Bulletin of the Iranian Mathematical Society Vol. 39 No. 3 (2013), pp 501-505.

# MAXIMUM SUM ELEMENT ORDERS OF ALL PROPER SUBGROUPS OF PGL(2,q)

#### S. M. JAFARIAN AMIRI

Communicated by Jamshid Moori

ABSTRACT. In this paper we show that if q is a power of a prime p, then the projective special linear group PSL(2, q) and the stabilizer of a point of the projective line have maximum sum element orders among all proper subgroups of projective general linear group PGL(2, q) for q odd and even respectively.

### 1. Introduction

Let G be a finite group. Define  $\psi(G) = \sum_{g \in G} o(g)$  where o(g) is the order of g in G. The function  $\psi$  was firstly defined by Amiri, Jafarian Amiri and Isaacs. In [1] authors proved that if G is a noncyclic group of order n (positive integer), then  $\psi(G) < \psi(C_n)$  where  $C_n$  is the cyclic group of order n. In [2] the authors proved that the alternating group  $A_n$  has maximum sum element orders among all proper subgroups of the symmetric group  $S_n$ . In this paper we obtain similar result on all proper subgroups of projective general linear group of dimension 2. Note that if  $H_1$  and  $H_2$  are maximal subgroups of G with  $|H_1| < |H_2|$ , then it is not necessary that  $\psi(H_1) < \psi(H_2)$ . For example consider  $G = C_{12} \times C_3$ ,  $H_1 = C_{12}$  and  $H_2 = C_6 \times C_3$ .

MSC(2010): Primary: 20G40; Secondary: 20E28.

Keywords: Linear groups, maximal subgroups, element order.

Received: 22 September 2011, Accepted: 27 April 2012

<sup>\*</sup>Corresponding author

<sup>© 2013</sup> Iranian Mathematical Society.

<sup>501</sup> 

## 2. Preliminaries

We recall that the group PGL(2, q) is the group of all fractional linear transformations

$$t_{a,b,c,d}: z \rightarrowtail \frac{az+b}{cz+d}$$

of the projective line  $X = \{\infty\} \cup GF(q)$  where  $a, b, c, d \in GF(q)$  with  $ad - bc \neq o$ .

First we quote two structural results which are used to deduce the main results of this paper.

**Proposition 2.1.** (Dickson, [3]) Let G = PGL(2,q) with  $q = p^f > 3$  for some odd prime p. Then the maximal subgroups of G not containing PSL(2,q) are:

- (1)  $C_p^f \rtimes C_{q-1}$ , that is, the stabilizer of a point of the projective line,
- (2)  $D_{2(q-1)}$  for  $q \neq 5$ ,
- (3)  $D_{2(q+1)}$ ,
- (4)  $S_4 \text{ for } q = p \equiv \pm 3 (mod 8),$
- (5)  $PGL(2, q_0)$  for  $q = q_0^r$  with r an odd prime.

**Proposition 2.2.** (Dickson, [3]) Let  $q = 2^f \ge 4$ . Then the maximal subgroup of PSL(2,q) are :

- (1)  $C_2^f \rtimes C_{q-1}$ , that is, the stabilizer of a point of the projective line, (2)  $D_{2(q-1)}$ ,
- (3)  $D_{2(q+1)}$ ,
- (4)  $PGL(2,q_0)$  for  $q = q_0^r$  with r a prime and  $q_0 \neq 2$ .

**Proposition 2.3.** (Huppert, [4]) Let G = PSL(2,q) where q is a ppower (p prime). Then

- (1) a Sylow p-subgroup P of G is an elementary abelian group of order q and the number of Sylow p-subgroup of G is q + 1,
- (2) G contains a cyclic subgroup A of order  $\frac{q-1}{2}$  such that  $N_G(\langle u \rangle)$ is a dihedral group of order q-1 for every nontrivial element  $u \in A$ ,
- (3) G contains a cyclic subgroup Bof order  $\frac{q+1}{2}$  such that  $N_G(\langle u \rangle)$  is a dihedral group of order q + 1 for every nontrivial element  $u \in B$ ,
- (4) the set  $\{P^x, A^x, B^x | x \in G\}$  is a partition of G.

502

### 3. Main results

Lemma 3.1.  $\psi(PSL(2,q)) > (q^2 - 1)(q + 1).$ 

*Proof.* It follows from Proposition 2.3 that

$$\psi(PSL(2,q)) = \frac{q(q+1)}{2}(\psi(C_{\frac{q-1}{2}}) - 1) + \frac{q(q-1)}{2}(\psi(C_{\frac{q+1}{2}}) - 1)$$

$$+(q+1)(q-1)p+1$$

Since  $\psi(C_n) > 2n$  for each positive integer n > 2,

$$\psi(PSL(2,q)) > \frac{q(q+1)}{2}(q-1) + \frac{q(q-1)}{2}(q+1) + (q-1)(q+1) = q(q^2-1) + (q^2-1) = (q^2-1)(q+1).$$

Here we state the main result about the proper subgroups of PGL(2,q) when q is odd.

**Theorem 3.2.** Let G = PGL(2,q) where q is a p-power and p is an odd prime. Then  $\psi(PSL(2,q) > \psi(H))$  for every proper subgroup H of G different from PSL(2,q), the projective special linear group.

*Proof.* We may clearly assume that H is a maximal subgroup of PGL(2, q) since every proper subgroup H of G is contained in a maximal subgroup M of G and  $\psi(H) \leq \psi(M)$ . If q = 3, then  $PGL(2,3) \cong S_4$  and  $PSL(2,3) \cong A_4$ . Therefore the result is valid by [2].

If q > 3, then H is isomorphic with one of the groups (1)-(5)in Proposition 2.1. Therefore we consider the following cases:

**Case 1**: Suppose that  $H = C_p^f \rtimes C_{q-1}$ . Then H is soluble and  $C_{q-1}$  is a Hall subgroup of H. Therefore subgroups of order q-1 are conjugate and each element that its order is a divisor of q-1, lies in one of such conjugates. Since  $C_H(x) = C_p^f$  for every nonidentity  $x \in C_p^f$ , we have

$$H = C_p^f \bigcup (\cup_{y \in C_p^f} K^y)$$

where K is a cyclic subgroup of H of order q - 1. Thus

$$\psi(H) \le (q-1)p + q\psi(C_{q-1}) < (q-1)p + q(q-1)^2$$

< (q-1)(q+1) + q(q-1)(q+1)and so  $\psi(H) < \psi(PSL(2,q))$  by Lemma 3.1.

**Case 2**: Suppose that  $H = D_{2(q-1)}$ . Then we have

$$\psi(H) = \psi(C_{q-1}) + 2(q-1) < (q-1)^2 + 2(q-1)$$

 $= q^2 - 1 < \psi(PSL(2,q)).$ 

**Case 3**: Suppose that  $H = D_{2(q+1)}$ . Then again we have

$$\psi(H) = \psi(C_{q+1}) + 2(q+1) < (q+1)^2 + 2(q+1)$$
  
=  $(q+1)(q+3) < (q+1)(q^2-1) < \psi(PSL(2,q)).$ 

**Case 4**: Suppose that  $H = S_4$ . Since every element of  $S_4$  has order at most 4,  $\psi(S_4) < 24.4 = 96$ . If  $q \ge 5$ , then  $|PSL(2,q)| = \frac{q(q^2-1)}{2} \ge \frac{5.24}{2}$  and so

$$\psi(PSL(2,q)) \ge 2|PSL(2,q)| \ge 2\frac{5.24}{2} > 96 > \psi(S_4).$$

**Case 5**: Suppose that  $H = PGL(2, q_0)$  for  $q = q_0^r$  with r an odd prime. Then the maximum order element in  $PGL(2, q_0)$  is  $q_0+1$ . Therefore

$$\psi(PGL(2,q_0)) < |PGL(2,q_0)|(q_0+1) = q_0(q_0^2-1)(q_0+1)$$
  
$$\leq (q_0^3-1)(q_0^3-1)(q_0^3+1)$$

$$= (q_0^6 - 1)(q_0^3 - 1).$$

Since  $q = q_0^r \ge q_0^3$ , we have

$$\psi(PGL(2,q_0) < (q^2 - 1)(q + 1) < \psi(PSL(2,q))$$

by Lemma 3.1. The proof is complete.

**Lemma 3.3.**  $\psi(C_2^f \rtimes C_{q-1}) \ge 3q^2 - 4q - 1$  where  $2^f = q$ .

Proof. We have

$$\psi(C_2^f \rtimes C_{q-1}) = 2(q-1) + 1 + q\psi((C_{q-1}) - 1)$$
  

$$\geq 2q - 1 + 3q(q-2) = 3q^2 - 4q - 1.$$

For q even we have  $SL(2,q) \cong PSL(2,q)$  and  $GL(2,q) \cong SL(2,q) \times C_{q-1}$ , hence  $PGL(2,q) \cong PSL(2,q)$ . Our main result in this case is :

**Theorem 3.4.** Let G = PSL(2,q) such that  $q = 2^f \ge 4$  for a positive integer f. If H is a proper subgroup of G, then  $\psi(H) \le \psi(C_2^f \rtimes C_{q-1})$ .

504

*Proof.* We may suppose that H is a maximal subgroup of PSL(2, q). By Proposition 2.2 we have following cases :

**Case 1:** If  $H = D_{2(q-1)}$ , then

$$\psi((D_{2(q-1)}) = \psi(C_{q-1}) + 2(q-1) \le (q-1)^2 + 2(q-1) = q^2 - 1 \le 2q^2 + (q^2 - 4q) - 1.$$

Thus the result follows from Lemma 3.3.

**Case 2**: If  $H = D_{2(q+1)}$ , then we have

$$\psi(H) = \psi(C_{q+1}) + 2(q+1) \le q(q+1) - 1 + 2q + 2$$
$$= q^2 + 3q + 1 \le q^2 + 8q - 4q - 1 \le 3q^2 - 4q - 1.$$

It follows from Lemma 3.3 that  $\psi(H) < \psi(C_2^f \rtimes C_{q-1})$ . **Case 3**: If  $H = PGL(2, q_0)$  as in case (3) in Proposition 2.2, then

 $\psi(PGL(2,q_0)) < |PGL(2,q_0)|(q_0+1)$ 

$$= q_0(q_0^2 - 1)(q_0 + 1) \le 2q(q - 1) \le 3q^2 - 4q - 1$$

and the proof is complete by Lemma 3.3.

### References

- H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs, Sums of element orders in finite groups, Comm. Algebra 37 (2009), no. 9, 2978–2980.
- [2] H. Amiri and S. M. Jafarian Amiri, Sum of element orders of maximal subgroups of the symmetric group, *Comm. Algebra* 40 (2012), no. 2, 770–778.
- [3] L. E. Dickson, Linear Groups: With an Exposition of the Galois Field Theory, W. Magnus Dover Publications, Inc., New York, 1958.
- [4] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin-New York, 1967.

#### Seyyed Majid Jafarian Amiri

Department of Mathematics, University of Zanjan, P.O. Box 45195-313, Zanjan, Iran Email: sm\_jafarian@znu.ac.ir, sm\_jaf@yahoo.ca and sm.jafariana110@gmail.com