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RINGS IN WHICH ELEMENTS ARE THE SUM OF AN IDEMPOTENT AND A REGULAR ELEMENT

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Abstract. Let R be an associative ring with unity. An element $a \in R$ is said to be r-clean if $a = e+r$, where e is an idempotent and r is a regular (von Neumann) element in R. If every element of R is r-clean, then R is called an r-clean ring. In this paper, we prove that the concepts of clean ring and r-clean ring are equivalent for abelian rings. Furthermore, we prove that if 0 and 1 are the only idempotents in R, then an r-clean ring is an exchange ring. Also we show that the center of an r-clean ring is not necessary r-clean, but if 0 and 1 are the only idempotents in R, then the center of an r-clean ring is r-clean. Finally, we give some properties and examples of r-clean rings.

1. Introduction

Let R be an associative ring with unity. An element $a \in R$ is said to be clean if $a = e + u$, where e is an idempotent and u is a unit in R. If every element of R is clean then R is called a clean ring. Clean rings were introduced by W. K. Nicholson in his fundamental paper [11]. He proved that every clean ring is an exchange ring, and a ring with central idempotents is clean if and only if it is an exchange ring. Semiperfect rings and unit-regular rings are examples of clean rings as shown in [6]

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and [5], respectively. Many authors have studied clean rings and their generalizations such as [2, 6, 10–12, 16] and [17].

Definition 1.1. *An element $a \in R$ is said to be r-clean if $a = e + r$, where e is an idempotent and r is a regular (von Neumann) element in R. If every element of R is r-clean, then R is called an r-clean ring.*

We introduced r-clean rings and gave some basic properties of r-clean rings in [1]. The trivial examples of r-clean rings, of course, are regular and clean rings. In [1], we gave some examples of r-clean rings that are not regular and one example of an r-clean ring which is not clean.
A ring is called abelian if all its idempotents are central. In this paper, we prove that every abelian r-clean ring is clean. Also we show that every r-clean ring with 0 and 1 as the only idempotents, is exchange.
Let A and B be rings, $M =_B M_A$ a bimodule and let the formal triangular matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ be r-clean, then both A and B are r-clean by [1, Theorem 16]. We show that when both A and B are clean or one of them is clean and the other one is r-clean, then T is r-clean. Furthermore, we show that the center of an r-clean ring is not necessarily r-clean, but if 0 and 1 are the only idempotents, then the center of an r-clean ring is r-clean. Also we give some properties and examples of r-clean rings.

Throughout this paper, R denotes an associative ring with unity, $U(R)$ the group of units, $Id(R)$ the set of idempotents, $J(R)$ the Jacobson radical of R and $M_n(R)$ the ring of all $n \times n$ matrices over R. Furthermore, $Reg(R) = \{a \in R |$ a is regular (von Neumann)$\}$.

2. **Main results**

In this section we first give some properties of r-clean rings.

Lemma 2.1. *Let R be an abelian ring. Let $a \in R$ be a clean element in R and $e \in Id(R)$. Then*

(1) *The element ae is clean.*
(2) *If $-a$ is clean, then $a + e$ is also clean.*

Proof. See [7, Proposition 3.5]. □

Theorem 2.2. *Let R be an abelian ring. Then R is r-clean if and only if R is clean.*

Proof. One direction is trivial. Conversely, let R be r-clean and $x \in R$. Then $x = e' + r$, where $e' \in Id(R)$ and $r \in Reg(R)$. So there exists $y \in R$ such that $ryr = r$. Clearly, $e = ry$ and yr are idempotents and (re+(1-e))(ye+(1-e))=1. Also since R is abelian, we have

$$(ye + (1-e))(re + (1-e)) = yre + 1 - e = eyr + 1 - e =$$

$$ry(yr) + 1 - e = r(yr)y + 1 - e = e + 1 - e = 1.$$

So $u = re + (1-e)$ is a unit. Furthermore, $r = eu$. Now, set $f = 1 - e$. Then $eu + f$ and hence, $-(eu + f)$ is a unit. Since f is an idempotent, so $-r = f + (-(eu + f))$ is clean. Then since $r \in Reg(R)$, it follows by Lemma 2.1(2) that x is clean, as required. □

Corollary 2.3. *Let R be an abelian ring. Then R is r-clean if and only if R is exchange.*

A ring is said to be reduced if it has no (nonzero) nilpotent elements. These rings are abelian. Therefore we have the following result.

Corollary 2.4. *Let R be a reduced ring. Then R is r-clean if and only if R is clean.*

Anderson and Camillo [2, Proposition 12], showed that no polynomial ring over a nonzero commutative ring is clean. Now, since concepts of clean ring and r-clean ring are equivalent for commutative rings, it follows that no polynomial ring over a nonzero commutative ring is r-clean. Therefore, we have a different proof for [1, Theorem 12].

Let R be a ring and α a ring endomorphism of R. Also let $R[[x, \alpha]]$ denote the ring of skew formal power series over R; that is, all formal power series in x with coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R.

Proposition 2.5. *Let R be an abelian ring and α an endomorphism of R. Then the following statements are equivalent:*

(1) *R is an r-clean ring.*
(2) *The formal power series ring $R[[x]]$ of R is an r-clean ring.*
(3) *The skew power series ring $R[[x;\alpha]]$ of R is an r-clean ring.*

Proof. (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are clear since R is a homomorphic image of $R[[x]]$ and $R[[x;\alpha]]$.
(1) $\Rightarrow$ (3). Let $f = a_0 + a_1x + ... \in R[[x;\alpha]]$. Since R is a clean ring

by Theorem 2.2, $a_0 = e_0 + u_0$, for some $e_0 \in Id(R)$ and $u_0 \in U(R)$. Hence $f = e_0 + (u_0 + a_1x + ...)$. But $U(R[[x;\alpha]]) = \{a_0 + a_1x + ...|a_0 \in U(R)\}$, without any assumption on the endomorphism α. Now, we have $Id(R) \subseteq Id(R[[x;\alpha]])$. Therefore, f is clean and so it is r-clean, as required.
(1) $\Rightarrow$ (2). It is enough to put $\alpha = 1_R$ in the proof of (1) $\Rightarrow$ (3). $\square$

A ring R is semiregular if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$.

Proposition 2.6. *Every abelian semiregular ring is r-clean.*

Proof. Let R be an abelian semiregular ring. As $R/J(R)$ is regular, so is r-clean. But R is abelian, so $R/J(R)$ is clean by Theorem 2.2. Hence R is clean by [10, Proposition 6]. $\square$

In the following we show that the converse of the Proposition 2.6 is not true.

Example 2.7. *Let $\mathbb{Q}$ be the field of rational numbers and L the ring of all rational numbers with odd denominators. Define*

$$R = \{(r_1, ..., r_n, s, s, ...)|\ n \geq 1, r_1, ..., r_n \in \mathbb{Q}, s \in L\}.$$

It is easy to check that R is a commutative exchange ring. So it is r-clean by Corollary 2.3*, while it is not semiregular.*

Let R be an r-clean ring and I an ideal of R. Then it is clear that R/I is r-clean. But in general, the converse of this result is not true (for example, if p is a prime number, then $\mathbb{Z}_p$ is r-clean, but $\mathbb{Z}$ is not r-clean). It was proved by Han and Nicholson [10, Proposition 6] that if R is a ring and $I \subseteq J(R)$ is an ideal of R, then R is clean if and only if R/I is clean and idempotents can be lifted modulo $J(R)$. It is clear that this result is true for abelian r-clean rings. The following result gives a partial converse to the fact that every homomorphic image of an r-clean ring is r-clean.

Theorem 2.8. *Let I be a regular ideal of the ring R and suppose that idempotents can be lifted modulo I. Then R is r-clean, if and only if R/I is r-clean.*

Proof. If R/I is r-clean, then for any $a \in R$, $a + I$ is r-clean. Thus there exists $e + I \in Id(R/I)$ such that $(a - e) + I \in Reg(R/I)$. Hence $((a - e) + I)(x + I)((a - e) + I) = (a - e) + I$ for some $x \in R$. So $(a-e)-(a-e)x(a-e) \in I$. Now, since I is regular, so $a-e \in Reg(R)$ by

[3, Lemma 1]. Since idempotents can be lifted modulo I, we may assume that e is an idempotent of R. Therefore a is r-clean, as required. □

Lemma 2.9. *Let R be a ring with no zero divisors. Then R is clean if R is r-clean.*

Proof. For every $x \in R$, we write $x = e + r$, where $e \in Id(R)$ and $r \in Reg(R)$. Then there exists $y \in R$ such that $ryr = r$. Now, if $r = 0$, then $x = e = (2e-1) + (1-e)$ is clean. But if $r \neq 0$, then since R is a ring with no zero divisors and $ryr = r$, so $r \in U(R)$. Hence x again is clean. □

Corollary 2.10. *Let R be a ring with no zero divisors. Then R is r-clean if and only if R is local.*

Proof. It follows from [13, Lemma 14] and Lemma 2.9. □

Theorem 2.11. *Let R be a ring. Then R is r-clean if and only if every element $x \in R$ can be written as $x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$.*

Proof. Let R be r-clean and $x \in R$. Then as R is r-clean, so $-x = r + e$, where $r \in Reg(R)$ and $e \in Id(R)$. Hence $x = (-r) - e$, where $-r \in Reg(R)$ and $e \in Id(R)$.
Conversely, suppose that every element $x \in R$ can be written as $x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$. So for every element $x \in R$, we can write $-x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$. Hence $x = (-r) + e$, where $-r \in Reg(R)$ and $e \in Id(R)$. □

A ring R is said to be von Neumann local if for any $a \in R$, either a or $1-a$ is regular. Some characterizations of von Neumann local rings have been studied in [8], [14] and [15]. In the following we give a relation between r-clean and von Neumann local rings.

Theorem 2.12. *Let R be a ring such that 0 and 1 are the only idempotents in R. Then R is r-clean if and only if it is von Neumann local.*

Proof. Let R be an r-clean ring and assume that 0 and 1 are the only idempotents in R. Then for any $a \in R$, either a or $a-1$ is regular. Hence R is von Neumann local.
Conversely, let R be von Neumann local. Then for any $a \in R$, either a or $1-a$ is regular. Now, if a is regular, then a is r-clean. If $1-a$ is regular, then so is $a-1$ and hence, a is r-clean. □

Corollary 2.13. *Let R be an r-clean ring with no nontrivial idempotents. Then R is exchange.*

Proof. As every regular element a of a ring R is an exchange element (in the sense that there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$), so von Neumann local rings are exchange rings. Therefore, the result is clear by Theorem 2.12. $\square$

Let A and B be rings, $M =_B M_A$ a bimodule and let the formal triangular matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ be r-clean, then both A and B are r-clean by [1, Theorem 16]. In the following theorem we have some conditions under which T is r-clean.

Theorem 2.14. *Let A and B be rings, $M =_B M_A$ a bimodule and assume that one of the following conditions holds:*

(1) *A and B are clean.*
(2) *one of the rings A and B is clean and the other one is r-clean.*

Then the formal triangular matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ is r-clean.

Proof. (1). It is proved in [10] that if A and B are clean, then T is clean. So it is r-clean.
(2). Case 1. Let A be r-clean and let B be clean. Then for every $t = \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \in T$, we have $a = e_1 + r$ and $b = e_2 + u$ for some $e_1, e_2 \in Id(R)$, $r \in Reg(R)$ and $u \in U(R)$. Assume that $ryr = r$ for some $y \in R$. Write $t = E + W$ where $E = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ and $W = \begin{pmatrix} r & 0 \\ m & u \end{pmatrix}$. Obviously, $E = E^2$ and the equality

$$\begin{pmatrix} r & 0 \\ m & u \end{pmatrix} \begin{pmatrix} y & 0 \\ -u^{-1}my & u^{-1} \end{pmatrix} \begin{pmatrix} r & 0 \\ m & u \end{pmatrix} = \begin{pmatrix} r & 0 \\ m & u \end{pmatrix}$$

implies that W is a regular matrix. Hence t is r-clean.
Case 2. Let A be clean and let B be r-clean. Then the proof is similar to Case 1. $\square$

Theorem 2.15. *Let R be a ring with no nontrivial idempotents and let $0 \neq a \in R$ be r-clean. Then for any $b \in R$, $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is r-clean in $M_2(R)$.*

Proof. If $a = 1$, then $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$ which shows that A is r-clean. Now, let $a \neq 1$. Then $a = r$ or $a = 1 + r$, where $r \in Reg(R)$. So there exists $y \in R$ such that $ryr = r$.
Case 1. If $a = r$, then

$$\begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & ryb \\ 0 & 0 \end{pmatrix}.$$

But 0 and 1 are the only idempotents in R, so $ry = 0$ or $ry = 1$. If $ry = 0$, then $a = r = ryr = 0$, which is a contradiction by hypothesis. Thus $ry = 1$. Hence $ryb = b$ and $A = \begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix}$ is regular. So A is r-clean.
Case 2. If $a = 1 + r$, then $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix}$. But as 0 and 1 are the only idempotent in R, $ry = 0$ or $ry = 1$. If $ry = 0$, then $r = 0$ and so $a = 1$, which is a contradiction by hypothesis. Thus $ry = 1$. Therefore similarly, we can see that $\begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix}$ is regular. Hence A is r-clean. □

Theorem 2.16. *Let R be a ring and let $diag(a_1, ..., a_n)$ be the $n \times n$ diagonal matrix with a_i in each entry on the main diagonal. If $a_1, ..., a_n \in R$ is r-clean, then $diag(a_1, ..., a_n)$ is r-clean in $M_n(R)$.*

Proof. Write $a_i = e_i + r_i$, where $e_i \in Id(R)$ and $r_i \in Reg(R)$ for every i, $1 \leq i \leq n$. Hence $diag(a_1, ..., a_n) = diag(e_1, ..., e_n) + diag(r_1, ..., r_n)$. But there exists $y_i \in R$ such that $r_i y_i r_i = r_i$ for every i, $1 \leq i \leq n$. Thus $diag(r_1, ..., r_n) diag(y_1, ..., y_n) diag(r_1, ..., r_n) = diag(r_1, ..., r_n)$. Hence $diag(r_1, ..., r_n)$ is regular and since $diag(e_1, ..., e_n)$ is idempotent, $diag(a_1, ..., a_n)$ is r-clean in $M_n(R)$. □

For more examples of r-clean rings, in the following we consider trivial extension and ideal-extension.

Let R be a ring and let ${}_R M_R$ be an R-R-bimodule. The trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

The trivial extension of R by a bimodule ${}_RM_R$ is isomorphic to the ring of all matrices $T = \begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. So a ring R is clean if and only if its trivial extension is a clean ring. It it is clear that this is true for abelian r-clean rings. But even if R is not abelian and the trivial extension of R is r-clean, then it is easy to check that R is r-clean.

Let R be a ring and let ${}_RV_R$ be an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then the ideal-extension $I(R; V)$ of R by V is defined to be the additive abelian group $I(R; V) = R \oplus V$ with the following multiplication

$$(r, v)(s, w) = (rs, rw + vs + vw).$$

Proposition 2.17. *Let R and V be as above. Then we have the following statements:*

(1) *If the ideal-extension $I(R; V)$ is r-clean, then R is r-clean.*
(2) *An ideal-extension $I(R; V)$ is clean if R is clean and for every $v \in V$, there exists $w \in V$ such that $v + w + vw = 0$.*
(3) *An ideal-extension $I(R; V)$ is r-clean if R is clean and for every $v \in V$, there exists $w \in V$ such that $v + w + vw = 0$.*
(4) *An ideal-extension $I(R; V)$ is r-clean if R is r-clean and for every $v \in V$ and $r, y \in R$, $vyr + vyv + ryv = v$.*

Proof. Let $E = I(R; V)$. For the proof of (1), let $x \in R$. Then $(x, 0) \in E$. Thus there exists $(e_1, e_2) \in Id(E)$ and $(r_1, r_2) \in Reg(E)$ such that $(x, 0) = (e_1, e_2) + (r_1, r_2)$. But $(r_1, r_2)(y_1, y_2)(r_1, r_2) = (r_1, r_2)$ for some $(y_1, y_2) \in E$. So $r_1y_1r_1 = r_1$. Hence $r_1 \in Reg(R)$ and since $e_1 \in Id(R)$, it follows that $x = e_1 + r_1$ is r-clean.

For the proof of (2), see [13, Proposition 7]. The assertion in (3) follows from (2). Finally, for the proof of (4), let $w = (a, v) \in E$. Then $a = e + r$, where $e \in Id(R)$ and $r \in Reg(R)$. But there exists $y \in R$ such that $ryr = r$. So $(r, v)(y, 0)(r, v) = (r, v)$ by hypothesis. Thus (r, v) is regular in E. Also $(e, 0)$ is an idempotent in E. Hence $w = (e, 0) + (r, v)$ is r-clean. Therefore, E is r-clean. □

It is well known that the center of a regular ring is regular (see [9, Theorem 1.14]). Also Burgess and Raphael [4, Proposition 2.5], have shown that the center of a clean ring is not necessarily clean. In fact, we have a clean (so r-clean) ring such that its center is not clean. Now, since the center of a ring is commutative, it follows that we have an r-clean ring such that its center is not r-clean. Therefore, the center of an r-clean ring is not necessarily r-clean. But we have the following theorem about the center of r-clean rings:

Theorem 2.18. *Let R be an r-clean ring with no nontrivial idempotents. Then the center of R is also an r-clean ring.*

Proof. Let $Z(R)$ be the center of R and $x \in Z(R)$. Then there exists $r \in Reg(R)$ such that $x = r$ or $x = r + 1$ by hypothesis. If $x = r$, then $r \in Reg(Z(R))$ and hence, $x = 0 + r$ is r-clean in $Z(R)$. But if $x = r + 1$, then $x - 1 = r$ is regular and $x - 1 \in Z(R)$. So similar to the previous case, $x - 1$ is regular in $Z(R)$. Hence x again is r-clean in $Z(R)$. Therefore, $Z(R)$ is r-clean. □

Theorem 2.19. *Let R be an abelian r-clean ring. Then R is indecomposable if and only if its center is a local ring.*

Proof. Let R be an indecomposable abelian r-clean ring, $Z(R)$ the center of R and $x \in Z(R)$. If $x = 0$ or $x = 1$, then it is clear that x or $x - 1$ are units in $Z(R)$. Now, let $x \neq 0, 1$. Then since R is r-clean, either x or $x - 1$ is regular in $Z(R)$ by the proof of Theorem 2.18.
Case 1. If x is regular in $Z(R)$, then there exists $y \in Z(R)$ such that $xyx = x$. Now, since $x \neq 0$ and xy, yx are idempotents in R, so $xy = yx = 1$ and hence, x is a unit.
Case 2. If $x - 1$ is regular in $Z(R)$, then (similar to Case 1) since $x \neq 1$, so $x - 1$ is a unit.
Therefore, either x or $x - 1$ is a unit in $Z(R)$, as required.
The converse is trivial. □

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