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STRONG CONVERGENCE THEOREM FOR FINITE FAMILY OF m-ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. The purpose of this paper is to propose a composite iterative scheme for approximating a common solution for a finite family of m-accretive operators in a strictly convex Banach space having a uniformly Gateaux differentiable norm. As a consequence, the strong convergence of the scheme for a common fixed point of a finite family of pseudocontractive mappings is also obtained.

1. Introduction

Let E be a real Banach space with dual E^*. The *normalized duality mapping* from E to 2^{E^*} is defined by

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x^*\|\},$$

where $\langle ., . \rangle$ denotes the duality pairing between elements of E and E^*.

Definition 1.1. [2] *A mapping $A\colon D(A) \subseteq E \to E$ is said to be accretive if for all $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle Ax - Ay, j(x-y) \rangle \geq 0.$$

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If E is a Hilbert space, accretive operators are also called monotone.
An operator A is called m-accretive if it is accretive and $\mathcal{R}(I+rA)$, the range of $(I+rA)$, is E for all $r>0$; and A is said to satisfy the range condition if $\overline{D(A)} \subseteq \mathcal{R}(I+rA), \forall\, r>0$.

Closely related to the class of accretive mappings is the class of pseudocontractive mappings.

Definition 1.2. [3] *The mapping $T : E \to E$ is called pseudocontractive if for all $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 .$$

The mapping T is pseudocontractive if and only if $(I-T)$ is accretive.
It is well known that if A is accretive [7], then $J_A := (I+A)^{-1}$ is a nonexpansive single-valued mapping from $\mathcal{R}(I+rA)$ to $D(A)$ and $\mathcal{F}(J_A) = \mathcal{N}(A)$, where $\mathcal{N}(\mathcal{A}) := \{x \in D(A) : Ax = 0\} = A^{-1}(0)$ and $\mathcal{F}(J_A) := \{x \in E : J_A x = x\}$. Here we also note that x^* is a zero of the accretive mapping A if and only if it is a fixed point of the pseudocontractive mapping $T := I - A$.
It is now well known that if A is accretive then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems [20]. Consequently, considerable research efforts, especially within the past 15 years or so, have been devoted to iterative methods for approximating the zeros of A, when A is accretive (e.g. [6] and references therein with many others).

Let K be a closed convex subset of a real Banach space E. A mapping $T : K \to E$ is called a contraction mapping if there exists $L \in [0,1)$ such that $\|Tx - Ty\| \leq L\|x-y\|$, for all $x, y \in K$. If $L = 1$, then T is called nonexpansive.
Clearly the class of nonexpansive mappings is a subset of the class of pseudocontractive mappings.

In 1976, Rockafellar [14] introduced a proximal point algorithm in a Hilbert space for a maximal monotone operator: For any $x_0 \in H$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = J_{r_n} x_n, \forall n \in N \tag{1.1}$$

where $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$, converges weakly to an element of $A^{-1}0 = \{x \in C : 0 \in Ax\}$. The weak and strong convergence

of the sequence $\{x_n\}$ have been extensively discussed in Hilbert and Banach spaces and in Banach spaces (see e.g. [16] and the references therein).
Whereas in 1967, one explicit iterative process was first introduced by Halpern [8] in the framework of Hilbert spaces. For any $u \in C, x_0 \in C$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \forall n \ \geq 0, \tag{1.2}$$

where $\{\alpha_n\} \subset [0,1]$. For T a nonexpansive mapping, the weak and strong convergence of the sequence $\{x_n\}$ have been investigated by several researchers (see [5, 10] and the references therein).
However, there remains an open question: For the real sequence $\{\alpha_n\}$, are the conditions (C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ and (C2) $\sum\limits_{n=0}^{\infty} \alpha_n = +\infty$ sufficient for the strong convergence of the sequence $\{x_n\}$ defined by the recursive algorithm (1.2) for nonexpansive mappings $T : C \to C$?
In 2000, Kamimura and Takahashi [9] showed a strong convergence theorem for a monotone operator in a Hilbert space: For A a maximal monotone operator and $J_r = (I + rA)^{-1}$ for all $r > 0$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_n x + (1-\alpha_n)J_{r_n}x_n, n \geq 0, \tag{1.3}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty]$ satisfy the conditions (C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ and (C2) $\sum\limits_{n=0}^{\infty} \alpha_n = +\infty$ and $\lim\limits_{n\to\infty} r_n = +\infty$. Then the iterative sequence $\{x_n\}$ converges strongly to some $A^{-1}0$.
In 2005, Kim and Xu [10] extended the result of Kamimura and Takahashi [9] to a uniformly smooth Banach space and that of Benavides-Acedoand-Xu [1] relaxing the condition of a weakly continuous duality map J_ψ with gauge ψ giving the result:
Suppose that A is an m-accretive operator, and $J_r := (I + rA)^{-1}$ for all $r > 0$, and the sequence $\{x_n\}$ is defined by (1.3), where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty]$ satisfy the following conditions: (C1), (C2) and (C3) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$; (C4) $\sum\limits_{n=1}^{\infty} \left|1 - \frac{r_{n+1}}{r_n}\right| < +\infty$. Then $\{x_n\}$ converges strongly to a zero of A.
This work was further extended by Xu [18] in the framework of Reflexive Banach space having weakly continuous duality map.
This gives rise naturally to the question that we are concerned with:

For the sequence $\{\alpha_n\}$, are the conditions sufficient for the strong convergence of the sequence $\{x_n\}$ defined by (1.3) for a finite family of m-accretive operators, when the conditions on $\{r_n\}$ are relaxed?
Motivated by (1.3), we introduce the following composite iterative algorithm to prove the strong convergence theorem for the sequence $\{x_n\}$ for a finite family of m-accretive operators:
For any $u, x_0 \in C$, let the sequence $\{x_n\}$ be generated by

$$\begin{cases} x_{n+1} &= (1-\beta_n)y_n + \beta_n S_r y_n, \\ y_n &= \alpha_n u + (1-\alpha_n) S_r x_n \end{cases} \tag{1.4}$$

where $S_r = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$, with $J_{A_i} = (I + A_i)^{-1}$, for $i = 0, 1, \cdots, r$, $a_i \in (0,1)$, $\sum_{i=1}^{r} a_i = 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences in $(0,1)$ satisfying appropriate conditions.

The purpose of the paper is to prove that the sequence $\{x_n\}$ defined by the composite iteration scheme (1.4) converges strongly to a common zero of a finite family of m-accretive operators in a strictly convex Banach space relaxing the restriction of the real sequence $\{r_n\}$, thus we generalize and extend the results of Ceng [4], Kamimura and Takahashi [9], Kim and Xu [10], Qin and Su [13], Xu [18], Zegeye and Shahzad [19] and the references therein.
We also shoe that the sequence $\{x_n\}$ converges strongly to a common fixed point of a finite family of pseudocontractive mappings provided that $(I - A_i)$ is m-accretive for each $i \in 1, 2, \cdots, r$. Consequently, we give an affirmative answer to the above question.

2. **Preliminaries**

Definition 2.1. *Let E be a real Banach space with dual E^*. The norm on E is said to be uniformly Gateaux differentiable if for each $y \in S_1(0) := \{x \in E : \|x\| = 1\}$, the limit*

$$\lim_{t\to\infty} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $x \in S_1(0)$.

It is well known that if E has a uniformly Gateaux differentiable norm, then the duality map is norm-to-weak* uniformly continuous on bounded subsets of E [15].

Definition 2.2. *A Banach space E is said to be strictly convex* [7] *if for $a_i \in (0,1), i = 1, 2, \cdots, r$, such that $\sum_{i=1}^{r} a_i = 1$ we have $\|a_1x - 1 + a_2x_2 + \cdots + a_rx_r\| < 1$ for $x_i \in E, i = 1, 2, \cdots, r$ with $\|x_i\| = 1, i = 1, 2, \cdots, r$ and $x_i \neq x_j$, for some $i \neq j$.*

In a strictly convex Banach space E, we have that if $\|x_1\| = \|x_2\| = \cdots = \|x_r\| = \|a_1x_1 + a_2x_2 + \cdots + a_rx_r\|$, for $x_i \in E$, $a_i \in (0,1), i = 1, 2, \cdots, r$, and such that $\sum_{i=1}^{r} a_i = 1$, then $x_1 = x_2 = \cdots = x_r$ [15].

We shall need the following lemmas to present our result.

Lemma 2.3. [12] *Let E be a real normed linear space. Then the following inequality holds: For each $x, y \in E$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \ \forall\, j(x + y) \in J(x + y).$$

Lemma 2.4. [17] *Let $\{a_n\}_{n=1}^{\infty}$ be a non-negative real sequence satisfying the inequality*

$$a_{n+1} \leq (1 - w_n)a_n + b_n, \ n \geq 0,$$

where $\{w_n\}_{n=1}^{\infty} \subset (0,1)$, $\sum_{n=1}^{\infty} w_n = \infty$, $\lim_{n\to\infty} w_n = 0$. Suppose either (i) $b_n = o(a_n)$, or $\sum_{n=1}^{\infty} |b_n| < \infty$, or $\limsup(\frac{b_n}{a_n}) \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.5. [11] *Let K be a nonempty closed convex subset of a Banach space E which has uniformly Gateaux differentiable norm and $T : K \to E$ a nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive mappings. Then there exists a continuous path $t \to z_t, 0 < t < 1$, satisfying $z_t = tu + (1 - t)Tz_t$, for arbitrary but fixed $u \in K$, which converges strongly to a fixed point of T.*

Lemma 2.6. [19] *Let K be a nonempty closed convex subset of a strictly convex Banach space E. Let $A_i : K \to E$, $i = 1, 2, \cdots, r$,*

be a finite family of m-accretive operators such that $\bigcap_{i=1}^{r} \mathcal{N}(A_i) \neq \emptyset$. *Let* $a_0, a_1, \cdots, a_r$ *be real numbers in* $(0,1)$ *such that* $\sum_{i=0}^{r} a_i = 1$ *and let* $S_r := a_0 I + a_1 J_{A_1} + \cdots + a_r J_{A_r}$, *where* $J_{A_i} := (I + A_i)^{-1}$. *Then* S_r *is nonexpansive and* $\mathcal{F}(S_r) = \bigcap_{i=1}^{r} \mathcal{N}(A_i)$.

3. Main Results

Theorem 3.1. *Let E be a strictly convex Banach space with a uniformly Gateaux differentiable norm and K be a closed convex subset of E. Assume that every nonempty closed bounded subset of E has the fixed point property for nonexpansive self-mappings. Let $A_i : K \to E, i = 1, \cdots, r$, be a finite family of m-accretive operators with $\bigcap_{i=1}^{r} \mathcal{N}(A_i) \neq \emptyset$.*
For any given $u, x_0 \in K$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated by the iterative algorithm

$$\begin{cases} x_{n+1} &= (1-\beta_n)x_n + \beta_n S_r x_n \\ \quad y_n &= \alpha_n u + (1-\alpha_n) S_r x_n, \quad n \geq 0, \end{cases} \tag{3.1}$$

where,
$S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$, *with* $J_{A_i} := (I + A_i)^{-1}$ *for* $0 < a_i < 1, i = 0, 1, \cdots, r, \sum_{i=0}^{r} a_i = 1$ *and both* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = +\infty$; *(iii)* $\beta_n \in [0, a)$, *for some* $a \in (0,1)$;
(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.
Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common solution of the equations $A_i x = 0$, for $i = 1, 2, \cdots, r$.

Proof. By Lemma 2.6, we have that S_r is well-defined, nonexpansive and $\mathcal{F}(S_r) = \bigcap_{i=1}^{r} \mathcal{N}(A_i)$, where $\mathcal{F}(S_r)$ is the fixed point set of S_r.
We shall first show that $\{x_n\}_{n=1}^{\infty}$ is bounded.

Let $x^* \in \mathcal{F}(S_r) = \bigcap_{i=1}^{r} \mathcal{N}(A_i)$, it follows from (3.1) that

$$\begin{aligned} \|y_n - x^*\| &= \|\alpha_n(u - x^*) + (1-\alpha_n)(S_r x_n - x^*)\| \\ &\le \alpha_n \|u - x^*\| + (1-\alpha_n)\|S_r x_n - x^*\| \\ &\le \alpha_n \|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\le \max\{\|u - x^*\|, \|x_n - x^*\|\}, \end{aligned} \tag{3.2}$$

also we have,

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\beta_n)\|y_n - x^*\| + \beta_n \|y_n - x^*\| \\ &= \|y_n - x^*\| \\ &\le \max\{\|u - x^*\|, \|x_n - x^*\|\} \end{aligned} \tag{3.3}$$

and (3.3) implies that

$$\|x_n - x^*\| \le \max\{\|u - x^*\|, \|x_0 - x^*\|\},$$

for all integers $n \ge 0$,
which implies that $\{x_n\}$ and hence $\{y_n\}$ is bounded.
Also since $\|S_r x_n - x^*\| \le \|x_n - x^*\|$ and $\|S_r y_n - x^*\| \le \|y_n - x^*\|$, so we get that $\{S_r x_n\}$ and $\{S_r y_n\}$ are also bounded. By using condition (i), we get

$$\|y_n - S_r x_n\| = \alpha_n \|u - S_r x_n\| \to 0 \text{ as } n \to \infty. \tag{3.4}$$

We now show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - S_r x_n\| = 0$.
From (3.1),we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|(1-\alpha_n)(S_r x_n - S_r x_{n-1}) + (u - S_r x_{n-1})(\alpha_n - \alpha_{n-1})\| \\ &\le (1-\alpha_n)\|S_r x_n - S_r x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\, \|u - S_r x_{n-1}\| \\ &\le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\, M_1, \end{aligned} \tag{3.5}$$

where M_1 is a constant such that $M_1 := \sup\{\|u - S_r x_{n-1}\|\}$
Moreover, again from (3.1), we have

$$\begin{aligned} x_{n+1} &= (1-\beta_n) y_n + \beta_n S_r y_n, \\ x_n &= (1-\beta_{n-1}) y_{n-1} + \beta_{n-1} S_r y_{n-1}, \end{aligned}$$

so that using (3.5),

$$\begin{aligned}
\|x_{n+1}-x_n\| &\leq (1-\beta_n)\|y_n-y_{n-1}\|+\beta_n\|S_ry_n-S_ry_{n-1}\| \\
&\quad +|\beta_n-\beta_{n-1}|\,\|S_ry_{n-1}-y_{n-1}\| \\
&\leq (1-\beta_n)\|y_n-y_{n-1}\|+\beta_n\|y_n-y_{n-1}\| \\
&\quad +|\beta_n-\beta_{n-1}|\,\|S_ry_{n-1}-y_{n-1}\| \\
&= \|y_n-y_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|S_ry_{n-1}-y_{n-1}\| \\
&\leq (1-\alpha_n)\|x_n-x_{n-1}\|+M_1\,|\alpha_n-\alpha_{n-1}| \\
&\quad +|\beta_n-\beta_{n-1}|\,\|S_ry_{n-1}-y_{n-1}\| \\
&\leq (1-\alpha_n)\|x_n-x_{n-1}\|+M_2(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|), \qquad (3.6)
\end{aligned}$$

where M_2 is a constant such that $M_2 > \max\{\|S_r - y_{n-1}\|, M_1\}$.
By assumptions (i)-(iv), we have
$\lim\limits_{n\to\infty}\alpha_n = 0$; $\sum\limits_{n=1}^{\infty}\alpha_n = +\infty$ and $\sum\limits_{n=0}^{\infty}(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|) < \infty$.
Thus, by applying Lemma 2.4 to (3.6), we get that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \qquad (3.7)$$

Again by (3.1),

$$\begin{aligned}
\|x_{n+1}-y_n\| &= \beta_n\|S_ry_n-y_n\| \\
&\leq \beta_n(\|S_ry_n-S_rx_n\|+\|S_rx_n-y_n\|) \\
&\leq \beta_n(\|y_n-x_n\|+\|S_rx_n-y_n\|) \\
&\leq a(\|y_n-x_{n+1}\|+\|x_{n+1}-x_n\|+\|S_rx_n-y_n\|)
\end{aligned}$$

which implies that

$$\|x_{n+1}-y_n\| \leq \frac{a}{1-a}(\|x_{n+1}-x_n\|+\|S_rx_n-y_n\|)$$

It is obvious from (3.4) and (3.7) that

$$\lim_{n\to\infty}\|x_{n+1}-y_n\| = 0, \qquad (3.8)$$

and (3.8) implies that

$$\lim_{n\to\infty}\|x_n-y_n\| = 0.$$

Since we have,

$$\begin{aligned}\|x_n - S_r x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - S_r x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \alpha_n \|u - S_r x_n\|\end{aligned}$$

Thus using equations (3.4), (3.7) and (3.8) above, it follows that

$$\|x_n - S_r x_n\| \to 0 \text{ as } n \to \infty. \tag{3.9}$$

We next show that

$$\limsup_{n\to\infty} \langle u - z, j(x_n - z)\rangle \leq 0,$$

for some $z \in \mathcal{F}(S_r) = \bigcap_{i=1}^{r} \mathcal{N}(A_i)$.

For $t \in (0,1)$, let $z_t \in E$ be the unique fixed point of the contraction mapping H_t given by

$$H_t x := tu + (1-t)S_r x,\ x \in E.$$

Then by Theorem 2.5, we get that

$z_t = tu + (1-t)S_r z_t \to z \in \mathcal{F}(S_r) = \bigcap_{i=1}^{r} \mathcal{N}(A_i)$ as $t \to 0$.

Applying Lemma 2.3 to $(z_t - x_n)$, we have

$$\begin{aligned}\|z_t - x_n\|^2 &= \|t(u - x_n) + (1-t)(S_r z_t - x_n)\|^2 \\ &\leq (1-t)^2 \|S_r z_t - x_n\|^2 + 2t\langle u - x_n, j(z_t - x_n)\rangle \\ &\leq (1-t)^2 (\|S_r z_t - S_r x_n\| + \|S_r x_n - x_n\|)^2 \\ &\quad + 2t(\langle u - z_t, j(z_t - x_n)\rangle + \|z_t - x_n\|^2) \\ &\leq (1+t^2)\|z_t - x_n\|^2 + \|S_r x_n - x_n\|[2\|z_t - x_n\| \\ &\quad + (1-t)^2 \|S_r x_n - x_n\|] - 2t\langle u - z_t, j(x_n - z_t)\rangle,\end{aligned}$$

hence,

$$\begin{aligned}\langle u - z_t, j(x_n - z_t)\rangle \leq \frac{t}{2}\|z_t - x_n\|^2 + \frac{\|S_r x_n - x_n\|}{2t}(2\|z_t - x_n\| \\ + \|S_r x_n - x_n\|)\end{aligned} \tag{3.10}$$

But by (3.9), $\|S_r x_n - x_n\| \to 0$ as $n \to \infty$.

Thus letting $n \to \infty$ in (3.10), we get that

$$\limsup_{n\to\infty} \langle u - z_t, j(x_n - z_t)\rangle \leq \frac{t}{2} M^*, \tag{3.11}$$

where M^* is a constant such that $\|z_t - x_n\|^2 \leq M^*$, $\forall\ t \in (0,1)$ and $n \geq 1$.
Since $z_t \to z$ and the duality mapping j is norm-to-weak* uniformly continuous on bounded subsets of E, thus letting $t \to 0$ in (3.11) follows that

$$\limsup_{n\to\infty}\langle u - z, j(x_n - z)\rangle \leq 0. \tag{3.12}$$

Again since $\{y_n\}$ is also bounded and

$$\begin{aligned}\|y_n - S_r y_n\| &\leq y_n - x_n + \|x_n - S_r x_n\| + \|S_r x_n - S_r y_n\| \\ &\leq y_n - x_n + \|x_n - S_r x_n\| + \|x_n - y_n\| \\ 2 &\leq y_n - x_n + \|x_n - S_r x_n\| \\ &\to 0.\end{aligned}$$

Thus from 3.12,

$$\limsup_{n\to\infty}\langle u - z, j(y_n - z)\rangle \leq 0. \tag{3.13}$$

Finally, we show that $\{x_n\}$ converges strongly to z.
From (3.1) and (3.3), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \|y_n - z\|^2 \\ &= \|\alpha_n(u - z) + (1 - \alpha_n)(S_r x_n - z)\|^2 \\ &\leq (1 - \alpha_n)^2\|S_r x_n - z\|^2 + 2\alpha_n\langle u - z, j(y_n - z)\rangle \\ &\leq (1 - \alpha_n)\|x_n - z\|^2 + \delta_n,\end{aligned}$$

where $\delta_n := 2\alpha_n c_n$ and $\limsup_{n\to\infty} c_n \leq 0$ for $c_n := \langle u - z, j(y_n - z)\rangle$.
Thus, applying Lemma 2.4 and using (3.13)above, we get that $\{x_n\}$ converges strongly to z, a common solution of the equations $A_i x = 0$, for $i = 1, 2, \cdots, r$. □

Corollary 3.2. *Let E, K be as defined in Theorem 3.1 and $A : K \to E$ be accretive operator satisfying the range condition with $\mathcal{N}(A) \neq \emptyset$.*
For any given $u, x_0 \in K$, let the iterative sequence $\{x_n\}_{n=1}^{\infty}$ be generated by the algorithm

$$\begin{cases} y_n &= \beta_n x_n + (1 - \beta_n) J_A x_n \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{3.14}$$

where $J_A := (I + A)^{-1}$ and both $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ are as defined in Theorem 3.1.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution of the equation $Ax = 0$.

Proof. Putting $A_1 = A_2 = \cdots A_r = A$, we get the required result from Theorem 3.1. □

Theorem 3.3. *Let E, K, $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ be as in Theorem3.1 and let $\{T_i\}_{i=1}^{r} : K \to E$ be a finite family of pseudocontractive mappings such that for each $i = 1, 2, \cdots, r$, $(I - T_i)$ is m-accretive on K with $\bigcap_{i=1}^{r} \mathcal{F}(T_i) \neq \emptyset$.*
For any given $u, x_0 \in K$, let the iterative sequence $\{x_n\}_{n=1}^{\infty}$ be generated by the algorithm

$$\begin{cases} y_n &= \beta_n x_n + (1-\beta_n) S_r x_n \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{3.15}$$

where,
$S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$, *with* $J_{A_i} := (I + (I - T_i))^{-1}$ *for* $0 < a_i < 1, i = 0, 1, \cdots, r, \sum_{i=0}^{r} a_i = 1$ *and both* $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0; (ii) \sum_{n=1}^{\infty} \alpha_n = +\infty$; *(iii)* $\beta_n \in [0, a)$, *for some* $a \in (0, 1)$;
(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.
Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{r}$.

Proof. Clearly, $\mathcal{F}(T_i) = \mathcal{N}(A_i)$ and hence $\bigcap_{i=1}^{r} \mathcal{F}(T_i) = \bigcap_{i=1}^{r} \mathcal{N}(A_i) \neq \emptyset$. Also, each A_i is m-accretive $(i = 1, 2, \cdots, r)$. Thus the proof follows from Theorem 3.1. □

Remark 3.4. *Theorem 3.1 and Theorem 3.3 are significant generalization and extension of the results of Ceng* [4], *Kim and Xu* [10], *Qin* [13], *Xu* [18], *Zegeye and Shahzad* [19] *in several aspects as:*

1. *For the sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$, satisfying the conditions (i), (ii) and (iii) is sufficient for the strong convergence of the algorithms (3.1) for a finite family of m-accretive operators.*

Thus the restriction on the sequence $\{\alpha_n\}_{n=1}^{\infty}$ *of satisfying conditions (iii) and (iii*)* [19] *can be dispensed with.*

2. *The use of the sequence* $\{r_n\}_{n=1}^{\infty}$ *with the restriction* [13] *is completely removed.*

3. *Also, our theorem extends the result of Ceng* [4] *to a finite family of* m*-accretive operators thus extending many other results with their references.*

4. *If* $\beta_n = 0$ *in (3.1), then we get the iterative sequence* $\{x_n\}$ *defined by (1.3), generalizing the results of Kim and Xu* [10]*, Xu* [18]*, Zegeye and Shahzad* [19] *and the references therein to a finite family of* m*-accretive operators in a more general space setting.*

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