RINGS IN WHICH ELEMENTS ARE THE SUM OF AN IDEMPOTENT AND A REGULAR ELEMENT

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ABSTRACT. Let $R$ be an associative ring with unity. An element $a \in R$ is said to be $r$-clean if $a = e + r$, where $e$ is an idempotent and $r$ is a regular (von Neumann) element in $R$. If every element of $R$ is $r$-clean, then $R$ is called an $r$-clean ring. In this paper, we prove that the concepts of clean ring and $r$-clean ring are equivalent for abelian rings. Furthermore, we prove that if 0 and 1 are the only idempotents in $R$, then an $r$-clean ring is an exchange ring. Also we show that the center of an $r$-clean ring is not necessary $r$-clean, but if 0 and 1 are the only idempotents in $R$, then the center of an $r$-clean ring is $r$-clean. Finally, we give some properties and examples of $r$-clean rings.

1. Introduction

Let $R$ be an associative ring with unity. An element $a \in R$ is said to be clean if $a = e + u$, where $e$ is an idempotent and $u$ is a unit in $R$. If every element of $R$ is clean then $R$ is called a clean ring. Clean rings were introduced by W. K. Nicholson in his fundamental paper [11]. He proved that every clean ring is an exchange ring, and a ring with central idempotents is clean if and only if it is an exchange ring. Semiperfect rings and unit-regular rings are examples of clean rings as shown in [6].

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and [5], respectively. Many authors have studied clean rings and their generalizations such as [2, 6, 10–12, 16] and [17].

**Definition 1.1.** An element \( a \in R \) is said to be \( r \)-clean if \( a = e + r \), where \( e \) is an idempotent and \( r \) is a regular (von Neumann) element in \( R \). If every element of \( R \) is \( r \)-clean, then \( R \) is called an \( r \)-clean ring.

We introduced \( r \)-clean rings and gave some basic properties of \( r \)-clean rings in [1]. The trivial examples of \( r \)-clean rings, of course, are regular and clean rings. In [1], we gave some examples of \( r \)-clean rings that are not regular and one example of an \( r \)-clean ring which is not clean. A ring is called abelian if all its idempotents are central. In this paper, we prove that every abelian \( r \)-clean ring is clean. Also we show that every \( r \)-clean ring with 0 and 1 as the only idempotents, is exchange. Let \( A \) and \( B \) be rings, \( M =_B M_A \) a bimodule and let the formal triangular matrix ring \( T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix} \) be \( r \)-clean, then both \( A \) and \( B \) are \( r \)-clean by [1, Theorem 16]. We show that when both \( A \) and \( B \) are clean or one of them is clean and the other one is \( r \)-clean, then \( T \) is \( r \)-clean. Furthermore, we show that the center of an \( r \)-clean ring is not necessarily \( r \)-clean, but if 0 and 1 are the only idempotents, then the center of an \( r \)-clean ring is \( r \)-clean. Also we give some properties and examples of \( r \)-clean rings.

Throughout this paper, \( R \) denotes an associative ring with unity, \( U(R) \) the group of units, \( Id(R) \) the set of idempotents, \( J(R) \) the Jacobson radical of \( R \) and \( M_n(R) \) the ring of all \( n \times n \) matrices over \( R \). Furthermore, \( Reg(R) = \{ a \in R | a \) is regular (von Neumann)\}.

### 2. Main results

In this section we first give some properties of \( r \)-clean rings.

**Lemma 2.1.** Let \( R \) be an abelian ring. Let \( a \in R \) be a clean element in \( R \) and \( e \in Id(R) \). Then

1. The element \( ae \) is clean.
2. If \( -a \) is clean, then \( a + e \) is also clean.

*Proof.* See [7, Proposition 3.5].

**Theorem 2.2.** Let \( R \) be an abelian ring. Then \( R \) is \( r \)-clean if and only if \( R \) is clean.
Proof. One direction is trivial. Conversely, let $R$ be $r$-clean and $x \in R$. Then $x = e' + r$, where $e' \in \text{Id}(R)$ and $r \in \text{Reg}(R)$. So there exists $y \in R$ such that $ryr = r$. Clearly, $e = ry$ and $yr$ are idempotents and $(re+(1-e))(ye+(1-e))=1$. Also since $R$ is abelian, we have
\[(ye + (1 - e))(re + (1 - e)) = yre + 1 - e = eyr + 1 - e = r(yr) + 1 - e = r(yr)y + 1 - e = e + 1 - e = 1.\]
So $u = re + (1 - e)$ is a unit. Furthermore, $r = eu$. Now, set $f = 1 - e$. Then $eu + f$ and hence, $-(eu + f)$ is a unit. Since $f$ is an idempotent, so $-r = f + (-eu + f))$ is clean. Then since $r \in \text{Reg}(R)$, it follows by Lemma 2.1(2) that $x$ is clean, as required. 

Corollary 2.3. Let $R$ be an abelian ring. Then $R$ is $r$-clean if and only if $R$ is exchange.

A ring is said to be reduced if it has no (nonzero) nilpotent elements. These rings are abelian. Therefore we have the following result.

Corollary 2.4. Let $R$ be a reduced ring. Then $R$ is $r$-clean if and only if $R$ is clean.

Anderson and Camillo [2, Proposition 12], showed that no polynomial ring over a nonzero commutative ring is clean. Now, since concepts of clean ring and $r$-clean ring are equivalent for commutative rings, it follows that no polynomial ring over a nonzero commutative ring is $r$-clean. Therefore, we have a different proof for [1, Theorem 12].

Let $R$ be a ring and $\alpha$ a ring endomorphism of $R$. Also let $R[[x, \alpha]]$ denote the ring of skew formal power series over $R$; that is, all formal power series in $x$ with coefficients from $R$ with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over $R$.

Proposition 2.5. Let $R$ be an abelian ring and $\alpha$ an endomorphism of $R$. Then the following statements are equivalent:

(1) $R$ is an $r$-clean ring.
(2) The formal power series ring $R[[x]]$ of $R$ is an $r$-clean ring.
(3) The skew power series ring $R[[x; \alpha]]$ of $R$ is an $r$-clean ring.

Proof. (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are clear since $R$ is a homomorphic image of $R[[x]]$ and $R[[x; \alpha]]$.

(1) $\Rightarrow$ (3). Let $f = a_0 + a_1 x + \ldots \in R[[x; \alpha]]$. Since $R$ is a clean ring
by Theorem 2.2, \( a_0 = e_0 + u_0 \), for some \( e_0 \in \text{Id}(R) \) and \( u_0 \in U(R) \). Hence \( f = e_0 + (u_0 + a_1x + ... \), \( U(R[[x; \alpha]]) = \{a_0 + a_1x + ... | a_0 \in U(R)\} \), without any assumption on the endomorphism \( \alpha \). Now, we have \( \text{Id}(R) \subseteq \text{Id}(R[[x; \alpha]]) \). Therefore, \( f \) is clean and so it is \( r \)-clean, as required.

(1) \( \Rightarrow \) (2). It is enough to put \( \alpha = 1_R \) in the proof of (1) \( \Rightarrow \) (3).

A ring \( R \) is semiregular if \( R/J(R) \) is regular and idempotents can be lifted modulo \( J(R) \).

**Proposition 2.6.** Every abelian semiregular ring is \( r \)-clean.

**Proof.** Let \( R \) be an abelian semiregular ring. As \( R/J(R) \) is regular, so is \( r \)-clean. But \( R \) is abelian, so \( R/J(R) \) is clean by Theorem 2.2. Hence \( R \) is clean by [10, Proposition 6].

In the following we show that the converse of the Proposition 2.6 is not true.

**Example 2.7.** Let \( \mathbb{Q} \) be the field of rational numbers and \( L \) the ring of all rational numbers with odd denominators. Define

\[
R = \{(r_1, ..., r_n, s, s, ...) | n \geq 1, r_1, ..., r_n \in \mathbb{Q}, s \in L\}.
\]

It is easy to check that \( R \) is a commutative exchange ring. So it is \( r \)-clean by Corollary 2.3, while it is not semiregular.

Let \( R \) be an \( r \)-clean ring and \( I \) an ideal of \( R \). Then it is clear that \( R/I \) is \( r \)-clean. But in general, the converse of this result is not true (for example, if \( p \) is a prime number, then \( \mathbb{Z}_p \) is \( r \)-clean, but \( \mathbb{Z} \) is not \( r \)-clean). It was proved by Han and Nicholson [10, Proposition 6] that if \( R \) is a ring and \( I \subseteq J(R) \) is an ideal of \( R \), then \( R \) is clean if and only if \( R/I \) is clean and idempotents can be lifted modulo \( J(R) \). It is clear that this result is true for abelian \( r \)-clean rings. The following result gives a partial converse to the fact that every homomorphic image of an \( r \)-clean ring is \( r \)-clean.

**Theorem 2.8.** Let \( I \) be a regular ideal of the ring \( R \) and suppose that idempotents can be lifted modulo \( I \). Then \( R \) is \( r \)-clean, if and only if \( R/I \) is \( r \)-clean.

**Proof.** If \( R/I \) is \( r \)-clean, then for any \( a \in R \), \( a + I \) is \( r \)-clean. Thus there exists \( e + I \in \text{Id}(R/I) \) such that \( (a - e) + I \in \text{Reg}(R/I) \). Hence \( ((a - e) + I)(x + I)((a - e) + I) = (a - e) + I \) for some \( x \in R \). So \( (a - e) - (a - e)x(a - e) \in I \). Now, since \( I \) is regular, so \( a - e \in \text{Reg}(R) \) by
[3, Lemma 1]. Since idempotents can be lifted modulo $I$, we may assume that $e$ is an idempotent of $R$. Therefore $a$ is $r$-clean, as required. □

**Lemma 2.9.** Let $R$ be a ring with no zero divisors. Then $R$ is clean if $R$ is $r$-clean.

**Proof.** For every $x \in R$, we write $x = e + r$, where $e \in Id(R)$ and $r \in Reg(R)$. Then there exists $y \in R$ such that $ryr = r$. Now, if $r = 0$, then $x = e = (2e - 1) + (1 - e)$ is clean. But if $r \neq 0$, then since $R$ is a ring with no zero divisors and $ryr = r$, so $r \in U(R)$. Hence $x$ again is clean. □

**Corollary 2.10.** Let $R$ be a ring with no zero divisors. Then $R$ is $r$-clean if and only if $R$ is local.

**Proof.** It follows from [13, Lemma 14] and Lemma 2.9. □

**Theorem 2.11.** Let $R$ be a ring. Then $R$ is $r$-clean if and only if every element $x \in R$ can be written as $x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$.

**Proof.** Let $R$ be $r$-clean and $x \in R$. Then as $R$ is $r$-clean, so $-x = r + e$, where $r \in Reg(R)$ and $e \in Id(R)$. Hence $x = (r) - e$, where $-r \in Reg(R)$ and $e \in Id(R)$.

Conversely, suppose that every element $x \in R$ can be written as $x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$. So for every element $x \in R$, we can write $-x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$. Hence $x = (-r) + e$, where $-r \in Reg(R)$ and $e \in Id(R)$. □

A ring $R$ is said to be von Neumann local if for any $a \in R$, either $a$ or $1 - a$ is regular. Some characterizations of von Neumann local rings have been studied in [8], [14] and [15]. In the following we give a relation between $r$-clean and von Neumann local rings.

**Theorem 2.12.** Let $R$ be a ring such that 0 and 1 are the only idempotents in $R$. Then $R$ is $r$-clean if and only if it is von Neumann local.

**Proof.** Let $R$ be an $r$-clean ring and assume that 0 and 1 are the only idempotents in $R$. Then for any $a \in R$, either $a$ or $a - 1$ is regular. Hence $R$ is von Neumann local.

Conversely, let $R$ be von Neumann local. Then for any $a \in R$, either $a$ or $1 - a$ is regular. Now, if $a$ is regular, then $a$ is $r$-clean. If $1 - a$ is regular, then so is $a - 1$ and hence, $a$ is $r$-clean. □
Corollary 2.13. Let $R$ be an $r$-clean ring with no nontrivial idempotents. Then $R$ is exchange.

Proof. As every regular element $a$ of a ring $R$ is an exchange element (in the sense that there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$), so von Neumann local rings are exchange rings. Therefore, the result is clear by Theorem 2.12. □

Let $A$ and $B$ be rings, $M = B M A$ a bimodule and let the formal triangular matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ be $r$-clean, then both $A$ and $B$ are $r$-clean by [1, Theorem 16]. In the following theorem we have some conditions under which $T$ is $r$-clean.

Theorem 2.14. Let $A$ and $B$ be rings, $M = B M A$ a bimodule and assume that one of the following conditions holds:

1. $A$ and $B$ are clean.
2. one of the rings $A$ and $B$ is clean and the other one is $r$-clean.

Then the formal triangular matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ is $r$-clean.

Proof. (1). It is proved in [10] that if $A$ and $B$ are clean, then $T$ is clean. So it is $r$-clean.

(2). Case 1. Let $A$ be $r$-clean and let $B$ be clean. Then for every $t = \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \in T$, we have $a = e_1 + r$ and $b = e_2 + u$ for some $e_1, e_2 \in Id(R)$, $r \in Reg(R)$ and $u \in U(R)$. Assume that $ryr = r$ for some $y \in R$. Write $t = E + W$ where $E = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ and $W = \begin{pmatrix} r & 0 \\ m & u \end{pmatrix}$.

Obviously, $E = E^2$ and the equality

$$\begin{pmatrix} r & 0 \\ m & u \end{pmatrix} \begin{pmatrix} y \\ -u^{-1}my \\ u^{-1} \end{pmatrix} \begin{pmatrix} r & 0 \\ m & u \end{pmatrix} = \begin{pmatrix} r & 0 \\ m & u \end{pmatrix}$$

implies that $W$ is a regular matrix. Hence $t$ is $r$-clean.

Case 2. Let $A$ be clean and let $B$ be $r$-clean. Then the proof is similar to Case 1. □

Theorem 2.15. Let $R$ be a ring with no nontrivial idempotents and let $0 \neq a \in R$ be $r$-clean. Then for any $b \in R$, $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is $r$-clean in $M_2(R)$. 
Proof. If \( a = 1 \), then \( A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} \) which shows that \( A \) is \( r \)-clean. Now, let \( a \neq 1 \). Then \( a = r \) or \( a = 1 + r \), where \( r \in \text{Reg}(R) \). So there exists \( y \in R \) such that \( ryr = r \).

Case 1. If \( a = r \), then
\[
\begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & ryb \\ 0 & 0 \end{pmatrix}.
\]
But 0 and 1 are the only idempotents in \( R \), so \( ry = 0 \) or \( ry = 1 \). If \( ry = 0 \), then \( a = r = ryr = 0 \), which is a contradiction by hypothesis. Thus \( ry = 1 \). Hence \( ryb = b \) and \( A = \begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix} \) is regular. So \( A \) is \( r \)-clean.

Case 2. If \( a = 1 + r \), then \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix} \). But as 0 and 1 are the only idempotent in \( R \), \( ry = 0 \) or \( ry = 1 \). If \( ry = 0 \), then \( r = 0 \) and so \( a = 1 \), which is a contradiction by hypothesis. Thus \( ry = 1 \). Therefore similarly, we can see that \( \begin{pmatrix} r & b \\ 0 & 0 \end{pmatrix} \) is regular. Hence \( A \) is \( r \)-clean. \( \square \)

**Theorem 2.16.** Let \( R \) be a ring and let \( \text{diag}(a_1, ..., a_n) \) be the \( n \times n \) diagonal matrix with \( a_i \) in each entry on the main diagonal. If \( a_1, ..., a_n \in R \) is \( r \)-clean, then \( \text{diag}(a_1, ..., a_n) \) is \( r \)-clean in \( M_n(R) \).

Proof. Write \( a_i = e_i + r_i \), where \( e_i \in \text{Id}(R) \) and \( r_i \in \text{Reg}(R) \) for every \( i \), \( 1 \leq i \leq n \). Hence \( \text{diag}(a_1, ..., a_n) = \text{diag}(e_1, ..., e_n) + \text{diag}(r_1, ..., r_n) \). But there exists \( y_i \in R \) such that \( r_i y_i r_i = r_i \) for every \( i \), \( 1 \leq i \leq n \). Thus \( \text{diag}(r_1, ..., r_n) \text{diag}(y_1, ..., y_n) \text{diag}(r_1, ..., r_n) = \text{diag}(r_1, ..., r_n) \). Hence \( \text{diag}(r_1, ..., r_n) \) is regular and since \( \text{diag}(e_1, ..., e_n) \) is idempotent, \( \text{diag}(a_1, ..., a_n) \) is \( r \)-clean in \( M_n(R) \). \( \square \)

For more examples of \( r \)-clean rings, in the following we consider trivial extension and ideal-extension.

Let \( R \) be a ring and let \( R M_R \) be an \( R-R \)-bimodule. The trivial extension of \( R \) by \( M \) is the ring \( T(R, M) = R \oplus M \) with the usual addition and the following multiplication
\[
(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).
\]
The trivial extension of $R$ by a bimodule $R_{M_R}$ is isomorphic to the ring of all matrices $T = \begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. So a ring $R$ is clean if and only if its trivial extension is a clean ring. It is clear that this is true for abelian $r$-clean rings. But even if $R$ is not abelian and the trivial extension of $R$ is $r$-clean, then it is easy to check that $R$ is $r$-clean.

Let $R$ be a ring and let $R_{V_R}$ be an $R$-$R$-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then the ideal-extension $I(R; V)$ of $R$ by $V$ is defined to be the additive abelian group $I(R; V) = R \oplus V$ with the following multiplication

$$(r, v)(s, w) = (rs, rw + vs + vw).$$

**Proposition 2.17.** Let $R$ and $V$ be as above. Then we have the following statements:

1. If the ideal-extension $I(R; V)$ is $r$-clean, then $R$ is $r$-clean.
2. An ideal-extension $I(R; V)$ is clean if $R$ is clean and for every $v \in V$, there exists $w \in V$ such that $v + w + vw = 0$.
3. An ideal-extension $I(R; V)$ is $r$-clean if $R$ is $r$-clean and for every $v \in V$, there exists $w \in V$ such that $v + w + vw = 0$.
4. An ideal-extension $I(R; V)$ is $r$-clean if $R$ is $r$-clean and for every $v \in V$ and $y, x \in R, vyr + vyv + ryv = v$.

**Proof.** Let $E = I(R; V)$. For the proof of (1), let $x \in R$. Then $(x, 0) \in E$. Thus there exists $(e_1, e_2) \in Id(E)$ and $(r_1, r_2) \in Reg(E)$ such that $(x, 0) = (e_1, e_2) + (r_1, r_2)$. But $(r_1, r_2)(y_1, y_2)(r_1, r_2) = (r_1, r_2)$ for some $(y_1, y_2) \in E$. So $r_1y_1r_1 = r_1$. Hence $r_1 \in Reg(R)$ and since $e_1 \in Id(R)$, it follows that $x = e_1 + r_1$ is $r$-clean.

For the proof of (2), see [13, Proposition 7]. The assertion in (3) follows from (2). Finally, for the proof of (4), let $w = (a, v) \in E$. Then $a = e + r$, where $e \in Id(R)$ and $r \in Reg(R)$. But there exists $y \in R$ such that $ryr = r$. So $(r, v)(y, 0)(r, v) = (r, v)$ by hypothesis. Thus $(r, v)$ is regular in $E$. Also $(e, 0)$ is an idempotent in $E$. Hence $w = (e, 0) + (r, v)$ is $r$-clean. Therefore, $E$ is $r$-clean.

It is well known that the center of a regular ring is regular (see [9, Theorem 1.14]). Also Burgess and Raphael [4, Proposition 2.5], have shown that the center of a clean ring is not necessarily clean. In fact, we have a clean (so $r$-clean) ring such that its center is not clean. Now,
Rings in which elements are the sum of an idempotent and a regular element

since the center of a ring is commutative, it follows that we have an
r-clean ring such that its center is not r-clean. Therefore, the center
of an r-clean ring is not necessarily r-clean. But we have the following
theorem about the center of r-clean rings:

**Theorem 2.18.** Let R be an r-clean ring with no nontrivial idempotents. Then the center of R is also an r-clean ring.

**Proof.** Let Z(R) be the center of R and x ∈ Z(R). Then there exists
r ∈ Reg(R) such that x = r or x = r + 1 by hypothesis. If x = r,
then r ∈ Reg(Z(R)) and hence, x = 0 + r is r-clean in Z(R). But if
x = r + 1, then x − 1 = r is regular and x − 1 ∈ Z(R). So similar to
the previous case, x − 1 is regular in Z(R). Hence x again is r-clean in
Z(R). Therefore, Z(R) is r-clean.

□

**Theorem 2.19.** Let R be an abelian r-clean ring. Then R is indecomposable if and only if its center is a local ring.

**Proof.** Let R be an indecomposable abelian r-clean ring, Z(R) the center
of R and x ∈ Z(R). If x = 0 or x = 1, then it is clear that x or x − 1
are units in Z(R). Now, let x ≠ 0, 1. Then since R is r-clean, either x
or x − 1 is regular in Z(R) by the proof of Theorem 2.18.

Case 1. If x is regular in Z(R), then there exists y ∈ Z(R) such that
xyx = x. Now, since x ≠ 0 and xy, yx are idempotents in R, so
xy = yx = 1 and hence, x is a unit.

Case 2. If x − 1 is regular in Z(R), then (similar to Case 1) since x ≠ 1,
so x − 1 is a unit.

Therefore, either x or x − 1 is a unit in Z(R), as required.

The converse is trivial. □

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