that $x = u, y = v$. Thus, $(x \circ y) \circ z = \{e, a\} \circ z$. Hence, $\{e, a\} \circ z = z$. On the other hand, if $y = z^{-1}$, then $x \circ (y \circ z) = x \circ \{e, a\} = x = y^{-1} = z$ and if $y \neq z^{-1}$ we have $x \circ (y \circ z) = x \circ (y \cdot z) = x \cdot (y \cdot z) = (x \cdot x^{-1}) \cdot z = e \cdot z = z$.

(ii) If $\{x, y, z\} \cap \{e, a\} \neq \emptyset$. Let $e \in \{x, y, z\}$. It is easy to see that the associativity condition holds. Now suppose that $\{x, y, z\} \cap \{e, a\} = \{a\}$. Without loss of generality let $x = a$. In this case we have

$$(x \circ y) \circ z = x \circ (y \circ z) = \begin{cases} a & \text{if } y = a, z = a \\ z & \text{if } y = a, z \neq a \\ y & \text{if } y \neq a, z = a \\ y \cdot z & \text{if } y \neq z^{-1}, y \neq a \neq z \\ \{e, a\} & \text{if } y = z^{-1}, y \neq a \neq z. \end{cases}$$

According to the structure of $\circ$ we conclude that e is the identity element of P_G and the other conditions for being polygroup hold too. □

Proposition 2.2. *If G is a group, then $P_G/\beta^* \cong G$.*

Definition 2.3. (See [1]) Let H be a hypergroup. We define

(1) $[x, y]_r = \{h \in H \mid x \cdot y \cap y \cdot x \cdot h \neq \emptyset\}$;
(2) $[x, y]_l = \{h \in H \mid x \cdot y \cap h \cdot y \cdot x \neq \emptyset\}$;
(3) $[x, y] = [x, y]_r \cup [x, y]_l$.

From now on we call $[x, y]_r$, $[x, y]_l$ and $[x, y]$ *right commutator* x and y, *left commutator* x and y and *commutator* x and y, respectively. Also, we will denote by $[H, H]_r$, $[H, H]_l$ and $[H, H]$, the set of all right commutators, left commutators and commutators, respectively.

Proposition 2.4. *If H is a commutative hypergroup, then $[x,y]_r = [x,y]_l = [x,y]$, for all $(x,y) \in H^2$.*

Definition 2.5. Let X be a nonempty subset of a polygroup $\langle P, \cdot, e, ^{-1} \rangle$. Let $\{A_i| \ i \in J\}$ be the family of all subpolygroups of P which contain X. Then, $\cap_{i \in J} A_i$ is called the *subpolygroup generated* by X. This subpolygroup is denoted by $< X >$ and we have $< X >= \cup\{x_1^{\varepsilon_1} \cdot \ldots \cdot x_k^{\varepsilon_k} \mid x_i \in X, k \in \mathbb{N}, \varepsilon_i \in \{-1, 1\}\}$. If $X = \{x_1, x_2, \ldots, x_n\}$. Then, the subpolygroup $< X >$ is denoted $< x_1, x_2, \ldots, x_n >$. In a special case $< [P,P]_r >$, $< [P,P]_l >$ and $< [P,P] >$ are shown by P'_r, P'_l and P', respectively.

Proposition 2.6. *(See [1]) Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup and $(x,y) \in P^2$. Then,*

(1) $[x,y]_r = [x^{-1}, y^{-1}]_l$;
(2) $P' = P'_r = P'_l$;
(3) $x \in P' \Rightarrow x^{-1} \in P'$.

Form now on we call P' the derived subpolygroup of P which is a subpolygroup of P.

Example 2.7. *Suppose that $P = \{e, a, b, c\}$. We consider the non-commutative polygroup $\langle P, \cdot, e, ^{-1} \rangle$, where $\cdot$ is defined on P as fallow:*

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	a	P	c
b	b	e, a, b	b	b, c
c	c	a, c	c	P

In this case we can see that $P' = P$.

Definition 2.8. A polygroup $\langle P, \cdot, e, ^{-1} \rangle$ is said to be *nilpotent* if $\ell_n(P) \subseteq \omega_P$ or equivalently $\ell_n(P) \cdot \omega_P = \omega_P$, for some integer n, where $\ell_0(P) = P$ and

$$\ell_{k+1}(P) = \langle \{h \in P \mid x \cdot y \cap h \cdot y \cdot x \neq \emptyset, \text{ such that } x \in \ell_k(P) \text{ and } y \in P\} \rangle.$$

The smallest integer c such that $\ell_c(P) \cdot \omega_P = \omega_P$ is called the *nilpotency class* or for simplicity the *class* of P.

Notice that $P = \ell_0(P) \supseteq \ell_1(P) \supseteq \ell_2(P) \supseteq \ldots$ that is $\{\ell_k(P)\}_{k \geq o}$ is a decreasing sequence which we call it the *generalized descending central series.*

Proposition 2.9. *Every commutative polygroup is nilpotent of class* 1.

Proof. Suppose that $\langle P, \cdot, e, ^{-1} \rangle$ is a commutative polygroup and $h \in \ell_1(P)$. Then, there exists $(x, y) \in P^2$ such that $x \cdot y \cap h \cdot y \cdot x \neq \emptyset$. Since P is commutative we have $x \cdot y \cap h \cdot x \cdot y \neq \emptyset$. So, $\bar{x}\bar{y} = \bar{h}\bar{x}\bar{y}$ and so $\bar{h} = e$ which means that $h \in \omega_P$. Therefore, $\ell_1(P) \cdot \omega_P = \omega_P$ □

Remark 2.10. The converse of the previous proposition holds for the class of groups but Example 2.7. shows that it is not valid for the class of polygroups.

A polygroup is called *proper* if it not a group.

Proposition 2.11. *Every proper polygroup of order less than* 7 *is nilpotent of class* 1.

Proof. Suppose that $\langle P, \cdot, e, ^{-1} \rangle$ is a polygroup of order less than 7. Then, P/β^* is an abelian group of order less that 6. Now, let $h \in \ell_1(P)$. Then, there exists $(x, y) \in P^2$ such that $x \cdot y \cap h \cdot x \cdot y \neq \emptyset$. Thus, $\bar{x}\bar{y} = \bar{h}\bar{y}\bar{x} = \bar{h}\bar{x}\bar{y}$ which implies that $h \in \omega_P$. Therefore, $\ell_1(P) \subseteq \omega_P$ and consequently $\ell_1(P) \cdot \omega_P = \omega_P$. □

Corollary 2.12. *The symmetric group S_3 is the smallest non-nilpotent polygroup.*

Example 2.13. *Let $P = \{e, a, b, c, d, f, g\}$. We consider the proper non-commutative polygroup $\langle P, \cdot, e, ^{-1} \rangle$, where $\cdot$ is defined on P as follows:*

$\cdot$	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	b	c	d	f	g
b	b	b	e, a	g	f	d	c
c	c	c	f	e, a	g	b	d
d	d	d	g	f	e, a	c	b
f	f	f	c	d	b	g	e, a
g	g	g	d	b	c	e, a	f

It is easy to see that $\omega_P = \{e, a\}$ while $\ell_n(P) = \{e, a, f, g\}$ and hence $\ell_n(P) \cdot \omega_P \neq \omega_P$ for all $n \in \mathbb{N}$. Thus, P is not a nilpotent polygroup of order 7.

Proposition 2.14. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup and consider the fundamental quotient as $G = \frac{P}{\beta^*}$. Then, for all $k \geq 1$*

$$\ell_k(G) = \langle \bar{t} \mid t \in \ell_k(P) \rangle.$$

Proof. Suppose that $\langle P, \cdot, e, ^{-1} \rangle$ is a polygroup and the fundamental quotient as $G = \frac{P}{\beta^*}$. Then, we do the proof by induction on k. For $k = 0$, we have $\langle \bar{t} \mid t \in \ell_0(P) = P \rangle = \ell_0(G)$. Now, suppose that $\bar{a} \in \langle \bar{t} \mid t \in \ell_{k+1}(P) \rangle$. Then, $a \in \ell_{k+1}(P)$ and so there exist $x \in \ell_k(P)$ and $y \in P$ such that $xy \cap ayx \neq \emptyset$. Thus, $\bar{x}\bar{y} = \bar{a}\bar{y}\bar{x}$. By the induction hypothesis we conclude that $\bar{a} \in \ell_{k+1}(G)$. Conversely, let $\bar{a} \in \ell_{k+1}(G)$. Without loss of generality suppose that $\bar{a} = \bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$, where $\bar{x} \in \ell_k(G), \bar{y} \in G$, which implies that $\bar{x}\bar{y} = \bar{a}\bar{y}\bar{x}$. Thus, there exist $c \in xy$ and $d \in ayx$ such that $\bar{c} = \bar{d}$. Since P is a polygroup there exists $u \in P$ such that $c \in xy \cap uyx$. From $\bar{x} \in \ell_k(G), \bar{y} \in G$ and the induction hypothesis we have $x \in \ell_k(P), y \in P$. Thus, $u \in \ell_{k+1}(P)$ and $\bar{a}\bar{y}\bar{x} = \bar{d} = \bar{c} = \bar{x}\bar{y} = \bar{u}\bar{y}\bar{x}$. Therefore, $\bar{a} = \bar{u} \in \langle \bar{t} \mid t \in \ell_{k+1}(P) \rangle$. □

Theorem 2.15. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. Then, P is nilpotent if and only if the fundamental quotient as $G = \frac{P}{\beta^*}$ is nilpotent.*

Proof. Suppose that P is a nilpotent polygroup so there exists $k \in \mathbb{N}$ such that $\ell_k(P) \subseteq \omega_p$. According to the previous proposition, we have $\ell_k(G) = \langle \bar{t} \mid t \in \ell_k(P) \subseteq \omega_p \rangle = \{e_G\} = \omega_G$, and so the fundamental quotient is a nilpotent group. Similarly, we can see the converse. □

Corollary 2.16. *Let G be a group. Then, P_G is nilpotent if and only if G is nilpotent.*

Theorem 2.17. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup and let N be a normal subpolygroup of P. Then, $\ell_n(\frac{P}{N}) = \frac{\ell_n(P) \cdot N}{N}$, for all $n \geq 0$.*

Proof. By induction on n we show that $\ell_n(\frac{P}{N}) \subseteq \frac{\ell_n(P) \cdot N}{N}$ and $\frac{\ell_n(P) \cdot N}{N} \subseteq \ell_n(\frac{P}{N})$. For $n = 0$, the inclusions are obvious. Now, suppose that $yN \in \ell_{n+1}(\frac{P}{N})$. Then, $yN \in [aN, bN]$, where $aN \in \ell_n(\frac{P}{N})$ and $bN \in \frac{P}{N}$. By the induction hypothesis we have $aN = a'N$, where $a' \in \ell_n(P)$. Therefore, $yN = y'N$, where $y' \in [a', b]$. Thus, $yN \in \frac{\ell_{n+1}(P) \cdot N}{N}$. If $yN \in \frac{\ell_{n+1}(P) \cdot N}{N}$, then $yN = y''N$, where $y'' \in \ell_{n+1}(P)$. So, there exist $a \in \ell_n(P)$ and $b \in P$ such that $y'' \in [a, b]$. Hence, $aN = yN \in [aN, bN]$, where $aN \in \ell_n(\frac{P}{N})$ which means that $yN \in \ell_{n+1}(\frac{P}{N})$ and our proof is completed. □

Corollary 2.18. *If $\langle P, \cdot, e, ^{-1} \rangle$ is a nilpotent polygroup, then*

(1) *every subpolygroup of P is nilpotent;*
(2) *if N is a normal subpolygroup of P, then $\frac{P}{N}$ is nilpotent.*

Definition 2.19. Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. We define $Z_0(P) = \omega_P$ and $Z_n(P) = \langle \{x \mid x \cdot y \cdot Z_{n-1}(P) = y \cdot x \cdot Z_{n-1}(P), \forall y \in P\} \rangle$, for all $n \in \mathbb{N}$.

Notice that $\omega_P = Z_0(P) \subseteq Z_1(P) \subseteq Z_2(P) \subseteq \dots$, that is, $\{Z_m(P)\}_{m \geq 0}$ is an increasing sequence which we call it *the generalized ascending central series*. Moreover, $Z_n(P)$ is a subpolygroup of P, for every $n \geq 0$.

Proposition 2.20. *If $\langle P, \cdot, e, ^{-1} \rangle$ is a polygroup and $n \geq 0$, then*

(1) *$Z_n(P)$ is a complete subpolygroup of P;*
(2) *$g \cdot g^{-1} \subseteq Z_n(P)$, for every $g \in P$;*
(3) *$Z_n(P)$ is a normal subpolygroup of P.*

Proof. (1) Since $\omega_P \subseteq Z_n(P)$, we conclude that $C(Z_n(P)) = Z_n(P) \cdot \omega_P = Z_n(P)$, which means that $Z_n(P)$ is complete.

(2) Let $g \in P$. Since $e \in g \cdot g^{-1} \cap Z_n(P)$ and $Z_n(P)$ is complete, $g \cdot g^{-1} \subseteq Z_n(P)$.

(3) Let $g \in P$ be an arbitrary element and $x \in Z_n(P)$. Then, $g \cdot x \cdot g^{-1} \cdot Z_{n-1}(P) = g \cdot g^{-1} \cdot x \cdot Z_{n-1}(P) \subseteq g \cdot g^{-1} \cdot Z_n(P) = Z_n(P)$. Hence, $g \cdot x \cdot g^{-1} \subseteq Z_n(P)$. □

Theorem 2.21. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. Then, P is nilpotent if and only if there exists $r \geq 0$ such that $Z_r(P) = P$.*

Proof. Suppose that there exists $r \geq 0$ such that $Z_r(P) = P$. In order to prove $\ell_r(P) \subseteq \omega_P$, by induction we show that $\ell_i(P) \subseteq Z_{r-i}(P)$. For $i = 0$ we have $\ell_0(P) = P \subseteq P = Z_r(P)$. Now, if $a \in \ell_{i+1}(P)$, then without loss of generality suppose that $x \cdot y \cap a \cdot y \cdot x \neq \emptyset$, where $x \in \ell_i(P)$ and $y \in P$. By the hypothesis of induction we conclude that $x \in Z_{r-i}(P)$. Hence, $x \cdot y \cdot Z_{r-i-1}(P) = y \cdot x \cdot Z_{r-i-1}(P)$ and so $a \in Z_{r-i-1}(P)$. Now if $i = r$, then $\ell_r(P) \subseteq Z_0(P) = \omega_P$. For the converse, suppose that $\ell_r(P) \subseteq \omega_P$. It is enough to show that $\ell_{r-i}(P) \subseteq Z_i(P)$, for all $0 \leq i \leq n$. For $i = 0$, we have $\ell_r(P) \subseteq \omega_P = Z_0(P)$. Let $a \in \ell_{r-i-1}(P)$ and $b \in P$. Then, $[a, b] \subseteq \ell_{r-i}(P)$. By using the induction hypothesis, we have $[a, b] \subseteq Z_i(P)$. Therefore, $a \cdot b \cdot Z_i(P) = b \cdot a \cdot Z_i(P)$ and so $a \in Z_{i+1}(P)$ as we need. If we take $i = r$, then we get our claim. □

Corollary 2.22. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. Then, $\ell_c(P) \subseteq \omega_P$ if and only if $Z_c(P) = P$, that is, P is nilpotent of class c if and only if $Z_c(P) = P$.*

Definition 2.23. Let $\langle P_1, \cdot, e_1, ^{-1} \rangle$ and $\langle P_2, \circ, e_2, ^{-I} \rangle$ be two polygroups. Then, on $P_1 \times P_2$ we can define a hyperproduct as follows: $(x_1, y_1) *$

$(x_2, y_2) = \{(x, y) \mid x \in x_1 \cdot x_2, y \in y_1 \circ y_2\}$. We call this the *direct hyperproduct* of P_1 and P_2. Clearly, $P_1 \times P_2$ equipped with the usual direct hyperproduct becomes a polygroup.

Proposition 2.24. *(See [8]) Let ω_{P_1}, ω_{P_2} and $\omega_{P_1 \times P_2}$ be the hearts of P_1, P_2 and $P_1 \times P_2$, respectively. Then, $\omega_{P_1 \times P_2} = \omega_{P_1} \times \omega_{P_2}$.*

Proposition 2.25. *Let P_1 and P_2 be two polygroups. Then, for all $k \geqslant 0$*

$$\ell_k(P_1 \times P_2) = \ell_k(P_1) \times \ell_k(P_2).$$

Proof. We prove our claim by induction on k. For $k = 0$, it is obvious. Now, suppose that $(a, b) \in \ell_{k+1}(P_1 \times P_2)$. Then, there exist $(u, v) \in \ell_k(P_1 \times P_2)$ and $(s, t) \in P_1 \times P_2$ such that $(u, v) * (s, t) \cap (a, b) * (s, t) * (u, v) \neq \emptyset$, that is, $u \cdot s \cap a \cdot s \cdot u \neq \emptyset$ and $v \circ t \cap b \circ t \circ v \neq \emptyset$. By using the induction hypothesis, we conclude that $a \in \ell_{k+1}(P_1)$ and $b \in \ell_{k+1}(P_2)$. Thus, $(a, b) \in \ell_{k+1}(P_1) \times \ell_{k+1}(P_2)$. Similarly, we obtain the converse. □

Proposition 2.26. *Let P_1, P_2 be two polygroups. Then, $P_1 \times P_2$ is nilpotent if and only if P_1 and P_2 are nilpotent.*

Proof. If P_1, P_2 are nilpotent, then there exist k_1 and k_2 such that $\ell_{k_1}(P_1) \subseteq \omega_{P_1}$ and $\ell_{k_2}(P_2) \subseteq \omega_{P_2}$. Suppose that $k = lcm(k_1, k_2)$. Hence, $\ell_k(P_1) \subseteq \ell_{k_1}(P_1)$ and $\ell_k(P_2) \subseteq \ell_{k_2}(P_2)$ and so we obtain $\ell_k(P_1 \times P_2) = \ell_k(P_1) \times \ell_k(P_2) \subseteq \ell_{k_1}(P_1) \times \ell_{k_2}(P_2) \subseteq \omega_{P_1} \times \omega_{P_2} = \omega_{P_1 \times P_2}$. Conversely, suppose that $P_1 \times P_2$ is nilpotent. Then, there exists k such that $\ell_k(P_1) \times \ell_k(P_2) = \ell_k(P_1 \times P_2) \subseteq \omega_{P_1 \times P_2} = \omega_{P_1} \times \omega_{P_2}$. Hence, $\ell_k(P_1) \subseteq \omega_{P_1}$ and $\ell_k(P_2) \subseteq \omega_{P_2}$. Therefore, P_1 and P_2 are nilpotent. □

Example 2.27. *Let $P_1 = \{e, a, b, c\}$ be the polygroup in Example 2.7. and let $P_2 = \{0, 1\}$ be the cyclic group of order two. Consider the non-commutative polygroup $P \cong P_1 \times P_2$, where $\cdot$ is defined on P as follows:*

$\cdot$	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	e	c	b	f	d	h	g
b	b	c	b	c	e, b, d, g	a, c, f, h	g	h
c	c	b	c	b	a, c, f, h	e, b, d, g	h	g
d	d	f	e, b, d	a, c, f	d	f	d, g	f, h
f	f	d	a, c, f	e, b, d	f	d	f, h	d, g
g	g	h	b, g	c, h	g	h	e, b, d, g	a, c, f, h
h	h	g	c, h	b, g	h	g	a, c, f, h	e, b, d, g

It is easy to see that $\ell_1(P_1) = P_1' = \omega_{P_1}$ *and* $\ell_1(P_2) = \omega_{P_2} = \{0\}$. *Hence,* P *is a nilpotent polygroup of class* 1.

Definition 2.28. (see [5]) Let H be a regular hypergroup. For $n \in \mathbb{N}$, let $a_1, \ldots, a_n$ be elements in H, and let $a_1', \ldots, a_n'$ be their inverses in H, respectively. The set $a_1 a_2 \ldots a_n a_n' a_{n-1}' \ldots a_1'$ is called a *product of type zero.* The union of all products of type 0 is denoted by $N(0)$.

Theorem 2.29. *(see [5]) If* H *is a regular and reversible hypergroup, then the heart is the union of the products of type zero (i.e.,* $\omega_H = N(0)$*).*

Proposition 2.30. *Let* $\langle P_1, \cdot, e_1, ^{-1} \rangle$ *and* $\langle P_2, \circ, e_2, ^{-I} \rangle$ *be two polygroups, and let* $\phi : P_1 \longrightarrow P_2$ *be a good homomorphism. Then, we have*

(1) *if* K_1 *is a subpolygroup of* P_1*, then* $\phi(K_1)$ *is a subpolygroup of* P_2;
(2) *if* K_2 *is a subpolygroup of* P_2*, then* $\phi^{-1}(K_2)$ *is a subpolygroup of* P_1;
(3) *if* ϕ *is one to one and* K_1 *is a nilpotent subpolygroup of* P_1*, then* $\phi(K_1)$ *is a nilpotent subpolygroup of* P_2.

Proof. (1) Suppose that $u, v \in \phi(K_1)$. Then, there exist $x, y \in K_1$ such that $\phi(x) = u$ and $\phi(y) = v$. We have $u \circ v = \phi(x) \circ \phi(y) = \phi(x \cdot y) \subseteq \phi(K_1)$. Now, suppose that $u \in \phi(K_1)$. Then, there exists $x \in K_1$ such that $\phi(x) = u$. We have $u^{-1} = \phi(x)^{-1} = \phi(x^{-1}) \in \phi(K_1)$. Thus, $\phi(K_1)$ is a subpolygroup of P_2.

(2) This is obvious.

(3) By induction on n we show that $\ell_n(\phi(K_1)) = \phi(\ell_n(K_1))$. For $n = 0$ it is obvious. Let $z \in \ell_{n+1}(\phi(K_1))$. Then, there exist $x \in \ell_n(\phi(K_1))$ and $y \in \phi(K_1)$ such that $x \circ y \cap z \circ y \circ x \neq \emptyset$. Since $\{x, y, z\} \subseteq \phi(K_1)$, there exist $a \in \ell_n(\phi(K_1)), b \in K_1$ and $c \in K_1$ such that $\phi(a) = x, \phi(b) = y, \phi(c) = z$. Therefore, we have

$$x \circ y \cap z \circ y \circ x = \phi(a) \circ \phi(b) \cap \phi(c) \circ \phi(b) \circ \phi(a) = \phi(a \cdot b) \cap \phi(c \cdot b \cdot a) \neq \emptyset.$$

Since ϕ is one to one, we conclude that $\phi(a \cdot b \cap c \cdot b \cdot a) \neq \emptyset$, so $a \cdot b \cap c \cdot b \cdot a \neq \emptyset$. By the induction hypothesis we have $c \in \ell_{n+1}(K_1)$. Thus, $z = \phi(c) \in \phi(\ell_{n+1}(K_1)$.

Conversely, let $z \in \phi(\ell_{n+1}(K_1))$. Then, there exists $c \in \ell_{n+1}(K_1)$ such that $\phi(c) = z$. So, from $c \in \ell_{n+1}(K_1)$ we conclude that there exist $a \in \ell_n(K_1)$ and $b \in K_1$ such that $a \cdot b \cap c \cdot b \cdot a \neq \emptyset$. Thus, $\phi(a) \circ \phi(b) \cap \phi(c) \circ \phi(b) \circ \phi(a) \neq \emptyset$. By the induction hypothesis we have $\phi(a) \in \phi(\ell_n(K_1) = \ell_n(\phi(K_1))$ and $\phi(b) \in \phi(K_1)$. So, $z = \phi(c) \in \ell_{n+1}(\phi(K_1))$.

Now, let K_1 be a nilpotent subpolygroup of P_1. Then, there exists m such that $\ell_m(K_1) \subseteq \omega_{K_1}$. By the previous theorem, we have $\ell_m(\phi(K_1)) = \phi(\ell_m(K_1) \subseteq \phi(\omega_{K_1}) \subseteq \omega_{\phi(K_1)}$, and the proof is completed. □

3. On solvable polygroups

In this section, we introduce and probe *solvable polygroups*.

Definition 3.1. A polygroup $\langle P, \cdot, e, ^{-1} \rangle$ is said to be *solvable* if $\imath_n(P) \subseteq \omega_P$ or equivalently $\imath_n(P) \cdot \omega_P = \omega_P$, for some integer n, where $\imath_0(P) = P$ and

$$\imath_{k+1}(P) = \langle \{h \in P \mid x \cdot y \cap h \cdot y \cdot x \neq \emptyset, \text{ such that } x, y \in \imath_k(P)\} \rangle.$$

The smallest integer c such that $\imath_c(P) \cdot \omega_P = \omega_P$ is called the *length* of P. Notice that $\{\imath_k(P)\}_{k \geq o}$ is a decreasing sequence.

Proposition 3.2. *Every commutative polygroup is solvable of length* 1.

Proposition 3.3. *Let P be a polygroup and $G = \frac{P}{\beta^*}$. Then, for all $k \geq 1$*

$$\imath_k(G) = \langle \bar{t} \mid t \in \imath_k(P) \rangle.$$

Proof. Suppose that P is a polygroup and $G = \frac{P}{\beta^*}$. Then, we do the proof by induction on k. For $k = 0$, we have $\langle \bar{t} \mid t \in \imath_0(P) = P \rangle = \imath_0(G)$. Now, suppose that $\bar{a} \in \langle \bar{t} \mid t \in \imath_{k+1}(P) \rangle$. Then, $a \in \imath_{k+1}(P)$ and so there exist $x, y \in \imath_k(P)$ such that $xy \cap ayx \neq \emptyset$. Thus, $\bar{x}\bar{y} = \bar{a}\bar{y}\bar{x}$. By the induction hypothesis we conclude that $\bar{a} \in \imath_{k+1}(G)$. Conversely, let $\bar{a} \in \imath_{k+1}(G)$. Without loss of generality suppose that $\bar{a} = \bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$, where $\bar{x}, \bar{y} \in \imath_k(G)$, which implies that $\bar{x}\bar{y} = \bar{a}\bar{y}\bar{x}$. Thus, there exist $c \in xy$ and $d \in ayx$ such that $\bar{c} = \bar{d}$. Since P is a polygroup, there exists $u \in P$ such that $c \in xy \cap uyx$. By the induction hypothesis we have $x, y \in \imath_k(P)$ which implies that $u \in \imath_{k+1}(P)$ and $\bar{a}\bar{y}\bar{x} = \bar{d} = \bar{c} = \bar{x}\bar{y} = \bar{u}\bar{y}\bar{x}$, so $\bar{a} = \bar{u} \in \langle \bar{t} \mid t \in \imath_{k+1}(P) \rangle$. □

Theorem 3.4. *Let P be a polygroup. Then, P is solvable if and only if $G = \frac{P}{\beta^*}$ is solvable.*

Proof. Suppose that P is a solvable polygroup. Then, there exists $k \in \mathbb{N}$ such that $\imath_k(P) \subseteq \omega_p$. According to the previous proposition, we have $\imath_k(G) = \langle \bar{t} \mid t \in \imath_k(P) \rangle = \{e_G\} = \omega_G$, and so $G = \frac{P}{\beta^*}$ is a nilpotent group. Similarly, we can see the converse. □

Corollary 3.5. *Every nilpotent polygroup is solvable.*

Proposition 3.6. *Every proper polygroup of order less than* 61 *is solvable.*

Proof. Suppose that $\langle P, \cdot, e, ^{-1} \rangle$ is a proper polygroup of order less than 61. Then, P/β^* is a group of order less that 60. Thus, P/β^* is not isomorphic to the smallest non-solvable group A_5 (alternating group of degree 5). Hence, P is solvable. □

In the following example, we introduce one of the smallest proper polygroups of order 61.

Example 3.7. *Let* A_5 *be the alternating group of degree* 5 *and* $P = A_5 \cup \{a\}$*, where* $a \notin A_5$*. We define on* P *the hyperoperations* $\circ$*, as follows:*

(1) $a \circ a = \{e, a\}$;
(2) $e \circ x = x \circ e = x$, *for every* $x \in P$;
(3) $a \circ x = x \circ a = x$, *for every* $x \in P - \{e, a\}$;
(4) $x \circ y = x \cdot y$, *for every* $x, y \in A_5$ *such that* $y \neq x^{-1}$;
(5) $x \circ x^{-1} = \{e, a\}$, *for every* $x \in P - \{e, a\}$.

It is easy to see that $(P, \circ)$ *is a polygroup. Moreover,* $P/\beta^* \cong A_5$ *and hence* P *is not solvable.*

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