

## RELATIVE N-TH NON-COMMUTING GRAPHS OF FINITE GROUPS

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ABSTRACT. Suppose  $n$  is a fixed positive integer. We introduce the relative  $n$ -th non-commuting graph  $\Gamma_{H,G}^n$ , associated to the non-abelian subgroup  $H$  of group  $G$ . The vertex set is  $G \setminus C_{H,G}^n$  in which  $C_{H,G}^n = \{x \in G : [x, y^n] = 1 \text{ and } [x^n, y] = 1 \text{ for all } y \in H\}$ . Moreover,  $\{x, y\}$  is an edge if  $x$  or  $y$  belong to  $H$  and  $xy^n \neq y^n x$  or  $x^n y \neq yx^n$ . In fact, the relative  $n$ -th commutativity degree,  $P_n(H, G)$  the probability that  $n$ -th power of an element of the subgroup  $H$  commutes with another random element of the group  $G$  and the non-commuting graph are the keys to construct such a graph. It is proved that two isoclinic non-abelian groups have isomorphic graphs under special conditions.

### 1. Introduction

Erdős associated a graph  $\Gamma$  to the group  $G$ , whose vertex set is  $G$  (as a set) and two vertices join whenever they do not commute. He asked whether there is a finite bound for the cardinalities of cliques in  $\Gamma$ , if  $\Gamma$  has no infinite clique. This problem was posed by Neumann in [13] and gave a positive answer to Erdős's question. Of course, there are some other ways to make a graph associated to a group or semigroup. One may refer to the works of Bertram et al. [4], Grunewald et al. [7],

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Moghadamfar et al. [11] and Williams [14] or recent papers on non-commuting graph, Engel graph and non-cyclic graph in [2], [1] and [3], respectively.

Let  $H$  be a subgroup of a finite group  $G$  and  $n$  a positive integer. Mohd Ali et al. in [12] introduced  $n$ -th commutativity degree which is the probability that the  $n$ -th power of a random element of  $G$  commute with another element. If  $n = 1$  then this probability coincide to the commutativity degree which was investigated by Erdős and Turan in [5]. Later Erfanian et al. improved their results to the relative case. They defined the relative  $n$ -th commutativity degree of a subgroup  $H$  of  $G$  as is the probability of commuting the  $n$ -th power of an element of the subgroup  $H$  with an element of group  $G$ . They generalized several facts which is valid for commutativity degree in [6].

By these facts about the graphs and the probability, we led to introduce the relative  $n$ -th non-commuting graph. We define the relative  $n$ -th non-commuting graph  $\Gamma_{H,G}^n$ , with vertex set  $G \setminus C_{H,G}^n$ , where  $C_{H,G}^n = \{x \in G : [x, y^n] = 1 \text{ and } [x^n, y] = 1 \forall y \in H\}$ . Moreover,  $\{x, y\}$  is an edge if  $x$  or  $y$  belong to  $H$  and  $xy^n \neq y^n x$  or  $x^n y \neq y x^n$ . We discuss about diameter, dominating set, domination number, clique number and planarity of the graph. Third section is organized to state connections between the  $n$ -th non-commuting graph and the  $n$ -th commutativity degree. We find a lower bound for the number of edges of the  $n$ -th non-commuting graph and also we show that this lower bound is sharp for nilpotent groups of class 2. Moreover, we prove that there is no relative  $n$ -th non-commuting star graph,  $n$ -th non-commuting complete graph and relative  $n$ -th noncommuting complete bipartite graph associated to an AC-group. We discuss about non-regularity of the relative  $n$ -th non-commuting graph in some special cases as well. Furthermore, it is proved that two isoclinic non-abelian groups are associated to the isomorphic graphs in special cases.

## 2. The $n$ -th Non-commuting Graph

Let us start with the definition of the relative  $n$ -th non-commuting graph.

**Definition 2.1.** *Let  $n$  be a fixed positive integer. We assign a graph to a non-abelian subgroup  $H$  of group  $G$ . The vertex set is  $G \setminus C_{H,G}^n$  where  $C_{H,G}^n = \{x \in G : [x, y^n] = 1 \text{ and } [x^n, y] = 1 \text{ for all } y \in H\}$  and*

two distinct vertices  $x$  and  $y$  join together whenever  $x$  or  $y$  belong to  $H$  and  $[x, y^n] \neq 1$  or  $[x^n, y] \neq 1$ . We call such a graph relative  $n$ -th non-commuting graph and is denoted by  $\Gamma_{H,G}^n$ . If  $H = G$  then the graph  $\Gamma_G^n$  is  $n$ -th non-commuting graph.

It is clear that for  $n = 1$  the graph  $\Gamma_G^1$  corresponds to the non-commuting graph denoted by  $\Gamma_G$  in [2]. Note that if  $H$  is an abelian subgroup of  $G$  or has exponent that divides  $n$  then there is no vertex in  $H$  but some vertices in  $G \setminus H$  may exist and hence  $\Gamma_{H,G}^n$  is an empty graph. Similarly, if the exponent of the group  $G$  divides  $n$  or  $G$  is an abelian then  $\Gamma_G^n$  is a null graph. Furthermore, if  $G$  is a group which its central factor is elementary abelian  $p$ -group of rank  $s$ . If  $m \equiv n \pmod{p}$  then  $\Gamma_G^m = \Gamma_G^n$ .

**Example 2.2.**

- (i) If  $n$  is an even or odd number and  $G = D_8$  then  $\Gamma_G^n$  has no vertices or  $\Gamma_G^n = \Gamma_G$ , respectively.
- (ii) For even or odd number  $n$ ,  $V(\Gamma_{S_3}^n) = V(\Gamma_{S_3})$ .
- (iii) Let  $G$  be a group of exponent  $m$  such that  $n \equiv r \pmod{m}$ , then  $\Gamma_G^n = \Gamma_G^r$ .

**Theorem 2.3.** For any non-abelian group  $G$  and  $n \in \mathbb{N}$ ,  $\text{diam}(\Gamma_G^n) \leq 4$ . Moreover, the girth of  $\Gamma_G^n$  is less than or equal to 4.

*Proof.* Let  $x$  and  $y$  be two distinct vertices of the graph such that they are not adjacent. We may assume  $xy^n = y^n x$  and  $x^n y = y x^n$ . Since  $x$  and  $y$  are not in  $C_G^n$ , there exist  $x_1, y_1 \in V(\Gamma_G^n)$  where  $\{x, x_1\}$  and  $\{y, y_1\}$  are edges. If  $x$  and  $y_1$  or  $y$  and  $x_1$  are adjacent then  $d(x, y) = 2$ . Otherwise,  $([x, y_1^n] = 1 \text{ and } [x^n, y_1] = 1)$  and  $([x_1, y^n] = 1 \text{ and } [x_1^n, y] = 1)$ . Moreover, (i)  $[x_1, x^n] \neq 1$  or (ii)  $[x_1^n, x] \neq 1$  and (iii)  $[y_1, y^n] \neq 1$  or (iv)  $[y_1^n, y] \neq 1$ . Thus, we have four cases. If (i) and (iii) then  $x_1 y_1$  is a vertex which is join  $x$  and  $y$ , hence  $d(x, y) = 2$ . Suppose (ii) and (iv). If  $x_1$  joins  $y_1$  then  $d(x, y) \leq 3$ . Assume  $x_1$  and  $y_1$  are not adjacent, then  $xy$  is a vertex which joins  $x_1$  and  $y_1$  so  $d(x, y) \leq 4$ . Now, suppose (i) and (iv). If  $x_1$  and  $y_1$  are adjacent then  $d(x, y) \leq 3$ . Otherwise we have  $x_1 y_1^n = y_1^n x_1$  and  $x_1^n y_1 = y_1 x_1^n$  then  $x_1 y_1$  is a vertex which is adjacent to  $y_1$  and  $x$  because  $[x_1 y_1, y_1^n] \neq 1$  and  $[x_1 y_1, x^n] \neq 1$  and this means  $d(x, y) \leq 3$ . Similarly, for the case (ii) and (iii) if  $x_1$  and  $y_1$  are adjacent then  $d(x, y) \leq 3$ . Otherwise,  $x y_1$  is a vertex adjacent to  $x_1$  and  $y$  so  $d(x, y) \leq 3$ .

In order to compute the girth, let  $\{x, y\}$  be an edge, if  $x^n y \neq y x^n$  then  $xy$

is a vertex which joins  $x$ . Consider two cases, if  $xy^n \neq y^n x$  then clearly  $xy$  and  $y$  are adjacent so we have cycle with length 3. If  $xy^n = y^n x$  then for vertices  $x$  and  $y$  there exist  $x_1$  and  $y_1$  respectively. Now, without loss of generality suppose  $y = x_1$  and  $[x^n, y] \neq 1$ . If  $x$  joins  $y_1$  then we have a triangle, so suppose not. Consider  $[y^n, y_1] \neq 1$  consequently  $yy_1$  is a vertex that joins  $y$  and  $x$ , a triangle obtained. Moreover, if  $[y, y_1^n] \neq 1$  then  $xy$  is a vertex which joins  $x$  and  $y_1$  so we have a square. Suppose  $y_1 \neq x$  and  $x_1 \neq y$  with  $xy^n = y^n x$ . If  $y$  joins  $x_1$  or  $x$  and  $y_1$  are adjacent then we get a triangle so suppose not. We have (i)  $[x_1, x^n] \neq 1$  or (ii)  $[x_1^n, x] \neq 1$  and (iii)  $[y_1, y^n] \neq 1$  or (iv)  $[y_1^n, y] \neq 1$ . Now, consider four cases similar as the argument for the diameter. Note that, if (ii) and (iii) then  $xy_1$  is a vertex joins  $y$ . Suppose  $x_1$  and  $y_1$  are not adjacent, so  $xy_1$  join  $x_1$  and we make a cycle of length 4. Furthermore, if (ii), (iv) and  $x_1$  and  $y_1$  are not adjacent, then  $\{x, y, y_1, xy\}$  is a cycle of length 4. For the case, (i) and (iv) in the worst conditions i.e.  $\{x_1, y_1\}$ ,  $\{x, y_1\}$  and  $\{x_1, y\}$  are not edges then  $xy$  is a vertex which joins  $y_1$  and  $x$  so we have a square. Case (i) and (iii), follows by similar method that  $x_1y_1$  is a vertex which joins  $x$  and  $y$ .  $\square$

In particular,  $\text{diam}(\Gamma_G) = 2$  and the girth of  $\Gamma_G$  is 3 (see [2]). We define  $C_G^n(S) = \{y \in G : [s, y^n] = 1 \text{ and } [s^n, y] = 1 \text{ for all } s \in S\}$ , where  $S$  is the subset of  $G$ .

Now, let us deal with the dominating set of the graph. The following results are generalizations of Remark 2.5, Proposition 2.12 part (1), Remark 2.13 and Proposition 2.14 in [2] so we prefer to omit some of the proofs.

**Proposition 2.4.** *If  $\{x\}$  is a dominating set for  $\Gamma_{H,G}^n$  where  $x \in H$  and  $n$  is an odd number, then  $x^2 = 1$ ,  $C_{H,G}^n = 1$  and  $C_G^n(x) = \langle x \rangle$ .*

*Proof.* If  $x^2 \neq 1$ , then  $x^{-1}$  is not adjacent to  $x$ . If  $C_{H,G}^n$  contains a non-trivial element  $g$  then  $[g, x^n] = [g, x] = 1$  and  $[g^n, x] = 1$ . Therefore  $t = gx$  is a vertex which does not join  $x$ . Because for vertex  $x$ , there exists  $y \in H$  such that (i)  $xy^n \neq y^n x$  or (ii)  $x^n y \neq yx^n$ . If (i) or (ii), then  $t \in V(\Gamma_{H,G}^n)$  because  $[t, y^n] \neq 1$  or  $[t^n, y] \neq 1$ , respectively. Furthermore,  $t$  is not adjacent to  $x$ , since clearly  $[g^n x^n, x] = 1$  and  $[t, x^n] = [gx, x^n] = 1$ . For the last part, if  $g \in C_G^n(x)$  is distinct from  $x$  and 1, then as it is not in  $C_{H,G}^n$ , it is a vertex not adjacent to  $x$  and we get a contradiction.  $\square$

In the above proposition if  $H = G$  we can deduce similar result for  $\Gamma_G^n$  when  $n$  is odd. Furthermore, if  $\{x\}$  is a dominating set for  $\Gamma_G^n$  when  $n$  is odd, then  $Z(G) = 1$ . For a subgroup  $H$  of  $G$ ,  $S \subseteq H \cap V(\Gamma_{H,G}^n)$  is a dominating set of  $\Gamma_{H,G}^n$  if and only if  $C_G^n(S) \subseteq C_{H,G}^n \cup S$ . Again, by considering  $H = G$  similar assertion follows for  $\Gamma_G^n$ . It is easy to verify that  $V(\Gamma_{H,G}^n)$  is a dominating set for  $\Gamma_G^n$  whenever  $C_{H,G}^n = 1$ . If  $S$  is an dominating set for  $\Gamma_H^n$  then it is a dominating set for  $\Gamma_{H,G}^n$  whenever  $C_{H,G}^n = C_G^n(S)$ .

**Proposition 2.5.** *Let  $G$  be a finite non-abelian group and  $\omega(\Gamma)$  denotes the clique number of the graph  $\Gamma$ .*

- (i) *For any non-abelian subgroup  $H$  of  $G$ ,  $\omega(\Gamma_H^n) \leq \omega(\Gamma_G^n)$ .*
- (ii) *For any non-abelian factor group  $G/N$  of  $G$ ,  $\omega(\Gamma_{G/N}^n) \leq \omega(\Gamma_G^n)$ .*

*Proof.* The first part follows easily. For (ii) assume that  $\{g_1N, \dots, g_sN\}$  are the vertices for the maximal complete subgraph of  $\Gamma_{G/N}^n$ . As  $[g_i, g_j^n] \notin N$  or  $[g_i^n, g_j] \notin N$  we conclude that  $g_i^n g_j \neq g_j g_i^n$  or  $g_j^n g_i \neq g_i g_j^n$ . Thus,  $\{g_1, \dots, g_s\}$  is a complete subgraph of  $\Gamma_G^n$ . □

By the above proposition it follows that

$$\omega(\Gamma_{PSL(m,R)}^n) \leq \omega(\Gamma_{PGL(m,R)}^n) \leq \omega(\Gamma_{GL(m,R)}^n).$$

Obviously,  $C_G^n \subseteq C_{H,G}^n$  and  $\Gamma_{H,G}^n$  is an induced subgraph of  $\Gamma_G^n$ . Moreover,  $\Gamma_H^n$  is an induced subgraph of  $\Gamma_{H,G}^n$  and  $\Gamma_{H,G}^n$  itself is an induced subgraph of  $\Gamma_{H,G}^1$ . It is clear that  $Z(G) \subseteq C_G^n$  and if  $x$  joins  $y$  in  $\Gamma_G^n$  they are adjacent in  $\Gamma_G$ , consequently  $\Gamma_G^n$  is an induced subgraph of  $\Gamma_G$ . Recall from [2] that  $\Gamma_G$  is planar if and only if  $G \cong S_3$  or  $D_8$  or  $Q_8$ . Hence, if  $G \cong S_3$  or  $D_8$  or  $Q_8$  then the n-th non-commuting graph is planar.

A group is called an AC-group if the centralizer of the set of non-central elements is abelian. Let  $G$  be an AC-group. As  $C_G^n \subseteq C_G^n(x) \subseteq C_G(x^n)$  for a vertex  $x$ , elements of  $C_G^n$  commute with each other, we deduce  $C_G^n, C_G^n(x)$  are subgroups of  $G$  such that  $C_G^n(x) = C_G(x^n)$  and  $C_G^n = Z(G)$ . Hence,  $\Gamma_G^n = \Gamma_G$  in this case. Thus  $\Gamma_G^n$  which is associated to the AC-group  $G$  has the exact properties of the non-commuting graph  $\Gamma_G$  (see [2]).

**Theorem 2.6.** *Let  $G$  be an AC-group. Non-null graph  $\Gamma_G^n$  is planar if and only if  $G \cong S_3$  or  $D_8$  or  $Q_8$ .*

**Example 2.7.** *The  $n$ -th non-commutating graph  $\Gamma_{S_3, S_4}^n$  is not planar. Because  $K_5$  with vertices  $\{(1\ 2), (1\ 2\ 3), (2\ 3\ 4), (1\ 2\ 4), (1\ 3\ 4)\}$  and  $\{(1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 2\ 3\ 4)\}$  are its subgraph whenever  $n$  is an even or odd number respectively. Thus,  $\Gamma_{S_3, S_m}^n$  is not planar, for  $m \geq 4$ .*

### 3. $n$ -th Non-commuting Graphs and $P_n(G)$

In this section, by the use of probability theory and graph theory obtain some results for some graphs. Therefore, we should recall some necessary definitions. For a non-abelian group  $G$  the relative  $n$ -th commutativity degree is the probability of commuting the  $n$ -th power of an element of a subgroup  $H$  with an element of the group  $G$ . It is defined by the following ratio

$$P_n(H, G) = \frac{|\{(x, y) \in H \times G : [x^n, y] = 1\}|}{|H||G|},$$

(see [6] for more details). If  $H = G$  then the above probability is called  $n$ -th commutativity degree and denoted by  $P_n(G)$ . Furthermore,  $P_1(G) = d(G)$  which is called the commutativity degree. Assume

$$A = \{(x, y) \in H \times G : [x^n, y] = 1\}$$

and  $B = \{(x, y) \in H \times G : [x^n, y] = 1 \text{ and } [y^n, x] = 1\}$ . In general  $B \subseteq A$  so we can deduce

$$|E(\Gamma_{H, G}^n)| \geq |H||G|(1 - P_n(H, G)) - \frac{|H|^2}{2}(1 - P_n(H)).$$

It is clear that if  $H = G$  then  $|E(\Gamma_G^n)| \geq (|G|^2 - |G|^2 P_n(G))/2$ .

An element  $x$  of a group  $G$  is called  $n$ -Bell, if  $[x^n, y] = [x, y^n]$  for all  $y \in G$ .  $G$  is called  $n$ -Bell if all its elements are  $n$ -Bell. In this case  $|A| = |B|$  so the bound for the number of edges is sharp. Furthermore, for all nilpotent groups of class 2 we have  $|A| = |B|$ .

Every graph has at most  $m(m-1)/2$  edges, where  $m$  is the number of vertices. Thus, for a nilpotent group of class 2 like  $G$  follows

$$P_n(G) \geq \frac{2|C_G^n|}{|G|} + \frac{1}{|G|} - \frac{|C_G^n|^2}{|G|^2} - \frac{|C_G^n|}{|G|^2}.$$

If  $G_1$  and  $G_2$  are two nilpotent groups of class 2,  $|C_{G_1}^n| = |C_{G_2}^n|$  and  $\Gamma_{G_1}^n \simeq \Gamma_{G_2}^n$  then clearly  $P_n(G_1) = P_n(G_2)$ .

**Proposition 3.1.** *Let  $G$  be a non-abelian group whose central factor is an elementary abelian  $p$ -group of rank  $s$  where  $p$  is prime. If  $p$  does not divide  $n$  then*

$$|E(\Gamma_G^n)| \geq \frac{(|G| - |C_G^n|)(p-1)|G|}{2p}$$

*Proof.* Degree-edge formula  $2|E(\Gamma_G^n)| = \sum_{x \in V(\Gamma_G^n)} \deg(x)$ , implies

$$2|E(\Gamma_G^n)| = (|G| - |C_G^n|)|G| - \sum_{x \in V(\Gamma_G^n)} |C_G^n(x)|.$$

Moreover for every non-central element  $t \in G$  we have

$$p^s = [G : Z(G)] = [G : C_G(t)][C_G(t) : Z(G)] \leq [G : C_G(t)]p^{s-1}$$

Therefore,  $|C_G(t)| \leq |G|/p$ . Clearly,  $C_G^n(x) \subseteq C_G(x^n)$  and the assertion follows.  $\square$

In the above theorem if  $p$  divides  $n$  there is no vertex for this graph. Easily one can define a one to one correspondence map between  $C_G^n(x)$  and  $C_G^n(x^g)$  for  $g \in G$ . Thus, the degree of vertices of  $\Gamma_G^n$  in the same conjugacy class are equal. By the fact that  $|C_G^n(x)| \leq |C_G(x^n)| \leq |G|/2$  for an arbitrary vertex  $x$ , it follows that  $\Gamma_G^n$  is Hamiltonian graph.

**Theorem 3.2.** *There is no relative  $n$ -th non-commuting complete graph associated to an AC-group  $G$ .*

*Proof.* Suppose  $\Gamma_{H,G}^n$  is a relative  $n$ -th non-commuting complete graph. Therefore  $|G| - |C_G^n(h)| = |G| - |C_{H,G}^n| - 1$  for a vertex  $h \in H$ . Consequently  $|C_G^n(h)| = 2$  and  $|h| = 2$ . Thus  $H$  is abelian so there is no vertices in  $H$  which is a contradiction.  $\square$

By the same argument it follows that there is no  $n$ -th non-commuting complete graph associated to an AC-group  $G$ .

**Theorem 3.3.** *There is no  $n$ -th non-commuting complete graph  $\Gamma_G^n$  associated to a non-abelian nilpotent group  $G$  of class 2 with  $C_G^n = 1$ .*

*Proof.* Assume that we have such a graph. Thus  $P_n(G) = 3/|G| - 2/|G|^2$ . On the other hand  $k(G)/|G| = d(G) \leq P_n(G)$  where  $k(G)$  is number of conjugacy classes (see [8]). Hence,  $k(G) \leq 2$  that implies  $G$  is an abelian group, which is a contradiction.  $\square$

**Theorem 3.4.** *There is no relative  $n$ -th non-commuting star graph associated to an AC-group  $G$  and its subgroup  $H$ .*

*Proof.* on the contrary suppose that  $\Gamma_{H,G}^n$  is the relative  $n$ -th non-commuting star graph. Assume  $h \in H$  is the unique vertex of degree  $(|V(\Gamma_{H,G}^n)| - 1)$ . Then we conclude that  $C_{H,G}^n = 1$ . On the other hand, for a vertex  $g \in G \setminus H$  we have  $|C_H^n(g)| = |H| - 1$  so  $[H : C_H^n(g)] = |H|/(|H| - 1)$  which is impossible. Secondly, assume  $g \in G \setminus H$  is the unique vertex of degree  $(|V(\Gamma_{H,G}^n)| - 1)$  and all the other vertices, for instance  $h \in H$ , have degree 1. Then we have  $|G| = |C_G^n(h)| + 1$ . Therefore,  $|C_{H,G}^n| = 1$  and similarly,  $[G : C_G^n(h)] = |G|/(|G| - 1)$  which is impossible. Hence such a graph does not exist.  $\square$

Furthermore, in Theorem 3.4 if  $G$  is an AC-group which is nilpotent of class 2, then not only the associated  $n$ -th non-commuting graph is not star but also there is no group with  $P_n(G) = 1 - 2/|G| + 4/|G|^2$  whenever  $|C_G^n| = 1$ .

**Theorem 3.5.** *There is no  $n$ -th non-commuting complete bipartite graph associated to an AC-group  $G$ .*

*Proof.* Suppose  $\Gamma_G^n$  is an  $n$ -th non-commuting complete bipartite graph, where vertices are partitioned in to two disjoint sets  $V_1$  and  $V_2$  such that  $|V_1| + |V_2| = |G| - |C_G^n|$ . We should have  $deg(x) = |G| - |C_G^n(x)| \leq (|G| - |C_G^n|)/2$ ,  $|C_G^n|q = |C_G^n(x)|$  for some  $q \in \mathbb{Z}$  and  $x \in V(\Gamma_G^n)$ . Hence  $|G| \leq |C_G^n|(2q - 1)$  and so  $[G : C_G^n(x)] \leq (2 - (1/q)) < 2$  which is a contradiction.  $\square$

We claim that there is no relative  $n$ -th non-commuting complete bipartite graph. Otherwise, the only possibility is to have two disjoint sets  $V_1$  and  $V_2$  such that any of the sets  $V_1, V_2$  contains vertices of  $H$ . Since all vertices of  $H$  are not adjacent so if  $h \in V_1$  then  $[h^n, x] = 1$  and  $[x^n, h] = 1$  for all  $x \in H \setminus C_{H,G}^n$  which implies that  $h \in C_{H,G}^n$  and it a contradiction.

We finish this section by some interesting results about the non-regularity of the  $n$ -th non-commuting graph. Note that  $H$  is a non-trivial proper subgroup of  $G$ .

**Theorem 3.6.** *There is no relative  $n$ -th non-commuting  $m$ -regular graph associated to an AC-group  $G$ , where  $m$  is an square free positive odd integer.*

*Proof.* Suppose  $\Gamma_{H,G}^n$  is a relative  $n$ -th non-commuting graph which is  $m$ -regular and  $P = \{p_1, p_2, \dots, p_k\}$  is the set of distinct odd primes which factorize  $m$ . If  $h \in H$  is a vertex of the relative  $n$ -th non-commuting  $\Gamma_{H,G}^n$  then  $m = deg(h) = |C_G^n(h)|([G : C_G^n(h)] - 1)$ ,  $|C_G^n(h)| = \prod_{p_i \in P} p_i$  and



$([G : C_G^n(h)] - 1) = \prod_{p_j \in S^c} p_j$ , where  $S$  and  $S^c$  are complement subsets of  $P^* = P \cup \{1\}$  such that  $|C_G^n(h)| \neq 1$ . Thus  $|G| = \prod_{p_i \in S} p_i (\prod_{p_j \in S^c} p_j + 1)$ . By similar argument about the degree of a vertex belongs to  $G \setminus H$  it follows that  $|H| = \prod_{p_i \in T} p_i (\prod_{p_j \in T^c} p_j + 1)$ , where  $T$  and  $T^c$  are complement subsets of  $P^*$ . Since  $H$  is a subgroup of  $G$  we get that  $\prod_{p_i \in T \setminus T \cap S} p_i (\prod_{p_j \in T^c} p_j + 1)$  divides  $(\prod_{p_j \in S^c} p_j + 1)$  which is impossible.  $\square$

**Theorem 3.7.** *The graph  $\Gamma_{H,G}^n$  associated to an AC-group  $G$  is not  $2k$ -regular graph, where  $k$  is a square free positive odd integer.*

*Proof.* By similar method of previous theorem, we obtain several cases for the orders of  $H$  and  $G$  which none of them is valid.  $\square$

If  $\Gamma_{H,G}^n$  is a graph associated to an AC-group  $G$  of odd order then the degrees of its vertices are even numbers. By Theorem 3.7 it is not  $2k$ -regular graph, for an square free positive odd integer  $k$ . It is not  $2^r$ -regular graph because otherwise we have  $2^r = |C_G^n(h)|([G : C_G^n(h)] - 1)$  for  $h \in H \cap V(\Gamma_{H,G}^n)$ . Since  $|C_G^n(h)| \neq 1$  it follows that  $|C_G^n(h)| = 2^\alpha$ ,  $1 \leq \alpha \leq r$  and it is a contradiction. We guess  $\Gamma_{H,G}^n$  is not regular at all when the order of  $G$  is odd.

#### 4. n-th Non-commuting Graph and Isoclinism

We are going to attempt on a known conjecture which states that if two isomorphic graphs are associated to the groups  $G$  and  $H$ , then the groups  $G$  and  $H$  are isomorphic as well or at least their orders are equal. We show that two non-abelian isoclinic groups under an extra condition, are associated to some isomorphic graphs.

**Proposition 4.1.** *Let  $G$  be a non-abelian AC-group and  $\Gamma_G^n \cong \Gamma_{S_3}^n$  where  $n \not\equiv 0 \pmod{6}$ . Then  $G \cong S_3$ .*

*Proof.* By graph isomorphism  $|G| - |C_G^n| = |S_3| - |C_{S_3}^n|$ . Obviously, if  $n \equiv i \pmod{6}$  for all  $1 \leq i \leq 5$  then  $C_{S_3}^n = 1$  and  $|G| - |C_G^n| = 5$ . Since  $G$  is an AC-group  $|C_G^n| \mid 5$  so  $|C_G^n|$  is 1 or 5. If  $|C_G^n| = 5$  then  $|G| = 10$  and this implies that  $G = D_{10}$ . As  $\Gamma_{S_3}^n$  is not null it follows that  $n \not\equiv 0 \pmod{10}$ , because otherwise it is against the graph isomorphism. If  $n \equiv j \pmod{10}$ , for all  $1 \leq j \leq 9$  then  $C_{D_{10}}^n = 1$  and this is the contradiction. Hence,  $|C_G^n| = 1$  and result follows.  $\square$

**Proposition 4.2.** *Let  $G$  be a non-abelian AC-group and  $\Gamma_G^n \cong \Gamma_{D_{2m}}^n$  where  $n$  is an odd number and  $m \geq 4$ . Then  $|G| = |D_{2m}|$ .*

*Proof.* Suppose  $D_{2m} = \langle a, b : a^m = b^2 = 1, a^b = a^{-1} \rangle$ . First, suppose that  $m$  is an odd number. By a simple computation on elements we can find  $C_{D_{2m}}^n = 1$  and so  $|G| - |C_G^n| = 2m - 1$ . Moreover,  $C_{D_{2m}}^n(b) = \{e, b\}$  implies  $|G| - |C_G^n(b')| = 2m - 2$  for some element  $b' \in G$ . Consequently,  $|C_G^n(b')| - |C_G^n| = 1$ . Hence,  $|C_G^n| = 1$  and this completes the proof in this case. Now, let  $m$  be an even number. By a similar method as in the previous case,  $C_{D_{2m}}^n = \{1, a^{m/2}\}$  and so  $2m - 2 = |G| - |C_G^n|$ . Furthermore, by the fact that  $C_{D_{2m}}^n(b) = \{e, b, a^{m/2}, a^{m/2}b\}$  follows  $|G| - |C_G^n(b')| = 2m - 4$ , for some element  $b' \in G$ . Consequently,  $|C_G^n(b')| - |C_G^n| = 2$ . Thus  $|C_G^n|$  is 1 or 2. It is easy to see that  $C_{D_{2m}}^n(a) = \{1, a, a^2, \dots, a^{m-1}\}$  and similarly we have  $m - 2 = |C_G^n(a')| - |C_G^n|$  for an element  $a' \in G$ . Now, if  $|C_G^n| = 1$  then  $|C_G^n(a')| = m - 1$ . Since  $G$  is an AC-group we should have  $m - 1 \mid |G| = 2m - 1$  which is a contradiction. Hence  $|G| = 2m$  as required.  $\square$

Hall [9] introduces the concept of isoclinism which is an equivalence relation on the class of all groups.

**Definition 4.3.** *Let  $G_1$  and  $G_2$  be two groups. Then the pair  $(\alpha, \beta)$  is called a  $n$ -isoclinism from  $G_1$  to  $G_2$  whenever*

- (i)  $\alpha : G_1/Z_n(G_1) \rightarrow G_2/Z_n(G_2)$  is an isomorphism, where  $Z_n(G_1)$  and  $Z_n(G_2)$  are the  $n$ -th term of the upper central series of  $G_1$  and  $G_2$ , respectively.
- (ii)  $\beta$  is an isomorphism from  $\gamma_{n+1}(G_1)$  to  $\gamma_{n+1}(G_2)$  with the law  $[g_{11}, \dots, g_{1, n+1}] \mapsto [g_{21}, \dots, g_{2, n+1}]$ , where  $g_{2j} \in \alpha(g_{1j}Z_n(G_1))$ . If there is such a pair  $(\alpha, \beta)$  with the above properties then we say that  $G_1$  and  $G_2$  are  $n$ -isoclinic and is denoted by  $G_1 \tilde{\sim}_n G_2$ .

If  $G_1$  and  $G_2$  are 1-isoclinic, then denote it by the abbreviate form  $G_1 \sim G_2$  and is it called isoclinism (see [10] for more details).

**Theorem 4.4.** *If  $G$  and  $H$  are two non-abelian groups which are isoclinic and  $|Z(H)| = |C_H^n| = |Z(G)| = |C_G^n|$ . Then  $\Gamma_G^n \simeq \Gamma_H^n$ .*

*Proof.* Since  $Z(H) \subseteq C_H^n$  and  $Z(G) \subseteq C_G^n$ , we conclude that  $Z(H) = C_H^n, Z(G) = C_G^n$ . There is an isomorphism  $\phi : G/Z(G) \rightarrow H/Z(H)$  such that  $g_i Z(G) \mapsto h_i Z(H)$  where  $\{g_1, \dots, g_t\}, \{h_1, \dots, h_t\}$  are transversal sets for  $Z(G)$  and  $Z(H)$  in  $G$  and  $H$ , respectively. Furthermore, one can define the bijection  $f : Z(H) = C_H^n \rightarrow Z(G) = C_G^n$  and by using

isomorphism  $\phi$  we deduce  $|V(\Gamma_G^n)| = |V(\Gamma_H^n)|$ . Hence the map  $\psi : G \setminus C_G^n \rightarrow H \setminus C_H^n$  can be constructed such that  $x = g_i z \mapsto h_i f(z)$  which is well-defined bijection and preserves edges.  $\square$

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