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FIBER BUNDLES AND LIE ALGEBRAS OF TOP SPACES

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Abstract. In this paper, by using the Frobenius theorem a relation between Lie subalgebras of the Lie algebra of a top space T and Lie subgroups of T (as a Lie group) is determined. As a result we can consider these spaces by their Lie algebras. We show that a top space with finite number of identity elements is a C^{∞} principal fiber bundle, by this method we can characterize top spaces.

1. Introduction

Lie groups were initially introduced as a tool to solve or simplify ordinary and partial differential equations, and found numerous applications in Physics. Top spaces are smooth manifolds, in contrast with the case of more general Lie groups. The notion of top spaces as a generalization of Lie groups (Examples 1.7 and 1.8) was introduced by Molaei in 1998. In this generalized setting, several authors (Araujo, Molaei, Mehrabi, Oloomi, Tahmoresi, Ebrahimi, etc.) studied various aspects and concepts of generalized groups and top spaces, [1, 4, 5, 8]. Each Lie group is a top space, but there are top spaces which are not Lie groups. Note that top spaces with finite number of identity elements have been characterized by diffeomorphic Lie groups, [10].

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There is a relation between Lie algebra of a top space T with finite number of identity elements and the one-parameter subgroups of T, in general this relation is not one to one, [11].

In the first section of this paper we introduce top spaces, which are generalizations of Lie groups [12], left invariant vector fields on top spaces, and Lie algebra of a top space.

Section 2 is devoted to the determination of Lie subalgebra of a Lie algebra of a top space T and Lie subgroups of T (as a Lie group, which is discussed in [11]).

In section three we study the principal fiber bundles whose bundle space is a top space. This generalization is different structure from principal fiber bundle with structural Lie groupoid which has introduced by A. Haefliger [7], and has studied by Gheorghe Ivan [6].

Now we recall the definition of a top space from [10].

Definition 1.1. *A non-empty Hausdorff smooth d-dimensional manifold T is called a top space if there is an action "." on T such that $t.s \in T$, for every $t, s \in T$ and satisfies the following conditions:*

From now on, we show $t.s$ by ts.

1. $(rs)t = r(st)$, *for all* $r, s, t \in T$;
2. *For each* $t \in T$, *there is a unique* $e(t) \in T$ *such that:* $te(t) = e(t)t = t$.
3. *For all* $t, s \in T$, $e(ts) = e(t)e(s)$.
4. *For each* $t \in T$, *there is an* $s \in T$ *such that* $ts = st = e(t)$, *s is shown by* t^{-1}.
5. *The mappings*

$$\begin{array}{rcl} m_1 : T \times T \to T & \quad \textit{and} \quad & m_2 : T \to T \\ (t, s) \mapsto ts & & t \mapsto t^{-1} \end{array}$$

are smooth mappings.

Remark 1.2. $e(t)$ *is called the identity of* t.

Remark 1.3. *In a Lie group the identity element is unique, but in a top space the identity element of an element is dependent on the given element.*

Remark 1.4. *The inverse of each element t of a top space, i.e.* $m_2(t)$, *is unique, then by* t^{-1} *we mean the inverse of* t.

Example 1.5. *The Euclidean subspace* $R^* = R - \{0\}$ *of the Euclidean space R with the product* $(a, b) \mapsto a|b|$ *is a top space.*

Example 1.6. *Each Lie group is top space.*

Example 1.7. *There exists a product on the $n-$torus $\mathbb{T}^n$ such that $\mathbb{T}^n$ is a top space, which is not a Lie group,* [11].

Example 1.8. *The d-dimensional Euclidean space R^d with the product:*

$$((a_1, \ldots, a_d), (b_1, \ldots, b_d)) \mapsto (\tfrac{da_1+\sum b_i}{d}, \ldots, \tfrac{da_d+\sum b_i}{d})$$

is a top space which is not a Lie group.

Note: In this paper the symbol $*$ shows the derivative of mapping. A C^∞-vector field Y on a top space T is called a left invariant vector field, if $(L_t)_{*t}(Y(s)) = (YoL_t)(s)$, where $t, s \in T$, $L_t : T \longrightarrow T$ is defined by $L_t(s) = ts$, and $(L_t)_{*s}$ is the derivative of L_t at s.

In fact Y is a left invariant vector field on a top space T if and only if $(L_t)_{*e(s)}(Y(e(s))) = Y(te(s))$, for all $t, s \in T$.

For example the subspace $R - \{0\}$ of the Euclidean space R with the product $(a, b) \mapsto a|b|$ is a top space, a C^∞-vector field Y on $R - \{0\}$ is a left invariant vector field if and only if $Y : R - \{0\} \to R$ is defined by $Y(u) = \alpha u$, for some constant number $\alpha \in R$.

The set of all left invariant vector fields on a top space T with the Lie bracket is called Lie algebra of T and usually denoted by $\mathcal{T}$. The Lie algebras of top spaces were considered in paper, [11].

Here we present two crucial theorems which characterize some top spaces, [10].

Theorem 1.9. *Suppose that $T$is a one-dimensional top space and the cardinality of $e(T)$ is finite, if $e^{-1}(e(t))$ is connected, for all $t\epsilon T$, then T is isomorphic to $\oplus_{card(e(t))}A_i$, where $A_i = \mathbb{R}^1 or A_i = \mathbb{S}^1$,* [10].

Theorem 1.10. *If T is a two-dimensional top space and $e^{-1}(e(t))$is a connected set, for all $t\epsilon T$, then T is isomorphic to $\oplus A_i$, where $A_i = \mathbb{R}^2, A_i = \mathbb{T}^2, A_i = \mathbb{R} \times S^1$ or the identity connected component T_0^t of the group of affine motions of the real line on Lie group $e^{-1}(e(t))$, where $e^{-1}(e(t)) = \{s \in T \mid e(t) = e(s)\}$,* [10].

Let T be a top space and G be a topological group, a covering map $P : T \to G$ is called a top space covering map of G, if P satisfies in the following condition:

$$P(t_1t_2) = P(t_1)P(t_2), \text{ for all } t_1, t_2 \in T,$$

see [11].

Theorem 1.11. *Let $P : T \to G$ be a top space covering for a topological group of G. Then there is a correspondence (but not necessarily one to one) between the one-parameter subgroups of G and the one-parameter subgroups of T,* [11].

This theorem, which has been proved in [11], determines the set of one-parameter subgroups in a top space with respect to a topological group.

The main corollary of theorem 1.8 is the following:

Corollary 1.12. *For a connected top space T, connected topological group G and a top space covering $P : T \to G$, there exists a unique Lie group structure on T such that $e(t_0)$ is the identity element, for some $t_0 \in T$, and the Lie algebra of T (as a Lie group) is equal to the Lie algebra of T (as a top space),* [11].

2. Lie algebras of top spaces

In this section by G we mean a topological group.

Definition 2.1. *Let T be a top space with the finite number of identities. A top space S is called a subtop space of T, if there is a one to one C^∞ map $i : S \to T$ with the following conditions:*

(1) *i_* is one to one;*
(2) *$i(s_1 s_2) = i(s_1)i(s_2)$, for all $s_1, s_2 \in S$.*

Example 2.2. *The Euclidean space R with the product $(a, b) \mapsto a$ is a top space. The subspace $(0, 1)$ with the same product, is a subtop space, because the inclusion $i : (0, 1) \to R$ satisfies the conditions of definition 2.1.*

By using the Frobenius theorem we present a theorem, which determines a relation between two structures of T, in the first structure T is a top space with finite number of identities and in the second one T is a Lie group, which is considered in Corollary 1.12. Note that these two structures depend on a top space covering.

Theorem 2.3. *Let $P : T \to G$ be a top space covering for a Lie group G. Every subtop space of T corresponds to a Lie subalgebra of the Lie algebra T, and every Lie subalgebra of the Lie algebra T corresponds to a Lie subgroup of T (as a Lie group, which is considered in Corollary 1.12).*

Proof. Let H be a subtop space of T, and h be the Lie algebra of H. We define $\tilde{h} = \{X \mid x \in h\}$, where $X(t) = (L_t)_{*t}(x(e))$, e is the identity element of T (as a Lie group in Corollary 1.12), it is easy to show that $\tilde{h}$ is a vector space and it is closed under the Lie bracket. Since the Lie algebra of T (as a top space) is equal to the Lie algebra of T (as a Lie group), $\tilde{h}$ is a Lie subalgebra of the Lie algebra T (as a top space).

Conversely, let $\mathcal{T}$ be a Lie algebra of T and h be a subalgebra of $\mathcal{T}$. Since h is a vector space, there are C^∞ left invariant vector fields $X_1, X_2, \ldots, X_d$ such that

$$h = span(X_1, \ldots, X_d).$$

So $\Theta(t) = span(X_1(t), \ldots, X_d(t))$ is an involutive C^∞ distribution on T, for all $t \in T$. Frobenius theorem implies that there is a unique maximal connected integral submanifold H of T (as a Lie group in Corollary 1.12) containing the identity element e. We know that H is a Lie subgroup, [2]. □

Corollary 2.4. *Let T be a top space with finite number of identities and $P : T \to G$ be a top space covering for a Lie group G. Let $\mathcal{T}$ be the Lie algebra of T, for every $X \in \mathcal{T}$ there is a one-parameter subgroup of T (as a Lie group in Corollary 1.12) generated by X; that is, there is a curve $\zeta : R \to T$ such that:*

(1) $\zeta(t_1 + t_2) = \zeta(t_1)\zeta(t_2)$;
(2) $X(\zeta(s)) = \zeta_*(s)$.

Note: In Corollary 2.4, if we consider T as a top space with finite number of identities, then for every left invariant vector field X we can find $|e(T)|$ one-parameter subgroups of T, [11].

Now, we recall a main lemma in the theory of manifolds, which characterizes the tangent space of the product of manifolds at a point.

Lemma 2.5. *Let M, N and P be manifolds, $\theta : M \times N :\longrightarrow P$ be a C^∞ map and θ_m, θ_n be defined by:*

$\theta_m : N \to P$ *by* $\theta_m(n) = \theta(m, n)$;
$\theta^n : M \to P$ *by* $\theta^n(m) = \theta(m, n)$.

Then $\theta_*(s+t) = (\theta^n)_*(s) + (\theta_m)_*(t)$, *for* $s+t \in (M \times N)_{(m,n)}$, $s \in M_m$ *and* $t \in N_n$, [2].

By the above lemma we deduce:

Note that, in this paper by $\overset{o}{\bigcup}$ we mean disjoint union.

Lemma 2.6. *Let $\kappa : T \to T$ be defined by $\kappa(t) = t^{-1}$, where T is a connected top space with finite number of identities. Moreover let $s \in T_t$. Then $(\kappa)_*(s) = -(R_{t^{-1}})_* o (L_{t^{-1}})_*(s)$, where R_t is the right multiplication by $t \in T$, and $(\kappa)_*$ is the derivative of κ.*

Proof. First we define two maps α and β by:

$$\begin{array}{lcr} \alpha : T \to T \times T & \text{by} & t \longmapsto (t,t) \\ \beta : T \times T \to T & \text{by} & (t_1, t_2) \longmapsto t_1 t_2. \end{array}$$

$\beta o (1 \times \kappa) o \alpha$ is constant, for all $t \in e^{-1}(e(s))$, $(\beta)_* o (1 \times \kappa)_* o (\alpha)_* = 0$, because:

$$Card(e(T)) < \infty \text{ and } T = \overset{o}{\bigcup_{e(t) \in e(T)}} (e^{-1}(e(t))), \text{ see [11].}$$

Let $s \in T_t$. Then

$$\begin{aligned} 0 &= (\beta)_* o (1 \times \kappa)_* o (\alpha)_* \\ &= (\beta)_* o ((1_*(s) + (\kappa)_*(s)) \\ &= (\beta)_*(s + (\kappa)_*(s)) \\ &= (R_{t^{-1}})_*(s) + (L_t)_*((\kappa)_*(s)). \qquad \text{(Lemma 2.5)} \end{aligned}$$

Then we have $(L_t)_*((\kappa)_*(s)) = -(R_{t^{-1}})_*(s)$.

By multiplying $(L_{t^{-1}})_*$ from the left side, we get $(\kappa)_*(s) = -(L_{t^{-1}})_* o (R_{t^{-1}})_*(s)$.

Since

$$\begin{aligned} (L_{t^{-1}})_* o (R_{t^{-1}})_*(s) &= (L_{t^{-1}} o R_{t^{-1}})_*(s) \\ &= (R_{t^{-1}} o L_{t^{-1}})_*(s) \\ &= (R_{t^{-1}})_* o (L_{t^{-1}})_*(s), \end{aligned}$$

$(\kappa)_*(s) = -(R_{t^{-1}})_* o (L_{t^{-1}})_*(s)$.

□

Corollary 2.7. *Let Y be an element of the Lie algebra T. Then $(\kappa)_* Y$ is right invariant.*

There is a parallel formulation of Lie algebra of a top space in terms of right invariant vector fields. $(\kappa)_*$ determines a connection with our formulation.

3. Fiber bundles of top spaces

In this section by constructing a fiber bundle we characterize top spaces. This fiber bundle has top space as a bundle space [10], and enable us to construct many different fiber bundles with this structure.

Now we present the main theorem of this section.

Theorem 3.1. *Let T be a top space with finite number of identity elements. Then there exists a C^∞ principal fiber bundle with the bundle space T, and the structural group $e^{-1}(e(t_0))$, where $t_0 \in T$.*

Proof. If $M = T$, then we show that $(T, e^{-1}(e(t_0)), M)$ is a C^∞principal fiber bundle, where T is the bundle space , M is the base bundle and $t_0 \in T$. We know that $e^{-1}(e(t_0))$ is a Lie group with identity element $e(t_0)$. Let $T \times e^{-1}(e(t_0)) \longrightarrow T$ be defined by $(t, s) \mapsto ts$. Then this C^∞ mapping is a free action, because :

If $s \in e^{-1}(e(t_0))$, $s : T \longrightarrow T$ is defined by $s(t) = ts$, then it is enough to show that s is diffeomorphism. If $t_1, t_2 \in T$ and $s(t_1) = s(t_2)$, then we have the following process:

$$\begin{aligned} s(t_1) &= s(t_2) \\ t_1 s &= t_2 s \quad \text{(by definition)} \\ (3.1)\ t_1 e(t_0) &= t_2 e(t_0) \quad \text{(by multiplying } s^{-1} \text{ from the right side)} \end{aligned}$$

by multiplying $e(t_1)$ from the right side we get

$$\begin{aligned} t_1 e(t_0) e(t_1) &= t_2 e(t_0) e(t_1) \\ (t_1 e(t_1)) e(t_0) e(t_1) &= t_2 e(t_0) e(t_1) \\ t_1 &= t_2 e(t_0) e(t_1) \quad \text{(because } e(t_1)e(t_0)e(t_1) = e(t_1)\text{)} \end{aligned}$$

by multiplying $e(t_2)$ from the right side we obtain that

$$\begin{aligned} t_1 e(t_2) &= t_2 e(t_0) e(t_1) e(t_2) \\ (3.2) \qquad t_1 e(t_2) &= t_2 \quad \text{(because } e(t_2)e(t_0)e(t_1)e(t_2) = e(t_2)\text{)} \end{aligned}$$

by replacing t_1 with t_2, we deduce $t_2 e(t_1) = t_1$. It is enough to show that $t_1, t_2 \in e^{-1}(e(t_1 t_2))$.

(3.1) implies $t_1 e(t_0) = t_2 e(t_0)$. So $(t_1 e(t_1 t_2)) e(t_0) = (t_2 e(t_1 t_2)) e(t_0)$.

Therefore

$$t_1(e(t_1 t_2) e(t_0) e(t_1 t_2)) = t_2(e(t_1 t_2) e(t_0) e(t_1 t_2)).$$

So

$$(t_1 e(t_1 t_2)) = (t_2 e(t_1 t_2)),$$

(3.2) and $t_2 e(t_1) = t_1$ imply:

$$t_1 = t_2.$$

Thus s is an injective map.

Since

$$e(t_1t_2)t_2 = e(t_1)(e(t_2)t_2) = e(t_1)t_2$$
$$\overset{by(3.2)}{=} e(t_1)(t_1e(t_2)) = t_1e(t_2) \overset{by(3.2)}{=} t_2$$

and

$$t_2e(t_1t_2) = (t_2e(t_2))e(t_1)e(t_2) = t_2(e(t_2)e(t_1)e(t_2)) = t_2,$$

$t_2 \in e^{-1}(e(t_1t_2))$.

With the same method $t_1 \in e^{-1}(e(t_1t_2))$.

It is easy to show that s is a surjective mapping. The inverse function theorem for manifolds implies that s is a diffeomorphism. Therefore $e^{-1}(e(t_0))$ acts (smoothly) on T from the right side.

Now we show that this action is a free action:

If $ts = t$, then $t^{-1}(ts) = t^{-1}t$. So $e(t)s = e(t)$.

Therefore $e(t_0)e(t)s = e(t_0)e(t)$.

So

$e(t_0)e(t)(e(t_0)s)e(t_0) = e(t_0)e(t)e(t_0)$ since $e(t_0)e(t)e(t_0) = e(t_0)$ and $s \in e^{-1}(e(t_0))$, $s = e(t_0)$.

Let $t \in T$ and $[t] = \{ts | s \in e^{-1}(e(t_0))\}$, it is easy to show that $[t] = e^{-1}(e(t_0))$.

Since $\bigcup_{t_0 \in T} e^{-1}e(t_0) = T$, T is a quotient space with the quotient map

$\pi : T \longrightarrow T$, defined by $t \mapsto tt_0$.

Now we show that T is locally trivial:

If $t \in T$, $U_t = e^{-1}(e(tt_0))$ and $Card(e(T)) < \infty$, then U_t is an open subset of T and $\pi^{-1}(U_t) = e^{-1}(e(t))$,

we define $F_{U_t} : e^{-1}(e(t)) \longrightarrow e^{-1}(e(t_0))$ by $t_0 \mapsto e(t_0)t_1e(t_0)$,

$R_s \circ F_U = se(t_0)t_1e(t_0) = e(t_0)st_1e(t_0) = F_U \circ R_s(t_1)$,

where $R_s : T \longrightarrow T, s \in e^{-1}(e(t_0))$, is defined by $R_s(t) = ts$.

Therefore $(T, e^{-1}(e(t_0)), T)$ is a C^∞principal fiber bundle. $\square$

Now we give new some easy proofs for corollaries of this theorem which have been proved in [10] by difficult proofs.

Corollary 3.2. *Let T be a top space with finite number of identity elements. Then $e^{-1}(e(t))$ are diffeomorphic, for all $t \in T$.*

Proof. Obviously, in principal fiber bundle, every fibers $e^{-1}(e(t))$ are diffeomorphic, then $e^{-1}(e(t))$ are diffeomorphic, for all $t \in T$. $\square$

Corollary 3.3. *If T is a top space with the finite number of identity elements, then $T = \overset{\circ}{\bigcup\limits_{t\in T}}(e^{-1}(e(t)))$, where $e^{-1}(e(t))$ are diffeomorphic Lie groups, for all $t \in T$.*

Proof. Since $(T, e^{-1}(e(t_0), T)$ is a C^∞ principal fiber bundle, every fibers are isomorphic to Lie group $e^{-1}(e(t))$ and so they are Lie groups diffeomorphic to $e^{-1}(e(t_0))$. □

Theorems 1.9 and 1.10 imply that

Corollary 3.4. *Every one-dimensional top space with finite number of identity elements is a C^∞principal fiber bundle, with the structural group $\mathbb{R}$ or $\mathbb{S}^1$, but not both of them.*

Corollary 3.5. *If T is a two-dimensional top space, then T is a C^∞ principal fiber bundle with structural group G, where $G = \mathbb{R}^2, \mathbb{T}^2, \mathbb{R}\times\mathbb{S}^1$ or identity connected component T_0^t of the group of affine motions of the real line on Lie group $e^{-1}(e(t))$, where $t \in T$.*

Conclusion: In this paper we present a relation between the Lie subalgebras of the Lie algebra of a top space T and subtop spaces of T. Moreover a relation between two structure of T, i.e. Lie subalgebras of the Lie algebra T and Lie subgroups of the Lie group T, is deduced.

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