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ON SOLUBILITY OF GROUPS WITH FINITELY MANY CENTRALIZERS

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ABSTRACT. For any group G, let $\mathcal{C}(G)$ denote the set of centralizers of G. We say that a group G has n *centralizers* (G is a $\mathcal{C}_n$-group) if $|\mathcal{C}(G)| = n$. In this note, we prove that every finite $\mathcal{C}_n$-group with $n \leq 21$ is soluble and this estimate is sharp. Moreover, we prove that every finite $\mathcal{C}_n$-group with $|G| < \frac{30n+15}{19}$ is non-nilpotent soluble. This result gives a partial answer to a conjecture raised by A. Ashrafi in 2000.

1. **Introduction**

For any group G, let $\mathcal{C}(G)$ denote the set of centralizers of G. We say that a group G has n *centralizers* ($G \in \mathcal{C}_n$, or G is a $\mathcal{C}_n$-group) if $|\mathcal{C}(G)| = n$. Also we say that G has a finite number of centralizers, written $G \in \mathcal{C}$, if $G \in \mathcal{C}_n$ for some $n \in \mathbb{N}$. Indeed $\mathcal{C} = \bigcup_{i\geq 1} \mathcal{C}_i$. It is clear that a group is a $\mathcal{C}_1$-group if and only if it is abelian. Belcastro and Sherman in [5], showed that there is no finite $\mathcal{C}_n$-group for $n \in \{2, 3\}$ (while Ashrafi in [2], showed that, for any positive integer $n \neq 2, 3$, there exists a finite group G such that $|\mathcal{C}(G)| = n$). Also they characterized all finite $\mathcal{C}_n$-groups for $n \in \{4, 5\}$. Tota (see Appendix of [10]) proved that every arbitrary $\mathcal{C}_4$-group is soluble. The author in [11] showed that the derived length of a soluble C_n-group (not necessarily finite) is $\leq n$.

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For more details concerning $\mathcal{C}_n$-groups see [1–4,11,12]. In this paper, we obtain a solubility criteria for C_n-groups in terms of $|G|$ and n.

Our main results are:

Theorem A. Let G be a finite $\mathcal{C}_n$-group with $n \leq 21$, then G is soluble. The alternating group of degree 5 has 22 centralizers.

Theorem B. If G is a finite $\mathcal{C}_n$-group, then the following hold:
(1) $|G| < 2n$, then G is a non-nilpotent group.
(2) $|G| < \frac{30n+15}{19}$, then G is a non-nilpotent soluble group.

Let G be a finite $\mathcal{C}_n$-group. In [5], Belcastro and Sherman raised the question whether or not there exists a finite $\mathcal{C}_n$-group G other than Q_8 and D_{2p} (p is a prime) such that $|G| \leq 2n$. Ashrafi in [2] showed that there are several counterexamples for this question and then Ashrafi raised the following conjecture (conjecture 2.4): If $|G| \leq 3n/2$, then G is isomorphic to $S_3, S_3 \times S_3$, or a dihedral group of order 10. Now by Theorem A, we can obtain that if $|G| \leq 3n/2$, then G is soluble. Therefore Theorem A give a partial answer to the conjecture put forward by Ashrafi.

2. **Proofs**

Let $n > 0$ be an integer and let $\mathcal{X}$ be a class of groups. We say that a group G satisfies the condition $(\mathcal{X}, n)$ (G is a $(\mathcal{X}, n)$-group) whenever in every subset with $n+1$ elements of G there exist distinct elements x, y such that $\langle x, y \rangle$ is in $\mathcal{X}$. Let $\mathcal{N}$ and $\mathcal{A}$ be the classes of nilpotent groups and abelian groups, respectively. Indeed, in a group satisfying the condition $(\mathcal{A}, n)$, the largest set of non-commuting elements (or the largest set of elements in which no two generate an abelian subgroup) has size at most n.
Here we give an interesting relation between groups that have n centralizers and groups that satisfy the condition $(\mathcal{A}, n-1)$.

Proposition 2.1. *Let n be a positive integer and let G be a $\mathcal{C}_n$-group (not necessarily finite). Then G satisfies the condition $(\mathcal{A}, n-1)$.*

Proof. Suppose, for a contradiction, that G does not satisfy the condition $(\mathcal{A}, n-1)$. Therefore, there exists a subset $X = \{a_1, a_2, \cdots, a_n\}$ of G such that $\langle a_i, a_j \rangle$ is not abelian, for every $1 \leq i \neq j \leq n$. It follows that $C_G(a_i) \neq C_G(a_j)$ for every $1 \leq i \neq j \leq n$. Now since $C_G(e) = G$, where e is the trivial element of G, we get $n = |\mathcal{C}(G)| \geq n+1$, which is impossible. $\square$

Note that by an easy computation we can see that the symmetric group of degree 4, S_4, satisfies the condition $(\mathcal{A}, 10)$, but S_4 is not a $\mathcal{C}_{11}$-group (in fact, S_4 is a $\mathcal{C}_{14}$-group). That is, the converse of the above Proposition is not true.

We can now deduce Theorem A.

Proof of Theorem A. Clearly every group that satisfies the condition $(\mathcal{A}, n)$ also satisfies the condition $(\mathcal{N}, n)$. Thus, by Proposition 2.1, G satisfies the condition $(\mathcal{N}, n)$ for some $n \leq 20$. Now this statement follows from the main result of [6]. By an easy computation we can obtain that the alternating group of degree 5, has 22 centralizers.

Note that Ashrafi and Taeri in [4], proved that, if G is a finite simple group and $|\mathcal{C}(G)| = 22$, then $G \cong A_5$. Then they, by this result, claimed that, if G is a finite group and $|\mathcal{C}(G)| \leq 21$, then G is soluble. Therefore, in view of Theorem A, we gave positive answer to their claim.

Tota in [10, Theorem 6.2]) showed that a group G belongs to $\mathcal{C}$ if and only if it is center-by-finite. Therefore, it is a natural problem to obtain bounds for $|G : Z(G)|$ in terms of n.

Theorem 2.2. *There is some constant $c \in \mathbf{R}_{>0}$ such that for any $\mathcal{C}_n$-group G*

$$n \leq |G : Z(G)| \leq c^{n-1}.$$

Proof. First, by the main result of [9] and Proposition 2.1 we have $|G : Z(G)| \leq c^{n-1}$, for some constant c. To complete the proof, we may assume that $Z(G) \neq 1$. Since elements in the same coset modulo $Z(G)$ have the same centralizer, it follows that $n \leq |G : Z(G)|$. $\square$

For the proof of Theorem B, we need the following lemma.

Lemma 2.3. *Let G be a finite $\mathcal{C}_n$-group. Then*

$$n \leq \frac{|G| + |I(G)|}{2},$$

where $I(G) = \{a \in G \mid a^2 = 1\} = \{a \in G \mid a = a^{-1}\}$.

Proof. Since $C_G(a) = C_G(a^{-1})$, we can obtain that

$$n \leq |I(G)| + |\frac{G - I(G)}{2}| \leq \frac{|G| + |I(G)|}{2},$$

as desired. □

Corollary 2.4. *Let G be a finite simple $\mathcal{C}_n$-group. Then $3n/2 < |G|$.*

Proof. It is well known that for every simple group we have $I(G) < |G|/3$. Now the result follows from Lemma 2.3. □

Here we show that a semi-simple $\mathcal{C}_n$-group has order bounded by a function of n. (Recall that a group G is semi-simple if G has no non-trivial normal abelian subgroups.)

Proposition 2.5. *Let G be a semi-simple $\mathcal{C}_n$-group. Then G is finite and $|G| \leq (n-1)!$.*

Proof. The group G acts on the set $A := \{C_G(x) \mid a \in G \setminus Z(G)\}$ by conjugation. By assumption $|A| = n-1$. Put $B = \bigcap_{x\in G} N_G(C_G(x))$. The subgroup B is the kernel of this action and so

$$G/B \hookrightarrow S_{n-1}. \qquad (*)$$

By definition of B, the centralizer $C_G(a)$ is normal in B for any element $a \in G$. Therefore, $a^{-1}a^b \in C_G(a)$ for any two elements $a, b \in B$. So B is a 2-Engel group (see [7]). Now it is well known that B is a nilpotent group of class at most 3. Now as G is a semi-simple group, we can obtain that $B = 1$. It follows from (*) that G is a finite group and $|G| \leq (n-1)!$, as desired. □

We need the following result for the proof of Theorem B.

Theorem 2.6. (Potter, 1988) *Suppose G admits an automorphism which inverts more than $4|G|/15$ elements. Then G is soluble.*

Proof of Theorem B. (1). Suppose, by contradiction, that G is a nilpotent group, so in particular, $Z(G) \neq 1$. Now it follows from Theorem 2.2 that $2n \leq |G|$, which is a contradiction.
(2). From part (1) we obtain that G is not nilpotent. Since $|G| < \frac{30n+15}{19}$ and so $2n > \frac{19|G|-15}{15}$, Lemma 2.3 implies that

$$|I(G)| \geq 2n - |G| > \frac{4|G|}{15} - 1.$$

On the other hand, since $I(G)$ is the set of all elements of G that are inverted by the identity automorphism, Theorem 2.6 completes the proof.

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