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SOME RESULTS ON THE POLYNOMIAL NUMERICAL HULLS OF MATRICES

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ABSTRACT. In this note we characterize polynomial numerical hulls of matrices $A \in M_n$ such that A^2 is Hermitian. Also, we consider normal matrices $A \in M_n$ whose k^{th} power are semidefinite. For such matrices we show that $V^k(A) = \sigma(A)$.

1. Introduction

Let M_n be the set of $n \times n$ complex matrices. The polynomial numerical hull of order k for a matrix $A \in M_n$ is defined and denoted by

$$V^k(A) = \{\xi \in \mathbb{C} : |p(\xi)| \leq \|p(A)\| \text{ for all } p(z) \in \mathcal{P}_k[\mathbb{C}]\},$$

where $\mathcal{P}_k[\mathbb{C}]$ is the set of complex polynomials with degree at most k. This notion was introduced by Nevanlinna [11] and further studied by several researchers; see, e.g., [1, 4, 5, 8–10]. The *joint numerical range* of $(A_1, A_2, \ldots, A_m) \in M_n \times \cdots \times M_n$ is denoted by

$$W(A_1, A_2, \ldots, A_m) = \{(x^*A_1x, x^*A_2x, \ldots, x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}.$$

By the result in [9] (see also [10])

$$V^k(A) = \{\zeta \in \mathbb{C} : (0, \ldots, 0) \in \text{conv}W((A-\zeta I), (A-\zeta I)^2, \ldots, (A-\zeta I)^k)\},$$

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where conv$X$ denotes the convex hull of $X \subseteq \mathbb{C}^k$.

In Section 2, we characterize polynomial numerical hulls of matrices $A \in M_n$ such that A^2 is Hermitian. Also, we show that [5, Theorem 4.4] is not formulated correctly, and we improve it in Theorem 2.3. In Section 3, we consider normal matrices $A \in M_n$ whose k^{th} power are semidefinite. For such matrices we show that $V^k(A) = \sigma(A)$.

2. Main results

In this section we consider matrices $A \in M_n$ such that A^2 is Hermitian. For such matrices, we give a complete description of $V^k(A), k \in \mathbb{N}$.

By [5, Theorem 4.1], if $A \in M_n$, then A^2 is Hermitian if and only if A is unitarily similar to a direct sum of a Hermitian matrix H, a skew-Hermitian matrix G, and 2-by-2 matrices as follows:

$$A = \operatorname{diag}(h_1, \ldots, h_p) \oplus i\operatorname{diag}(g_1, \ldots, g_q) \oplus A_1 \oplus \ldots \oplus A_r, \tag{2.1}$$

where $g_1 \geq \cdots \geq g_q, h_1 \geq \cdots \geq h_p$ and

$$A_j = \begin{bmatrix} \mu_j & i\nu_j \\ i\nu_j & -\mu_j \end{bmatrix}, \quad \text{with } \mu_j, \nu_j > 0, j = 1, \ldots, r.$$

The following example shows that the statement of [5, Theorem 4.4] is not formulated correctly.

Example 2.1. *Let* $A = [i] \oplus \begin{bmatrix} \sqrt{6} & i\sqrt{3} \\ i\sqrt{3} & -\sqrt{6} \end{bmatrix}$*. By [5, Theorem 4.4],* $V^2(A) \cap \mathbb{R} = \{-\sqrt{3}, \sqrt{3}\}$*.*
But, we will show that $\pm 1 \in V^2(A) \cap \mathbb{R}$*.*

Observe that $\mu \in V^2(A) \cap \mathbb{R}$ if and only if $(\mu, 0, \mu^2) \in W := W(\Re(A), \Im(A), A^2)$, where $\Re(A) = \frac{A+A^*}{2}, \Im(A) = \frac{A-A^*}{2i}$. By [5, Theorem 4.3],

$$W = \operatorname{conv}\left(\{(0, 1, -1)\} \bigcup \left\{(x, y, 3) : (x, y) : \tfrac{x^2}{6} + \tfrac{y^2}{3} = 1\right\}\right).$$

Since W is convex and $\{(\pm 2, -1, 3), (0, 1, -1)\} \subset W, (\pm 1, 0, 1) \in W$, we see that $\pm 1 \in V^2(A) \cap \mathbb{R}$.

Recall that an extreme point of a convex set S in a real vector space is a point in S which does not lie in any open line segment joining two points of S. The Krein Milman Theorem says that if S is convex and compact in a locally convex space, then S is the convex hull of its extreme points. Also, by Carathéodory Theorem, we know that if Q is a nonempty subset of $\mathbb{R}^n$, then every vector $x \in \operatorname{conv}Q$ is a convex combination of at most $n + 1$ vectors in Q (see [7, Theorem 22.16]).

Lemma 2.2. *[3, Theorem III.9.2] Let P be an arbitrary m-dimensional subspace in $\mathbb{R}^n$ and let K be a convex subset in $\mathbb{R}^n$. Then every extreme point of the intersection $P \cap K$ can be expressed as a convex combination of at most $n-m+1$ extreme points of K. Moreover, if K is a compact set, then*

$$P \cap K = \operatorname{conv}\left(\bigcup_{a_1,\ldots,a_{n-m+1} \in \operatorname{ext}(K)} \left[P \cap \operatorname{conv}\left(\{a_1, \ldots, a_{n-m+1}\} \right) \right] \right),$$

where $\operatorname{ext}(K)$ *is the set of all extreme points of K.*

Now, we state the main theorem in this section.

Theorem 2.3. *Assume $A \in M_n$ satisfies (2.1). Let K_1 be the convex hull of the union of the sets:*
(a.1) $\{(h_j, h_j^2) : 1 \le j \le p\}$,
(a.2) $\{(\pm\mu_j, \mu_j^2 - \nu_j^2) : 1 \le j \le r\}$,
(a.3) $\{(0, g_1 g_q), (0, \tilde{g})\}$ *if* $g_1 g_q \le 0$, *where*

$$\tilde{g} = \max\{g_i g_j : g_i g_j \le 0,\ 1 \le i < j \le q\},$$

(a.4) $$\bigcup_{k=1}^{r} \bigcup_{i=1}^{q} \left\{ (\pm x, z) : \begin{array}{l} x = \left(\frac{g_i}{g_i + \operatorname{sgn}(g_i)\nu_k t} \right) \mu_k \sqrt{1-t^2}, \\ z = \frac{-\operatorname{sgn}(g_i)\nu_k g_i^2 t + g_i\left(\mu_k^2 - \nu_k^2\right)}{g_i + \operatorname{sgn}(g_i)\nu_k t}, \\ g_i \ne 0, t \in [0,1] \end{array} \right\}.$$

Let K_2 be the convex hull of the union of the sets:
(b.1) $\{(g_j, -g_j^2) : 1 \le j \le q\}$,
(b.2) $\{(\pm\nu_j, \mu_j^2 - \nu_j^2) : 1 \le j \le r\}$,
(b.3) $\{(0, -h_1 h_p), (0, -\tilde{h})\}$ *if* $h_1 h_p \le 0$, *where*

$$\tilde{h} = \max\{h_i h_j : h_i h_j \le 0,\ 1 \le i < j \le p\},$$

(b.4) $$\bigcup_{k=1}^{r} \bigcup_{i=1}^{p} \left\{ (\pm y, z) : \begin{array}{l} y = \frac{h_i \nu_k \sqrt{1-t^2}}{h_i + \operatorname{sgn}(h_i)\mu_k t}, \\ z = \frac{\operatorname{sgn}(h_i)\mu_k h_i^2 t + h_i\left(\mu_k^2 - \nu_k^2\right)}{h_i + \operatorname{sgn}(h_i)\mu_k t}, \\ h_i \ne 0, t \in [0,1] \end{array} \right\}.$$

Then $V^2(A) = \{\mu \in \mathbb{R} : (\mu, \mu^2) \in K_1\} \cup \{i\mu \in i\mathbb{R} : (\mu, -\mu^2) \in K_2\}$.

Proof. Without loss of generality we assume that $g_1 > \cdots > g_q$, and $h_1 > \cdots > h_p$. By [5, Theorem 4.3], the joint numerical range $W(A, A^2)$ of a matrix A whose square is Hermitian, is convex. Then [5, Theorem

5.3] implies that $\mu \in V^2(A)$ if and only if $(\mu, \mu^2) \in W(A, A^2)$. Whereas A^2 is Hermitian, $\mu^2 \in \mathbb{R}$. Thus, $V^2(A) \subseteq \mathbb{R} \cup i\mathbb{R}$. Now, we consider $V^2(A) \cap \mathbb{R}$. Also, it is readily seen that $\mu \in V^2(A) \cap \mathbb{R}$ if and only if $\left(\mu, 0, \mu^2\right) \in W := W\left(\Re(A), \Im(A), A^2\right)$. Since $\left\{\left(\mu, 0, \mu^2\right) : \mu \in \mathbb{R}\right\} \subseteq P_{xz} := \{(x, 0, z) : x, z \in \mathbb{R}\}$ and W is convex, we obtain that $\mu \in V^2(A) \cap \mathbb{R}$ if and only if $\left(\mu, 0, \mu^2\right) \in E := \operatorname{conv}(P_{xz} \cap W)$. By [5, Theorem 4.3], $W = \operatorname{conv}\{S_1, S_2, S_3\}$, where

$$\begin{aligned} S_1 &= \left\{(h_i, 0, h_i^2) : 1 \leq i \leq p\right\}, \\ S_2 &= \left\{(0, g_i, -g_i^2) : 1 \leq i \leq q\right\}, \\ S_3 &= \bigcup_{j=1}^{r} \left\{\left(x, y, \mu_j^2 - \nu_j^2\right) : \frac{x^2}{\mu_j^2} + \frac{y^2}{\nu_j^2} = 1\right\}, \end{aligned}$$

and hence $E = \operatorname{conv}\left[P_{xz} \cap \operatorname{conv}(S_1 \cup S_2 \cup S_3)\right]$. By Carathéodory Theorem,

$$E = \operatorname{conv}\left(P_{xz} \cap \bigcup_{\{a_i\}_{i=1}^4 \subset S_1 \bigcup S_2 \bigcup S_3} \operatorname{conv}\left(\{a_i\}_{i=1}^4\right)\right).$$

Now Lemma 2.2 shows that

$$E = \operatorname{conv}\left(\bigcup_{\{a_i\}_{i=1}^4 \subset S_1 \bigcup S_2 \bigcup S_3} \operatorname{conv}\left(\bigcup_{1 \leq i < j \leq 4} \left[P_{xz} \cap \operatorname{conv}\left(\{a_i, a_j\}\right)\right]\right)\right),$$

and therefore,

$$\begin{aligned} E &= \operatorname{conv}\left(\bigcup_{\{a_i\}_{i=1}^4 \subset S_1 \bigcup S_2 \bigcup S_3} \bigcup_{1 \leq i < j \leq 4} \left[P_{xz} \cap \operatorname{conv}\left(\{a_i, a_j\}\right)\right]\right) \\ &= \operatorname{conv}\left(\bigcup_{\{a,b\} \subset S_1 \bigcup S_2 \bigcup S_3} \left[P_{xz} \cap \operatorname{conv}\left(\{a, b\}\right)\right]\right) \\ &= \operatorname{conv}\left(\bigcup_{i=1}^3 \bigcup_{j=1}^3 \bigcup_{a \in S_i} \bigcup_{b \in S_j} \left[P_{xz} \cap \operatorname{conv}\left(\{a, b\}\right)\right]\right). \end{aligned}$$

We set $D_{ij} = \bigcup_{a \in S_i} \bigcup_{b \in S_j} \left[P_{xz} \cap \operatorname{conv}\left(\{a, b\}\right)\right]$. Since $S_1 \subseteq P_{xz}$,

$$S_1 \subset D_{11} \subset \operatorname{conv}(S_1), D_{12} = D_{21} \subset \operatorname{conv}(S_1 \cup D_{22}), D_{13} = D_{31} \subset \operatorname{conv}(S_1 \cup D_{33}),$$

it follows that

$$E = \operatorname{conv}(S_1 \cup D_{22} \cup D_{23} \cup D_{33}).$$

Therefore,

$$\mu \in V^2(A) \cap \mathbb{R} \Leftrightarrow \left(\mu, 0, \mu^2\right) \in \operatorname{conv}(S_1 \cup D_{22} \cup D_{23} \cup D_{33}).$$

Direct calculation shows that

$$\begin{aligned}
D_{22} &= \{(0,0,g_ig_j) : g_ig_j \le 0\},\\
\operatorname{conv}(D_{33}) &= \operatorname{conv}\left(\left\{(\pm\mu_i, 0, \mu_i^2-\nu_i^2),\ 1\le i\le r\right\}\right).
\end{aligned}$$

Now, we consider D_{23}. Fix $1 \le j \le r$ and $1 \le i \le q$.
Let $a = (0, g_i, -g_i^2) \in S_2$ and

$$b = \left(\alpha, \beta, \mu_j^2 - \nu_j^2\right) \in B_j := \left\{ \left(x, y, \mu_j^2-\nu_j^2\right) : \frac{x^2}{\mu_j^2} + \frac{y^2}{\nu_j^2} = 1 \right\} \subseteq S_3.$$

In the case $g_i = 0$, we have

$$\bigcup_{b\in B_j} \left[P_{xz} \cap \operatorname{conv}\left(\{a,b\}\right)\right] = \operatorname{conv}\left\{(0,0,0), (\pm\mu_j, 0, \mu_j^2-\nu_j^2)\right\}.$$

Thus, if there exists $1 \le i \le q$ such that $g_i = 0$, define

$$D_0 := \bigcup_{j=1}^{r} \operatorname{conv}\left\{(0,0,0), (\pm\mu_j, 0, \mu_j^2-\nu_j^2)\right\} \subseteq \operatorname{conv}\left(D_{22}\cup D_{33}\right), \tag{2.2}$$

and otherwise define $D_0 = \emptyset$.

Now, we assume that $g_i \neq 0$. It is readily seen that, if $\beta g_i > 0$, then $P_{xz} \cap \operatorname{conv}(\{a,b\}) = \emptyset$. Therefore, we consider $\beta g_i \le 0$.

$$P_{xz} \cap \operatorname{conv}\left(\{a,b\}\right) = \left\{ (x,0,z) : \begin{array}{l} \left(\frac{g_i}{g_i-\beta}\right)\alpha = x, \\ z = \frac{\beta}{g_i-\beta} g_i^2 + \left(\frac{g_i}{g_i-\beta}\right)\left(\mu_j^2-\nu_j^2\right). \end{array} \right\}.$$

Since $\frac{\alpha^2}{\mu_j^2} + \frac{\beta^2}{\nu_j^2} = 1$, and $\beta g_i \le 0$, we obtain that $\beta = -\operatorname{sgn}(g_i)\nu_j\sqrt{1-\frac{\alpha^2}{\mu_j^2}}$.

Consider $t = \sqrt{1-\frac{\alpha^2}{\mu_j^2}} \in [0,1]$. Then

$$D_{23} = D_0 \cup \bigcup_{j=1}^{r}\bigcup_{i=1}^{q} \left\{ (\pm x, 0, z) : \begin{array}{l} x = \left(\frac{g_i}{g_i+\operatorname{sgn}(g_i)\nu_j t}\right)\mu_j\sqrt{1-t^2}, \\ z = \frac{-\operatorname{sgn}(g_i)\nu_j g_i^2 t + g_i\left(\mu_j^2-\nu_j^2\right)}{g_i+\operatorname{sgn}(g_i)\nu_j t}, \\ g_i \neq 0, t\in[0,1]. \end{array} \right\}$$

where D_0 is given in (2.2). Therefore,

$$V^2(A) \cap \mathbb{R} = \{\lambda \in \mathbb{R} : (\lambda, \lambda^2) \in K_1\}.$$

Similarly, we can show that

$$V^2(A) \cap i\mathbb{R} = \{\lambda \in \mathbb{R} : (\lambda, -\lambda^2) \in K_2\}.$$

□

Example 2.4. *Let* $A = [3] \oplus [-2i] \oplus \begin{bmatrix} 4 & 3i \\ 3i & -4 \end{bmatrix}$.

By the notations as in Theorem 2.3, it is readily seen that

$$K_1 = conv\left(\{(3,9)\} \cup \{(\pm 4, 7)\} \cup \left\{\left(\pm\tfrac{8\sqrt{1-t^2}}{2+3t}, \tfrac{14-12t}{2+3t}\right) : t \in [0,1]\right\}\right),$$
$$K_2 = conv\left(\{(-2,-4)\} \cup \{(\pm 3, 7)\} \cup \left\{\left(\pm\tfrac{9\sqrt{1-t^2}}{3+4t}, \tfrac{21+36t}{3+4t}\right) : t \in [0,1]\right\}\right).$$

Therefore, by Theorem 2.3 we have

$$\begin{aligned} V^2(A) \cap \mathbb{R} &= \left\{\mu \in \mathbb{R} : \left(\mu, \mu^2\right) \in K_1\right\} \\ &= \left[\tfrac{-23}{7}, \tfrac{-8\sqrt{370+27\sqrt{37}}}{29+9\sqrt{37}}\right] \cup \left[\tfrac{8\sqrt{370+27\sqrt{37}}}{29+9\sqrt{37}}, 3\right], \\ V^2(A) \cap i\mathbb{R} &= \left\{i\mu \in i\mathbb{R} : \left(\mu, -\mu^2\right) \in K_2\right\} = i\left[-2, \tfrac{-2}{10}\right]. \end{aligned}$$

The following figures illustrate how Theorem 2.3 characterize $V^2(A)$.

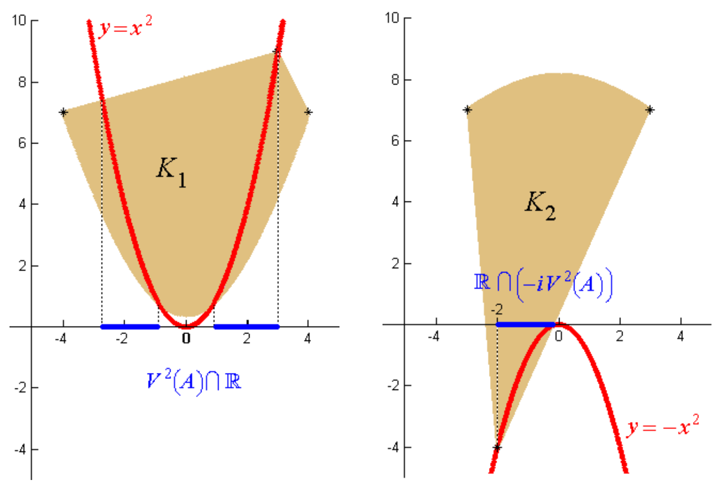

FIGURE 1. Characterizing $V^2(A)$ via Theorem 2.3

Therefore,

$$V^2(A) = \left[\frac{-23}{7}, \frac{-8\sqrt{370+27\sqrt{37}}}{29+9\sqrt{37}}\right] \cup \left[\frac{8\sqrt{370+27\sqrt{37}}}{29+9\sqrt{37}}, 3\right] \cup i\left[-2, \frac{-2}{10}\right].$$

Remark 2.5. *Assume that $A \in M_n$ is such that A^2 is Hermitian. By [11, Theorem 2.1.5] and [5, Theorem 4.2], we know that $V^1(A) = W(A)$, and $V^k(A) = \sigma(A), k \geq 4$. Also Theorem 2.3, characterizes $V^2(A)$. So, it is enough to study $V^3(A)$ to characterize $V^k(A)$, $\forall k \in \mathbb{N}$.*

By a similar method as followed in the proof of Theorem 2.3 we state the following theorem.

Theorem 2.6. *Let*

$$A = \operatorname{diag}(h_1, \cdots, h_p) \oplus i\operatorname{diag}(g_1, \cdots, g_q), \tag{2.3}$$

where $h_1 \geq \cdots \geq h_p$, and $g_1 \geq \cdots \geq g_q$.

Define

$$S_1 := \operatorname{conv}\left(\begin{array}{l} \left\{\left(h_j, h_j^2, h_j^3\right) : 1 \leq j \leq p\right\} \bigcup \\ \left\{\left(0, -g^2, 0\right) : \{g, -g\} \subset \{g_i : 1 \leq i \leq q\}\right\} \bigcup \\ \bigcup\limits_{\substack{\{a,b,c\} \subset \{g_i\} \\ a<-b<c, ac>0}} \left\{\left(0, \frac{abc(ac^2-ab^2+a^2b-bc^2+b^2c-a^2c)}{(c-b)(b-a)(a-c)(a+b+c)}, 0\right)\right\} \end{array} \right),$$

and

$$S_2 := \operatorname{conv}\left(\begin{array}{l} \left\{\left(g_j, -g_j^2, -g_j^3\right) : 1 \leq j \leq q\right\} \bigcup \\ \left\{\left(0, h^2, 0\right) : \{h, -h\} \subset \{h_i : 1 \leq i \leq p\}\right\} \bigcup \\ \bigcup\limits_{\substack{\{a,b,c\} \subset \{h_i\} \\ a<-b<c, ac>0}} \left\{\left(0, \frac{abc(ac^2-ab^2+a^2b-bc^2+b^2c-a^2c)}{(c-b)(b-a)(c-a)(a+b+c)}, 0\right)\right\} \end{array} \right).$$

Then

$$V^3(A) = \{\mu \in \mathbb{R} : (\mu, \mu^2, \mu^3) \in S_1\} \cup \{i\mu \in i\mathbb{R} : (\mu, -\mu^2, -\mu^3) \in S_2\}.$$

Corollary 2.7. *Let $A \in M_n$ be a normal matrix such that $A = A_1 \oplus iA_2$, $A_1^* = A_1$ and let A_2 be a semi-definite matrix. Then $V^3(A) = \sigma(A)$.*

Proof. Without loss of generality, we assume that $\sigma(A_2) \subseteq (0, \infty)$. We consider the set S_2 as in Theorem 2.6 and let a, b and c be real numbers such that $a < -b < c$ and $ac > 0$. It is readily seen that

$$\frac{abc\left(ac^2 - ab^2 + a^2b - bc^2 + b^2c - a^2c\right)}{(c-b)(b-a)(c-a)(a+b+c)} > 0.$$

Therefore, $(0,0,0) \notin S_2$ and hence

$$\begin{aligned}
&V^3(A) \cap \mathbb{R} \\
&\subset \left\{\mu \in \mathbb{R} : \left(\mu, \mu^2\right) \in \operatorname{conv}\left(\left\{\left(\lambda, \lambda^2\right) : \lambda \in \sigma(A_1)\right\}\right)\right\} \\
&= \sigma(A_1), \\
&V^3(A) \cap i\mathbb{R} \setminus \{0\} \\
&\subset \left\{i\mu \in i\mathbb{R} : \left(\mu, -\mu^3\right) \in \operatorname{conv}\left(\begin{array}{l} \left\{\left(\lambda, -\lambda^3\right) : \lambda \in \sigma(A_2)\right\} \\ \cup \{(0,0)\} \end{array}\right)\right\} \\
&= \sigma(A_2) \cup \{0\}.
\end{aligned}$$

So $V^3(A) = \sigma(A)$. □

3. Additional results

In this section we consider normal matrices whose k^{th} power are semidefinite. By [6, Theorem 2.1], we know that if $A \in M_n$ is a normal matrix such that A^2 is semidefinite, then $V^2(A) = \sigma(A)$. In the following, we extend this result.

Theorem 3.1. *Let $A \in M_n$ be a normal matrix such that $A^k,\ k \geq 2$ is semi-definite. Then $V^k(A) = \sigma(A)$.*

Proof. Without loss of generality, we can assume that

$$A = A_1 \oplus e^{\frac{i2\pi}{k}} A_2 \oplus e^{\frac{i4\pi}{k}} A_3 \oplus \cdots \oplus e^{\frac{i2(k-1)\pi}{k}} A_k,$$

such that $A_j \in M_{m_j},\ j = 1, \ldots k$ are positive semidefinite. Whereas $V^k\left(e^{i\theta} A\right) = e^{i\theta} V^k(A)$, it is enough to prove that

$$V^k(A) \cap \mathbb{R} = \begin{cases} \sigma(A_1) & \text{if } k \text{ is odd} \\ \sigma(A_1) \cup \sigma\left(A_{\frac{k}{2}+1}\right) & \text{if } k \text{ is even} \end{cases} \tag{3.1}$$

Suppose that $A_j = \operatorname{diag}\left(a_{1,j}, \cdots, a_{m_j,j}\right)$, $1 \leq j \leq k$.

We know that $\eta \in V^k(A)$ if and only if there exists $\mathbf{x} = (x_1, \ldots, x_k)^T, x_j \in \mathbb{C}^{m_j}, \|\mathbf{x}\| = (\sum_{j=1}^{k} x_j^* x_j)^{1/2} = 1$ such that $\eta^r = \sum_{j=1}^{k} e^{\frac{2i(j-1)\pi}{k}} x_j^* A_j^r x_j, r = 1, \ldots, k$.

Direct calculation shows that

$$V^k(A) \cap \mathbb{R} \subset \left\{ \eta \in \mathbb{R} : \left[\begin{array}{l} \eta = \sum_{j=1}^{k} \cos\left(\frac{2(j-1)\pi}{k}\right) x_j^* A_j x_j, \\ \eta^k = \sum_{j=1}^{k} x_j^* A_j^k x_j \\ x_j \in \mathbb{C}^{m_j}, \ \sum_{j=1}^{k} x_j^* x_j = 1. \end{array} \right. \right\}.$$

Define

$$p_{i,j} := \left(\cos\left(\frac{2(j-1)\pi}{k}\right) a_{i,j}, a_{i,j}^k \right), 1 \le j \le k, 1 \le i \le m_j.$$

So

$$V^k(A) \cap \mathbb{R} \subset \left\{ \eta \in \mathbb{R} : \left(\eta, \eta^k\right) \in \operatorname{conv}\left(\{p_{i,j}\}_{\substack{1 \le j \le k \\ 1 \le i \le m_j}} \right) \right\}$$

We know that $a_{ij} \ge 0$ and $\left|\cos\left(\frac{2(j-1)\pi}{k}\right)\right| < 1, \ j \in \{2, \ldots, k\} \backslash \{\frac{k}{2}+1\}$. By the convexity of $f(x) = |x|^k$ we obtain that

$$V^k(A) \cap \mathbb{R} \subset \begin{cases} \sigma(A_1) & \text{if } k \text{ is odd} \\ \sigma(A_1) \cup \sigma\left(A_{\frac{k}{2}+1}\right) & \text{if } k \text{ is even.} \end{cases}$$

Whereas $\sigma(A) \subseteq V^k(A)$ and $\sigma(A_i) \subseteq \mathbb{R}, \ i = 1, \ldots, k$, the equation (3.1) holds. □

It will be interesting to characterize the polynomial numerical hulls of matrices whose k^{th} power are Hermitian.

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