LINEAR PRESERVERS OF G-ROW AND G-COLUMN MAJORIZATION ON \( M_{n,m} \)

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Abstract. Let \( A \) and \( B \) be \( n \times m \) matrices. The matrix \( B \) is said to be g-row majorized (respectively g-column majorized) by \( A \), denoted by \( B \precrow g A \) (respectively \( B \preccol g A \)), if every row (respectively column) of \( B \), is g-majorized by the corresponding row (respectively column) of \( A \). In this paper all kinds of g-majorization are studied on \( M_{n,m} \), and the possible structure of their linear preservers will be found. Also all linear operators \( T : M_{n,m} \to M_{n,m} \) preserving (or strongly preserving) g-row or g-column majorization will be characterized.

1. Introduction

An \( n \times n \) matrix \( R \) (not necessarily nonnegative) is called g-row stochastic if \( Re = e \), where \( e = (1, 1, \ldots, 1)^t \). A matrix \( D \) is called g-doubly stochastic if both \( D \) and \( D^t \) are g-row stochastic matrices. The collection of all \( n \times n \) g-row stochastic matrices, and \( n \times n \) g-doubly stochastic matrices are denoted by \( GR_n \) and \( GD_n \) respectively. Throughout the paper, \( M_{n,m} \) is the set of all \( n \times m \) matrices with entries in \( \mathbb{F} \) (\( \mathbb{R} \) or \( \mathbb{C} \)), and \( M_n := M_{n,n} \). The set of all \( n \times 1 \) column vectors is denoted by \( \mathbb{F}^n \), and the set of all \( 1 \times n \) row vectors is denoted by \( \mathbb{F}_n \). The symbol \( \mathbb{N}_k \) is used for the set \( \{1, \ldots, k\} \). The symbol \( e_i \) is the row (or
column) vector with 1 as $i^{th}$ component and 0 elsewhere. The summation of all components of a vector $x$ in $\mathbb{F}^n$ or $\mathbb{F}_n$ is denoted by $\text{tr}(x)$. The symbol $[x_1/x_2/\ldots/x_n]$ (resp. $[x_1 \mid x_2 \mid \ldots \mid x_m]$) is used for the $n \times m$ matrix whose rows (resp. columns) are $x_1, x_2, \ldots, x_n \in \mathbb{F}_m$ (resp. $x_1, x_2, \ldots, x_m \in \mathbb{F}^n$). For a matrix $X = [x_{ij}] \in \mathbb{M}_{n,m}$, its average (column) vector $\bar{X} = [x_1/\ldots/x_n] \in \mathbb{F}^n$ is defined by the components $\bar{x}_i = m^{-1}(x_{i1} + x_{i2} + \cdots + x_{im})$, for $i \in \mathbb{N}_n$. The letter $J$ stands for the $(\text{rank}-1)$ square matrix all of whose entries are 1.

For $A, B \in \mathbb{M}_{n,m}$, it is said that $A$ is lgs-majorized (resp. rgs-majorized) by $B$ and denoted by $A \prec_{\text{lgs}} B$ (resp. $A \prec_{\text{rgs}} B$) if there exists an $n \times n$ (resp. $m \times m$) g-doubly stochastic matrix $D$ such that $A = DB$ (resp. $A = BD$), see [4, 6].

Let $A, B \in \mathbb{M}_{n,m}$. The matrix $A$ is said to be lgw-majorized (resp. rgw-majorized) by $B$ and denoted by $A \prec_{\text{lgw}} B$ (resp. $A \prec_{\text{rgw}} B$) if there exists an $n \times n$ (resp. $m \times m$) g-row stochastic matrix $R$ such that $A = RB$ (resp. $A = BR$), for more details see [2, 5].

Let $\prec$ be a relation on $\mathbb{M}_{n,m}$. A linear operator $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ is said to be a linear preserver (resp. strong linear preserver) of $\prec$ if $X \prec Y$ implies $TX \prec TY$ (resp. $X \prec Y$ if and only if $TX \prec TY$).

The linear preservers and strong linear preservers of lgs-majorization are characterized in [6] as follows:

**Proposition 1.1.** [6, Theorem 3.3] Let $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator that preserves lgs-majorization. Then one of the following statements holds:

(i) There exist $A_1, A_2, \ldots, A_m \in \mathbb{M}_{n,m}$ such that $TX = \sum_{j=1}^m \text{tr}(x_j)A_j$, where $X = [x_1 \ldots x_m]$;

(ii) There exist $S \in \mathbb{M}_m$, $a_1, \ldots, a_m \in \mathbb{F}_m$ and invertible matrices $B_1, B_2, \ldots, B_m \in \mathbb{GD}_n$, such that $TX = [B_1 X a_1 \mid \ldots \mid B_m X a_m] + JXS$.

**Proposition 1.2.** [6, Theorem 3.7] Let $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ strongly preserves $\prec_{\text{lgs}}$ if and only if $TX = AXR + JXS$, for some $R, S \in \mathbb{M}_m$ and invertible matrix $A \in \mathbb{GD}_n$ such that $R(R + nS)$ is invertible.

In [2, 5], the authors proved that a linear operator $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ strongly preserves lgw-majorization (resp. rgw-majorization) if and only if $TX = AXM$ (resp. $TX = MXA$), for some invertible matrices $M \in \mathbb{M}_m$ (resp. $M \in \mathbb{M}_n$) and $A \in \mathbb{GR}_n$ (resp. $A \in \mathbb{GR}_m$).
In the present paper, we find the possible structure of linear operators that preserve lgw, rgw or rgs-majorization. Also, all linear preservers and strong linear preservers of g-row and g-column majorization will be characterized. To see some kinds of majorization and their linear preservers we refer the readers to [1], [3] and [7]-[11].

2. LGS-COLUMN (RGS-ROW) MAJORIZATION ON $\textbf{M}_{n,m}$

In this section we characterize all linear operators on $\textbf{M}_{n,m}$ that preserve or strongly preserve lgs-column (rgs-row) majorization.

**Definition 2.1.** Let $A, B \in \textbf{M}_{n,m}$. It is said that $B$ is lgs-column (resp. rgs-row) majorized by $A$, written as $B \prec_{\text{column}} \text{lgs} A$ (resp. $B \prec_{\text{row}} \text{rgs} A$), if every column (resp. row) of $B$ is lgs- (resp. rgs-) majorized by the corresponding column (resp. row) of $A$.

We use the following statements to prove the main result of this section.

**Proposition 2.2.** [6, Theorem 2.4] Let $T : F^n \rightarrow F^n$ be a linear operator. Then $T$ preserves gs-majorization if and only if one of the following statements holds:

(a) $Tx = \text{tr}(x)a$, for some $a \in F^n$;
(b) $Tx = \alpha Dx + \beta Jx$, for some $\alpha, \beta \in F$ and invertible matrix $D \in GD_n$.

**Proposition 2.3.** [6, Lemma 3.1] Let $A \in GD_n$ be invertible. Then the following conditions are equivalent:

(a) $A = \alpha I + \beta J$, for some $\alpha, \beta \in F$;
(b) $(Dx + ADy) \prec_{\text{gs}} (x + Ay)$, for all $D \in GD_n$ and for all $x, y \in F^n$.

**Proposition 2.4.** [6, Lemma 3.2] Let $T_1, T_2 : F^n \rightarrow F^n$ satisfy $T_1(x) = \alpha Ax + \beta Jx$ and $T_2(x) = \text{tr}(x)a$, for some $\alpha, \beta \in F$, $\alpha \neq 0$, invertible matrix $A \in GD_n$ and $a \in (F^n \setminus \text{Span}\{e\})$. Then there exists a g-doubly stochastic matrix $D$ and a vector $x \in F^n$ such that $T_1(Dx) + T_2(Dx) \prec_{\text{gs}} T_1(x) + T_2(x)$.

**Lemma 2.5.** Let $a \in F^m$. The linear operator $T : \textbf{M}_{n,m} \rightarrow \textbf{M}_{n,m}$ defined by $TX = [Xa | \ldots | Xa]$, preserves lgs-column majorization if and only if $a \in \bigcup_{i=1}^m \text{Span}\{e_i\}$.

**Proof.** If $a \in \bigcup_{i=1}^m \text{Span}\{e_i\}$, it is easy to show that $T$ preserves $\prec_{\text{column}} \text{lgs}$. Conversely, let $T$ preserve $\prec_{\text{column}} \text{lgs}$. Assume that $a = (a_1, \ldots, a_m)^t \notin \bigcup_{i=1}^m \text{Span}\{e_i\}$. Then there exist distinct $i, j \in \mathbb{N}_m$ such that $a_i, a_j \neq 0$. 
Without loss of generality assume that $a_1, a_2 \neq 0$. Put
\[ X := \begin{pmatrix} -a_2 & -a_1 \\ a_2 & a_1 \end{pmatrix} \oplus 0, \ Y := \begin{pmatrix} a_2 & -a_1 \\ -a_2 & a_1 \end{pmatrix} \oplus 0 \in M_{n,m}. \]

It is clear that $X \prec_{\text{column}} Y$, so $Xa \prec_{\text{lgS}} Ya$. But $Ya = 0$ and $Xa \neq 0$, which is a contradiction. □

For every $i, j \in \mathbb{N}_m$, consider the embedding $E^j : \mathbb{F}^m \rightarrow M_{n,m}$ by $E^j(x) = xe_j$ and projection $E_i : M_{n,m} \rightarrow \mathbb{F}^n$ by $E_i(A) = Ae_i$. It is easy to show that for every linear operator $T : M_{n,m} \rightarrow M_{n,m}$,
\[ TX = \left[ \sum_{j=1}^m T^j_i x_j \right] \cdots \left[ \sum_{j=1}^m T^j_m x_j \right], \]

where $T^j_i = E_i \circ T \circ E^j$ and $X = [x_1 | \ldots | x_m]$. If $T$ preserves $\prec_{\text{column}}$, it is clear that $T^j_i : \mathbb{F}^m \rightarrow \mathbb{F}^m$ preserves $\prec_{\text{lgS}}$.

Now, we state the main theorem of this section.

**Theorem 2.6.** Let $T : M_{n,m} \rightarrow M_{n,m}$ be a linear operator. Then $T$ preserves lgs-column majorization if and only if there exist $A_1, \ldots, A_m \in M_{n,m}, b_1, \ldots, b_m \in \cup_{i=1}^m \text{Span}\{e_i\}$, invertible matrices $B_1, \ldots, B_m \in GD_n$, and $S \in M_m$ such that for every $i \in \mathbb{N}_m$, $b_i = 0$ or $A_1 e_i = \cdots = A_m e_i = 0$ and for all $X = [x_1 | \ldots | x_m] \in M_{n,m}$,
\begin{equation}
(2.1) \quad TX = \sum_{j=1}^m \text{tr}(x_j) A_j + [B_1 X b_1 | \ldots | B_m X b_m] + JXS.
\end{equation}

**Proof.** First, assume that the condition (2.1) holds. Suppose $X = [x_1 | \ldots | x_m], Y = [y_1 | \ldots | y_m] \in M_{n,m}$ and $X \prec_{\text{column}} Y$. Since for every $i \in \mathbb{N}_m$, $b_i = 0$ or $A_1 e_i = \cdots = A_m e_i = 0$, it is easy to see that $TXe_i \prec_{\text{lgS}} TYe_i$ and hence $TX \prec_{\text{column}} TY$. Conversely, assume that $T$ preserves $\prec_{\text{column}}$. For every $i, j \in \mathbb{N}_m$, $T^j_i : \mathbb{F}^m \rightarrow \mathbb{F}^n$ preserves $\prec_{\text{lgS}}$.

Then, each $T^j_i$ is of the form (a) or (b) in Proposition 2.2. Let
\[ I = \{ k \in \mathbb{N}_m : \exists l \in \mathbb{N}_m \text{ such that } T^j_k \text{ is of the form (b) with } \alpha^l_k \neq 0 \}. \]

For every $k \in I$ there exists $l \in \mathbb{N}_m$ such that $T^j_k x = \alpha^l_k B_k x + \beta^l_k Jx$ for some invertible matrix $B_k \in GD_n$ and $\alpha^l_k \neq 0, \beta^l_k \in \mathbb{F}$.

We show that if $k \in I$, then $T^j_k$ is of form (b) with same invertible matrix $B_k \in GD_n$, for every $j \in \mathbb{N}_m$. 

Suppose $k \in \mathbf{I}$, then there exist $l \in \mathbb{N}_n$, $\alpha^l_k \neq 0, \beta^l_k \in \mathbb{F}$, invertible matrix $B_k \in \mathbf{GD}_n$ such that $T^j_k x = \alpha^l_k B_k x + \beta^l_k J x$. For every $x, y \in \mathbb{F}^m$ define $X = x e_j + y e_l \in M_{n,m}$. It is clear that $D X \prec_{\text{lgs\_column}} X$, and hence $D T X \prec_{\text{lgs\_column}} T X$, for all $D \in \mathbf{GD}_n$. This implies that $T^j_k D x + T^j_k D y \prec_{\text{lgs}} T^j_k x + T^j_k y$. Then by Propositions 2.3 and 2.4, there exist $\alpha^j_k, \beta^j_k \in \mathbb{F}$ such that $T^j_k x = \alpha^j_k B_k x + \beta^j_k J x$. For $k \in \mathbf{I}$, set $b_k := (\alpha^1_k, \ldots, \alpha^m_k)^t$, $s_k := (\beta^1_k, \ldots, \beta^m_k)^t \in \mathbb{F}^m$ and for $k \in (\mathbb{N}_m \setminus \mathbf{I})$ set $b_k = s_k := 0 \in \mathbb{F}^m$. Define $S := [s_1 \mid \ldots, s_m] \in M_{m,m}$.

If $k \notin \mathbf{I}$, then $T^j_k$ is of form (a) for every $j \in \mathbb{N}_m$ and hence $T^j_k x = (\text{tr} x) a^j_k$, for some $a^j_k \in \mathbb{F}^n$. For $k \in \mathbf{I}$, put $a^j_k = 0$ and define $A_j := [a^1_k \mid \ldots \mid a^m_k] \in M_{n,m}$. It is clear that for every $i \in \mathbb{N}_m, b_i = 0$ or $A_1 e_i = \cdots = A_m e_i = 0$ and by a straightforward calculation one may show that for any $X = [x_1 \mid \ldots \mid x_m] \in M_{n,m},$

$$TX = \sum_{j=1}^m \text{tr}(x_j) A_j + [B_1 X b_1 \mid \ldots \mid B_m X b_m] + J X S.$$ 

If $b_j \notin \bigcup_{i=1}^m \text{Span}\{e_i\}$ for some $j \in \mathbb{N}_m$, then Lemma 3.7 implies that $T$ is not a linear preserver of $\prec_{\text{lgs\_column}}$ which is a contradiction. Therefore $b_1, \ldots, b_m \in \bigcup_{i=1}^m \text{Span}\{e_i\}$, as desired. \hfill \Box

The structure of strong linear preservers of lgs-column majorization is characterized as follows:

**Theorem 2.7.** Let $T : M_{n,m} \rightarrow M_{n,m}$ be a linear operator. Then $T$ strongly preserves lgs-column majorization if and only if there exist invertible matrices $B_1, \ldots, B_m \in \mathbf{GD}_n$, $S \in M_m$ and, $b_1, \ldots, b_m \in \bigcup_{i=1}^m \text{Span}\{e_i\}$ such that $D (D + n S)$ is invertible and

$$TX = [B_1 X b_1 \mid \ldots \mid B_m X b_m] + J X S,$$

where $D = [b_1 \mid \ldots \mid b_m]$.

**Proof.** The fact that the condition (2.2) is sufficient for $T$ to be a strong linear preserver of $\prec_{\text{lgs\_column}}$ is easy to prove. So, we prove the necessity of the conditions. Assume that $T$ is a strong linear preserver of $\prec_{\text{lgs\_column}}$. It can be easily seen that $T$ is invertible. By Theorem 2.6, there exist $A_1, \ldots, A_m \in M_{n,m}, b_1, \ldots, b_m \in \bigcup_{i=1}^m \text{Span}\{e_i\}$, $S \in M_m$, and invertible matrices $B_1, \ldots, B_m \in \mathbf{GD}_n$ such that for all $X = [x_1 \mid \ldots \mid x_m] \in M_{n,m},$

$$TX = \sum_{j=1}^m \text{tr}(x_j) A_j + [B_1 X b_1 \mid \ldots \mid B_m X b_m] + J X S$$ and for
every $i \in \mathbb{N}_m$, $b_i = 0$ or $A_1 e_i = \cdots = A_m e_i = 0$. We show that for every $j \in \mathbb{N}_m$, $A_j = 0$. Assume that there exists $j \in \mathbb{N}_m$, such that $A_j \neq 0$. Without loss of generality suppose that $A_je_1 \neq 0$, then $b_1 = 0$. Set $V := \text{Span}\{b_2, \ldots, b_m\}$, so $\dim V \leq m - 1$. It follows that there exists $0 \neq s \in V\perp$. Set $X := [s^t/-s^t/0/\ldots/0] \in \mathbb{M}_{n,m}$. Then $X$ is nonzero and $TX = 0$, which is a contradiction. Therefore $A_j = 0$, for every $j \in \mathbb{N}_m$.

Now, we prove (by contradiction) that $D$ is invertible. Indeed, assume that $D$ is not invertible. Choose a nonzero $s \in (\text{Span}\{b_1, \ldots, b_m\})^\perp$ and put $X := [s^t/-s^t/0/\ldots/0] \in \mathbb{M}_{n,m}$. Then $X$ is nonzero and $TX = 0$, which is a contradiction. Therefore $D$ is invertible.

Finally, we show that $D + nS$ is invertible. Assume, by contradiction, that $D + nS$ is not invertible. Choose a nonzero $x \in \mathbb{F}_m$ such that $(D + nS)x = 0$ and put $X := [x/\ldots/x] \in \mathbb{M}_{n,m}$. Then $X$ is nonzero and

$$TX = [B_1XB_1 \mid \ldots \mid B_mXB_m] + JSX = X(D + nS) = 0,$$

which is a contradiction. Therefore $D + nS$ is invertible and the proof is complete. □

Let $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Define $\tau : \mathbb{M}_{m,n} \to \mathbb{M}_{m,n}$ by $\tau X = (TX^t)^t$. It is easy to see that $T$ is a (strong) linear preserver of $\prec_{\text{row}}$ if and only if $\tau$ is a (strong) linear preserver of $\prec_{\text{column}}$. Combining this fact and previous theorems, we have the following corollaries:

**Corollary 2.8.** Let $T : \mathbb{F}_n \to \mathbb{F}_n$ be a linear operator. Then $T$ preserves rgs-majorization if and only if one of the following statements holds:

(a) $Tx = \text{tr}(x)a$, for some $a \in \mathbb{F}_n$;

(b) $Tx = \alpha XD + \beta xJ$, for some $\alpha, \beta \in \mathbb{F}$ and invertible matrix $D \in GD_n$.

**Corollary 2.9.** Let $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ preserves rgs-row majorization if and only if there exist $A_1, \ldots, A_n \in \mathbb{M}_{n,m}$, $b_1, \ldots, b_n \in \cup_{i=1}^n \text{Span}\{e_i\}$, invertible matrices $B_1, \ldots, B_n \in GD_m$, and $S \in \mathbb{M}_n$ such that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $e_i A_1 = \cdots = e_i A_n = 0$ and for all $X = [x_1/\ldots/x_n] \in \mathbb{M}_{n,m}$,

$$TX = \sum_{j=1}^n \text{tr}(x_j)A_j + [b_1XB_1 / \ldots / b_nXB_n] + SXJ.$$

**Corollary 2.10.** Let $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ be a linear operator. Then $T$ strongly preserves rgs-row majorization if and only if there exist
B_1, \ldots, B_n \in GD_m, S \in M_n and b_1, \ldots, b_n \in \cup_{i=1}^n \text{Span}\{e_i\} such that D(D + mS) is invertible and
\[ TX = [b_1 XB_1 / \ldots / b_n XB_n] + SXJ, \]
where D = [b_1 / \ldots / b_n].

3. RGW AND LGW-MAJORIZATION ON M_{n,m}

In this section, we begin to study the structure of linear preservers of rgw and lgw-majorization on M_{n,m}, and then the linear operators T : M_{n,m} → M_{n,m} preserving or strongly preserving rgw-row (lgw-column) majorization will be characterized.

In the following theorems we state some results from [2].

Proposition 3.1. [2, Theorem 2.3] Let T : \mathbb{F}_m → \mathbb{F}_m be a linear operator. Then, T preserves \textless_{rgw} if and only if one of the following statements holds:

(i) Tx = \alpha xB, for some \alpha \in \mathbb{F} and some invertible B \in GR_n;
(ii) Tx = \alpha xB, for some \alpha \in \mathbb{F} and some B \in GR_n such that \{x : xB = 0\} = \{x : \text{tr}(x) = 0\}.

Proposition 3.2. [2, Lemma 2.6] Let A \in M_n and \alpha be a nonzero scalar in \mathbb{F}. Then A = \gamma I for some \gamma \in \mathbb{F} if and only if we have
\[ axRA + yR <_{rgw} ax + y, \forall x, y \in \mathbb{F}_m, \forall R \in GR_m. \]

Lemma 3.3. Let A \in GR_m be invertible and 0 \neq \alpha \in \mathbb{F}. Define T_1 : \mathbb{F}_m → \mathbb{F}_m by T_1x = \alpha xA and suppose T_2 : \mathbb{F}_m → \mathbb{F}_m is a linear preserver of \textless_{rgw} such that
\[ T_1 xR + T_2 yR <_{rgw} T_1 x + T_2 y, \]
for all x, y \in \mathbb{F}_m and R \in GR_m. Then there exists \lambda \in \mathbb{F} such that T_2x = \lambda xA.

Proof. Since T_2 preserves \textless_{rgw}, T_2 is of form (i) or (ii) in Proposition 3.1. Assume that T_2 is of form (ii), then T_2x = \text{tr}(x)a for some nonzero a \in \mathbb{F}_m. Let \[ x = -\frac{1}{\alpha} a A^{-1}, \] and set y := e_1. Then we have
\[ axRA + \text{tr}(yR)a <_{rgw} axA + \text{tr}(y)a, \]
for all R \in GR_m. It follows that
\[ \alpha(-\frac{1}{\alpha} a A^{-1}) RA + \text{tr}(e_1 R)a <_{rgw} \alpha(-\frac{1}{\alpha} a A^{-1}) A + \text{tr}(e_1)a = -a + a = 0. \]
So \[ -a A^{-1} RA + a = 0, \] for all R \in GR_m. Thus aR = a, for all R \in GR_m, and hence a = 0, a contradiction. Therefore, T_2x = \beta xA_2, for some
$\beta \in \mathbb{F}$ and invertible matrix $A_2 \in \text{GR}_m$. Now, by Proposition 3.2, $T_2x = \lambda xA$, for some $\lambda \in \mathbb{F}$. □

For every $i,j \in \mathbb{N}_n$ consider the embedding $E^j : \mathbb{F}_m \to \mathbb{M}_{n,m}$ and the projection $E_i : \mathbb{M}_{n,m} \to \mathbb{F}_m$, where $E^j(x) = e_jx$ and $E_i(A) = e_iA$. It is easy to prove that for every linear operator $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$,

$$TX = T[x_1/\cdots/x_n] = \left[\sum_{j=1}^n T^j_1x_j/\cdots/\sum_{j=1}^n T^j_nx_j\right],$$

where $x_i$ is the $i$th row of $X$ and $T^j_i = E_i \circ T \circ E^j$. If $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ preserves rgw-majorization, then it's easy to see that $T^j_i : \mathbb{F}_m \to \mathbb{F}_m$ preserves rgw-majorization.

Now, we find the possible structure of linear operators preserving $\prec_{rgw}$ on $\mathbb{M}_{n,m}$.

**Theorem 3.4.** If a linear operator $T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ preserves rgw-majorization, then there exist $A \in \mathbb{M}_n(\mathbb{F}_m)$, $b_1, \ldots, b_n \in \mathbb{F}_n$, and invertible matrices $A_1, \ldots, A_n \in \text{GR}_m$, such that

$$TX = mA\overline{X} + \left[b_1XA_1/\cdots/b_nXA_n\right], \forall X \in \mathbb{M}_{n,m}.$$

**Proof.** For every $p \in \mathbb{N}_n$, one of the following cases holds:

Case 1: there exists $q \in \mathbb{N}_n$ such that $T^q_p x = \alpha x A_p$ for some $0 \neq \alpha \in \mathbb{F}$ and invertible $A_p \in \text{GR}_m$. We show that for all $j \in \mathbb{N}_n$, $T^j_p x = \lambda^j_p x A_p$, for some $\lambda^j_p \in \mathbb{F}$. For $x, y \in \mathbb{F}_m$ put $X = e_p x + e_j y$. It is clear that $XR \prec_{rgw} X$, for all $R \in \text{GR}_m$, therefore $TXR \prec_{rgw} TX$, for all $R \in \text{GR}_m$ and hence,

$$T^q_p x R + T^j_p y R \prec_{rgw} T^q_p x R + T^j_p y R, \forall x,y \in \mathbb{F}_m, \forall R \in \text{GR}_m.$$

Use Lemma 3.3 to conclude that $T^j_p x = \lambda^j_p x A_p$, for some $\lambda^j_p \in \mathbb{F}$. Put

$$b_p := (\lambda^1_p, \ldots, \lambda^n_p) \in \mathbb{F}_n$$

and $A_{(p)} = 0 \in \mathbb{F}_n(\mathbb{F}_m)$.

Case 2: For every $q \in \mathbb{N}_n$, $T^q_p$ is of form $(ii)$ in Proposition 3.1. Then $T^q_p x = \text{tr}(x) a^q_p$ for some $a^q_p \in \mathbb{F}_m$. Put $A_{(p)} = [a^1_p \ldots a^n_p] \in \mathbb{F}_n(\mathbb{F}_m)$ and $b_p = 0 \in \mathbb{F}_m$. Now, Let $A = [A_{(1)}/\ldots/A_{(n)}]$. Then
Proof. Without loss of generality we can assume that \( XR \prec \) early independent set. Let \( X \) exist. Assume that \( A \in \text{Corollary 3.5.} \) Let \( \square \) where \( A \in \mathbb{M}_n(\mathbb{F}_n) \), \( b_1, \ldots, b_n \in \mathbb{F}_n \), and \( A_1, \ldots, A_n \in \text{Gr}_m \) are invertible matrices. 

\[
TX = T[x_1/ \ldots/x_n] = \left[ \sum_{j=1}^{n} T^j x_j / \ldots / \sum_{j=1}^{n} T^j x_j \right] = [b_1XA_1/ \ldots/b_nXA_n] + mA X,
\]

where \( A \in \mathbb{M}_n(\mathbb{F}_m) \), \( b_1, \ldots, b_n \in \mathbb{F}_n \), and \( A_1, \ldots, A_n \in \text{Gr}_m \) are invertible matrices. \( \square \)

**Corollary 3.5.** Let \( \{b_1, \ldots, b_n\} \subset \mathbb{F}_n \) and \( \dim(\text{Span}\{b_1, \ldots, b_n\}) \geq 2 \). Assume that \( A_1, \ldots, A_n \in \text{Gr}_m \) are invertible and define \( T : \mathbb{M}_{n,m} \to \mathbb{M}_n(\mathbb{F}_n) \) by \( TX = [b_1XA_1/ \ldots/b_nXA_n] \). If \( T \) preserves \( \prec_{rgw} \), then there exist \( B \in \mathbb{M}_n(\mathbb{F}_n) \) and invertible \( A \in \text{Gr}_m \) such that \( TX = BXA \).

**Proof.** Without loss of generality we can assume that \( \{b_1, b_2\} \) is a linearly independent set. Let \( X \in \mathbb{M}_{n,m} \), \( R \in \text{Gr}_m \) be arbitrary. Then \( XR \prec_{rgw} X \), and hence \( TXR \prec_{rgw} TX \). It follows that

\[
[b_1XRA_1/ \ldots/b_nXRA_n] \prec_{rgw} [b_1XA_1/ \ldots/b_nXA_n]
\]

\[
\Rightarrow b_1XRA_1 + b_2XRA_2 \prec_{rgw} b_1XA_1 + b_2XA_2
\]

\[
\Rightarrow b_1XR + b_2XR(A_2^{-1}) \prec_{rgw} b_1X + b_2X(A_2^{-1}).
\]

Since \( \{b_1, b_2\} \) is linearly independent, for every \( x, y \in \mathbb{F}^n \), there exists \( B_{x,y} \in \mathbb{M}_{n,m} \) such that \( b_1B_{x,y} = x \) and \( b_2B_{x,y} = y \). Put \( X = B_{x,y} \) in the above relation. Thus,

\[
xR + yR(A_2^{-1}) \prec_{rgw} x + y(A_2^{-1}), \forall R \in \text{Gr}_m, \forall x, y \in \mathbb{F}_n.
\]

Then by Proposition 3.2, \( (A_2^{-1}) = \alpha I \) and hence \( A_2 = \alpha A_1 \), for some \( 0 \neq \alpha \in \mathbb{F} \). For every \( i \geq 3 \), if \( b_i = 0 \) we can choose \( A_i = A_1 \); if \( b_i \neq 0 \) then \( \{b_1, b_i\} \) or \( \{b_2, b_i\} \) is linearly independent. By the same argument as above, we conclude that \( A_i = \gamma_iA_1 \), for some \( 0 \neq \gamma_i \in \mathbb{F} \), or \( A_i = \lambda_iA_2 \), for some \( 0 \neq \lambda_i \in \mathbb{F} \). Define \( A = A_1 \). Then for every \( i \geq 2 \), \( A_i = \alpha_i A \), for some \( \alpha_i \in \mathbb{F} \) and we get

\[
TX = [b_1XA/(r_2b_2)XA/(r_nb_nXA)] = BXA,
\]

where \( B = [b_1 | r_2b_2/ \ldots/r_nb_n] \), for some \( r_2, \ldots, r_m \in \mathbb{F} \). \( \square \)

If \( A \in \text{Gr}_m \) is invertible and \( B \in \mathbb{M}_n \), it is easy to see that \( X \mapsto BXA \) is a linear preserver of \( \prec_{rgw} \). But the following example shows that there exist linear preservers of \( \prec_{rgw} \) which are not of this form.
Example 3.6. Let \( T : M_2 \rightarrow M_2 \) be such that
\[
TX = \begin{pmatrix}
x_{11} & x_{12} \\
x_{11} - x_{12} & x_{11} + x_{22}
\end{pmatrix}
\]
where \( X = [x_{ij}] \).

We show that \( T \) preserves \( \prec_{rgw} \), but \( T \) is not of the form \( X \mapsto MXA \).

Let \( X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \) and \( Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \), and suppose that
\( X \prec_{rgw} Y \). If \( y_{11} + y_{12} = 0 \), so \( x_{11} + x_{12} = 0 \), and \( TX \prec_{rgw} TY \).

Let \( y_{11} + y_{12} \neq 0 \). Without loss of generality assume that \( y_{11} + y_{12} = 1 \).

Since \( X \prec_{rgw} Y \), there exists \( R \in GR_2 \), such that \( X = YR \).

Let \( R = \begin{pmatrix} a & 1 - a \\ b & 1 - b \end{pmatrix} \) and \( y = (\lambda, 1 - \lambda) \).

Put \( S := \begin{pmatrix} \alpha & 1 - \alpha \\ \alpha - 1 & 2 - \alpha \end{pmatrix} \),
where \( \alpha = \lambda(a - b) + b - \lambda + 1 \).

Therefore \( S \in GR_2 \) and \( TYS = TX \).

So \( TX \prec_{rgw} TY \). By a straightforward calculation one may show that \( T \) is not of the form \( X \mapsto BXA \).

The proof of the following lemma is similar to the proof of Lemma 2.5.

Lemma 3.7. Let \( a \in \mathbb{F}_n \). The linear operator \( T : M_{n,m} \rightarrow M_{n,m} \) defined by \( TX = [aX/\ldots/aX] \) preserves \( \prec_{rgw} \) if and only if \( a \in \cup_{i=1}^{n} \text{Span}\{e_i\} \).

The structure of linear preservers and strong linear preservers of rgw-row majorization is characterized as follows:

Theorem 3.8. Let \( T : M_{n,m} \rightarrow M_{n,m} \) be a linear operator. Then \( T \) preserves rgw-row majorization if and only if there exist \( A \in M_n(\mathbb{F}) \), \( b_1, \ldots, b_n \in \cup_{i=1}^{n} \text{Span}\{e_i\} \), and invertible matrices \( A_1, \ldots, A_n \in GR_m \) such that for every \( i \in \mathbb{N}_n \), \( b_i = 0 \) or \( A_i = 0 \), where \( A = [A_{(1)}/\ldots/A_{(n)}] \) and
\[
(3.1) \quad TX = mAX + [b_1XA_1/\ldots/b_nXA_n].
\]

Proof. The fact that the condition (3.1) is sufficient for \( T \) to be a linear preserver of \( \prec_{rgw} \) is easy to prove. So, we prove the necessity of the condition. Therefore, assume that \( T \) preserves \( \prec_{rgw} \). For every \( i, j \in \mathbb{N}_n \), \( T^j_i : \mathbb{F}_m \rightarrow \mathbb{F}_m \) preserves \( \prec_{rgw} \). Then, each \( T^j_i \) is of the form (i) or (ii) in Proposition 3.1. Let
\[
I = \{k \in \mathbb{N}_n : \exists l \in \mathbb{N}_n \text{ such that } T^j_k \text{ is of the form (ii) with } \alpha_k^j \neq 0\}.
\]

We show that if \( k \in I \), then \( T^j_k \) is of form (ii) of Proposition 3.1, with the same invertible matrix \( A_k \in GR_m \), for every \( j \in \mathbb{N}_n \). Suppose \( k \in I \), then there exist \( l \in \mathbb{N}_n \), \( 0 \neq \alpha_k^j \in \mathbb{F} \) and invertible matrix \( A_k \in GR_m \)
It is clear that for every \( j \)

\[
B_j = \begin{bmatrix}
A_1 & \cdots & A_n
\end{bmatrix}
\]

is a preserver of \( \text{rgw-row majorization} \) if and only if there exist invertible \( T_1, \ldots, T_n \) and hence \( \text{rgw-row majorization} \). So by Proposition 3.2, there exists \( \alpha_k \) such that

\[
T_k^j x + T_k^j y R \prec \text{rgw} \quad \forall x, y \in F_n, \forall R \in \text{GR}_m.
\]

So by Proposition 3.2, there exists \( \alpha_k \) such that \( T_k^j x = \alpha_k^j x A_k \). Set \( b_k := (\alpha_k^1, \ldots, \alpha_k^n) \). Then \( b_k = 0 \) if \( k \notin I \).

If \( k \notin I \), then \( T_k^j \) is of form (i) of Proposition 3.1, for every \( j \in N_n \).

\[
T_k^j x = ma_k^j x \quad \forall x \in F_n, \quad \text{if} \quad k \in I, \text{put} \quad a_k^j = 0 \quad \forall j \in N_n.
\]

For \( k \in N_n \) define \( A_k = (a_k^1, \ldots, a_k^n) \).

It is clear that for every \( i \in N_n \), \( b_i = 0 \) or \( A_i = 0 \). Let \( A = [A_1, \ldots, A_n] \).

Then \( TX = [\sum_{j=1}^n T_k^j x_j, \ldots, \sum_{j=1}^n T_n^j x_j] = m A X + [b_1 X A_1, \ldots, b_n X A_n] \).

To complete the proof we must apply Lemma 3.7 to conclude that \( b_i \in \text{Span} \{ e_i \} \) for every \( i \in N_n \).

\[\square\]

**Theorem 3.9.** A linear operator \( T : M_{n,m} \rightarrow M_{n,m} \) is a strong linear preserver of \( \text{rgw-row majorization} \) if and only if there exist invertible matrices \( A_1, \ldots, A_n \in \text{GR}_m \) and \( b_1, \ldots, b_n \in \cup_{i=1}^n \text{Span} \{ e_i \} \) such that \( B = [b_1, \ldots, b_n] \) is invertible and

\[
TX = [b_1 X A_1, \ldots, b_n X A_n].
\]

**Proof.** Assume that there exists a \( k \in (N_n \setminus I) \). Without loss of generality let \( 1 \in (N_n \setminus I) \), so \( b_1 = 0 \). Set \( V := \text{Span} \{ b_2, \ldots, b_n \} \), then \( \dim V \leq n - 1 \). It follows that \( \dim V^\perp \geq 1 \) and there exists \( 0 < s \in V^\perp \).

Set \( X := [s | -s | 0 | \ldots | 0] \). Therefore \( X \) is nonzero and for every \( i \in N_n \), \( b_i X = 0 \) so \( TX = 0 \), which is a contradiction. Then \( I = N_n \) and \( TX = [b_1 X A_1, \ldots, b_n X A_n] \).

Now, we show that \( B \) is invertible. If \( B \) is not invertible, set \( V := \text{Span} \{ b_1, \ldots, b_n \} \). So \( \dim V \leq n - 1 \). Therefore \( \dim V^\perp \geq 1 \) and there exists \( 0 < s \in V^\perp \).

Set \( X := [s | -s | 0 | \ldots | 0] \). Then \( X \) is nonzero and \( TX = 0 \), which is a contradiction. \[\square\]

In the remainder of this section we characterize linear operators that preserve or strongly preserve \( \text{lgw} \) or \( \text{lgw-column majorization} \). We begin with a theorem of [5].

**Theorem 3.10.** [5, Theorem 2.4] A linear operator \( T : F^n \rightarrow F^n \) preserves \( \text{lgw-majorization} \) if and only if one of the following assertions holds:

(i) There exists \( R \in M_n \) such that \( \text{Ker}(R) = \text{Span} \{ e \}, e \notin \text{Im}(R) \), and \( Tx = Rx \) for every \( x \in F^n \);
There exist an invertible matrix \( R \in GR_n \) and \( \alpha \in \mathbb{F} \) such that \( Tx = \alpha Rx \) for every \( x \in \mathbb{F}^n \).

**Corollary 3.11.** A linear operator \( T : \mathbb{F}^n \to \mathbb{F}^n \) preserves \( \text{lgw-majorization} \) if and only if one of the following assertions holds:

(i) there exists an invertible matrix \( D \in GR_n \), such that 
\[
Tx = \left(D - \frac{1}{n} J\right)x \text{ for every } x \in \mathbb{F}^n;
\]

(ii) There exist an invertible matrix \( R \in GR_n \) and \( \alpha \in \mathbb{F} \) such that 
\[
Tx = \alpha Rx \text{ for every } x \in \mathbb{F}^n.
\]

**Proof.** Let \( R \in M_n \). We show that \( \text{Ker}(R) = \text{Span}\{e\} \) and \( e \notin \text{Im}(R) \) if and only if \( R = (D - \frac{1}{n} J) \) for some invertible matrix \( D \in GR_n \).

First, Let \( R = (D - \frac{1}{n} J) \) for some invertible matrix \( D \in GR_n \). It is clear that \( \text{Span}\{e\} \subset \text{Ker}(R) \). If \( x \in \text{Ker}(R) \), then 
\[
Dx = \frac{1}{n} \text{tr}(x)e.
\]
Therefore \( \text{Ker}(R) = \text{Span}\{e\} \). Assume that \( e \in \text{Im}(R) \), then \( (D - \frac{1}{n} J)x = e \) for some \( x \in \mathbb{F}^n \). It implies that 
\[
Dx = \left(1 - \frac{1}{n} \text{tr}(x)\right)e \text{ and hence } x \in \text{Span}\{e\},
\]
which is a contradiction. So \( e \notin \text{Im}(R) \).

Conversely. Let \( \text{Ker}(R) = \text{Span}\{e\} \) and \( e \notin \text{Im}(R) \). Put \( D := R + \frac{1}{n} J \). Since \( Re = 0, D \in GR_n \). It is enough to show that \( D \) is invertible. If 
\[
Dx = 0 \text{ then } Rx = \left(-\frac{1}{n} \text{tr}(x)\right)e.
\]
If \( \text{tr}(x) \neq 0 \), then \( e \in \text{Im}(R) \) which is a contradiction, so \( \text{tr}(x) = 0 \) and \( Rx = 0 \). Therefore \( x \in \text{Span}\{e\} \), which implies that \( x = 0 \). \( \square \)

**Lemma 3.12.** Let \( A \in GR_n \) be invertible. Then the following conditions are equivalent:

(a) \( A = \alpha I + \beta J \), for some \( \alpha, \beta \in \mathbb{R} \);

(b) \( D x + A D y \triangleleft_{\text{lgw}} x + Ay \), for all \( D \in GD_n \) and for all \( x, y \in \mathbb{F}^n \).

**Proof.** (a \( \Rightarrow \) b) If \( A = \alpha I + \beta J \), it is easy to show that \( D x + A D y \triangleleft_{\text{lgw}} x + Ay \), for all \( D \in GD_n \) and for all \( x, y \in \mathbb{F}^n \).

(b \( \Rightarrow \) a) The matrix \( A \) is invertible, so condition (b) can be written as follows:
\[
D x + ADA^{-1} y \triangleleft_{\text{lgw}} x + y, \forall D \in GD_n, \forall x, y \in \mathbb{F}^n.
\]

Put \( x = e - e_i \) and \( y = e_i \) in the above relation. Thus, \( [e - (D - ADA^{-1})e_i] \triangleleft_{\text{lgw}} e_i \), for every \( i \in \mathbb{N} \). So \( (D - ADA^{-1})e_i = 0 \), for every
$i \in \mathbb{N}_n$, and $DA = AD$, for every $D \in \text{GD}_n$. Therefore, $A = \alpha I + \beta J$, for some $\alpha, \beta \in \mathbb{F}$.

\[\begin{array}{l}
\text{Theorem 3.13. Let } T : M_{n,m} \to M_{n,m} \text{ be a linear operator that preserves lgw-majorization. Then, there exist invertible matrices } A_1, \ldots, A_m \in \text{GR}_n, b_1, \ldots, b_m \in \mathbb{F}^m \text{ and } S \in M_m \text{ such that}

TX = [A_1Xb_1 \mid \ldots \mid A_mXb_m] + JXS.
\end{array}\]

Proof. Suppose that $T$ preserves lgw-majorization. It is easy to prove that $T_f^i : \mathbb{F}^n \to \mathbb{F}^m$ preserves lgw-majorization. Then by Corollary 3.11, for every $i, j \in \mathbb{N}_n$, $T_f^i x = (\alpha_i^J A_i^J - \frac{1}{n} \gamma_i^J J)x$, for some invertible matrices $A_i^J \in \text{GR}_n$, $\alpha_i^J \in \mathbb{F}$ and $\gamma_i^J \in \{0, 1\}$. Then

\[TX = T[x_1| \ldots |x_m] = \left[\sum_{j=1}^m T_f^i x_j \mid \ldots \mid \sum_{j=1}^m T_f^i m x_j\right] = \left[\sum_{j=1}^m (\alpha_i^J A_i^J - \frac{1}{n} \gamma_i^J J)x_j \mid \ldots \mid \sum_{j=1}^m (\alpha_i^J A_i^J - \frac{1}{n} \gamma_i^J J)x_j \right].\]

For every $x, y \in \mathbb{F}^m$, define $X = E^j(x) + E^q(y) \in M_{n,m}$. If $\alpha_i^q = 0$ for every $i \in \mathbb{N}_m$, then put $A_i^q = I$. Now, suppose that there exists some $p \in \mathbb{N}_m$ such that $\alpha_i^q \neq 0$. Then for every $D \in \text{GD}_n$, $DX \ll_{tgw} X$, and hence $[\alpha_i^q A_i^q D x + \alpha_i^q A_i^q D y| \ldots |\alpha_i^q A_i^q m x + \alpha_i^q A_i^q m y] \ll_{tgw}$

\[\Rightarrow \alpha_p A_p D x + \alpha_p A_p D y \ll_{tgw} \alpha_i^q A_i^q p x + \alpha_i^q A_i^q p y \Rightarrow Dx + (A_p^q)^{-1} A_p^q D (\alpha_p^q \alpha_q y) \ll_{tgw} x + (A_p^q)^{-1} A_p^q D (\alpha_p^q \alpha_q y).\]

So by Lemma 3.12, $(A_p^q)^{-1} A_p^q = \lambda_p^J I + \beta_p^J J$. Set $A_p := A_p^q$, then $A_p^q = \lambda_p^J I + \beta_p^J J$. Therefore for some $\mu_i^q \in \mathbb{F}$ we have

\[TX = \left[\begin{array}{l}
A_1 \sum_{j=1}^m \mu_i^q x_j \mid \ldots \mid A_p \sum_{j=1}^m \mu_i^q x_j \mid \ldots \mid A_m \sum_{j=1}^m \mu_i^q x_j
\end{array}\right] + JXS,\]
where
\[ S = \begin{pmatrix}
-\frac{1}{n} \gamma_1 + \beta_1 & \cdots & -\frac{1}{n} \gamma_m + \beta_m \\
\vdots & \ddots & \vdots \\
-\frac{1}{n} \gamma_1 + \beta_1 & \cdots & -\frac{1}{n} \gamma_m + \beta_m
\end{pmatrix}. \]

Now, for every \( i \in \mathbb{N}_m \), define
\[ b_i = \begin{pmatrix} \mu_1^i \\ \mu_2^i \\ \vdots \\ \mu_m^i \end{pmatrix}. \]

Then,
\[ TX = [A_1Xb_1 | \ldots | A_mXb_m] + JXS. \]

**Corollary 3.14.** Let \( T \) satisfy the condition of Theorem 3.13 and let \( \text{rank}[b_1 | \ldots | b_m] \geq 2 \). Then \( TX = AXR + JXS \), for some \( R, S \in M_n \), and invertible matrix \( A \in GR_n \).

**Proof.** Without loss of generality we can assume that \( \{b_1, b_2\} \) is a linearly independent set. Let \( X \in M_{n,m} \), \( D \in GD_n \) be arbitrary. Then \( DX <_{lgw} X \) and hence, \( TDX <_{lgw} TX \). It follows that
\[ [A_1 DXb_1 | \ldots | A_m DXb_m] <_{lgw} [A_1 Xb_1 | \ldots | A_m Xb_m] \]
\[ \Rightarrow A_1 DXb_1 + A_2 DXb_2 <_{lgw} A_1 Xb_1 + A_2 Xb_2 \]
\[ \Rightarrow DXb_1 + (A_1^{-1}A_2) DXb_2 <_{lgw} Xb_1 + (A_1^{-1}A_2) Xb_2. \]

Since \( \{b_1, b_2\} \) is linearly independent, for every \( x, y \in \mathbb{R}^n \), there exists \( B_{x,y} \in M_{n,m} \) such that \( B_{x,y}b_1 = x \) and \( B_{x,y}b_2 = y \). Put \( X := B_{x,y} \) in the above relation. Thus,
\[ DB_{x,y}b_1 + (A_1^{-1}A_2) DB_{x,y}b_2 <_{lgw} B_{x,y}b_1 + (A_1^{-1}A_2) B_{x,y}b_2 \]
\[ \Rightarrow D x + (A_1^{-1}A_2) D y <_{lgw} x + (A_1^{-1}A_2) y, \forall D \in GD_n. \]

Then by Lemma 3.12, \( A_1^{-1}A_2 = \alpha I + \beta J \) and hence \( A_2 = \alpha A_1 + \beta J \), for some \( \alpha, \beta \in \mathbb{F} \), \( \alpha \neq 0 \). For every \( i \geq 3 \), if \( b_i = 0 \) we can choose \( A_i = A_1 \). If \( b_i \neq 0 \) then \( \{b_1, b_2\} \) or \( \{b_2, b_i\} \) is linearly independent. Then by the same argument as above, \( A_i = \gamma_i A_1 + \delta_i J \), for some \( \gamma_i, \delta_i \in \mathbb{F} \), \( \gamma_i \neq 0 \), or \( A_i = \lambda_i A_2 + \mu_i J \), for some \( \lambda_i, \mu_i \in \mathbb{F} \), \( \lambda_i \neq 0 \).

Define \( A := A_1 \). Then for every \( i \geq 2 \), \( A_i = \alpha_i A_2 + \beta_i J \), for some \( \alpha_i, \beta_i \in \mathbb{F} \) and hence
\[ TX = [AXb_1 | AX(r_2b_2) | \ldots | AX(r_mb_m)] + JXS = AXR + JXS, \]
where, \( R = [b_1 | r_2 b_2 | \ldots | r_m b_m] \), for some \( r_2, \ldots, r_m \in \mathbb{F} \) and \( S \) is as in Theorem 3.13.

**Lemma 3.15.** Let \( b_1, \ldots, b_m \in \mathbb{F}^m \). The linear operator \( T : M_{n,m} \to M_{n,m} \) defined by \( TX = [Xb_1 | \ldots | Xb_m] \) preserves \( \prec_{\text{lgw}} \) column majorization if and only if \( b_j \in \bigcup_{i=1}^{n} \text{Span}\{e_i\} \), for every \( j \in \mathbb{N}_m \).

The following theorems give the structure of linear and strong linear preserver of \( \prec_{\text{lgw}} \) column majorization on \( M_{n,m} \). Since the proofs are similar to the proofs of Theorems 2.6 and 2.7, we leave the proofs to the readers.

**Theorem 3.16.** Let \( T : M_{n,m} \to M_{n,m} \) be a linear operator. Then \( T \) preserves \( \prec_{\text{lgw}} \) column majorization if and only if there exist invertible matrices \( A_1, \ldots, A_m \in \text{GR}_n \), \( b_1, \ldots, b_m \in \bigcup_{i=1}^{m} \text{Span}\{e_i\} \) and \( D \in M_m \) such that for every \( i \in \mathbb{N}_n \), \( b_i = 0 \) or \( A_1 e_i = \ldots = A_m e_i = 0 \) and for all \( X = [x_1 | \ldots | x_n] \in M_{n,m} \), \( TX = [A_1 X b_1 | \ldots | A_m X b_m] + JXD \).

**Theorem 3.17.** Let \( T : M_{n,m} \to M_{n,m} \) be a linear operator. Then \( T \) strongly preserves \( \text{lgw-column} \) majorization if and only if there exist invertible matrices \( A_1, \ldots, A_m \in \text{GR}_n \) and \( b_1, \ldots, b_m \in \bigcup_{i=1}^{m} \text{Span}\{e_i\} \) such that \( B := [b_1 | \ldots | b_m] \) is invertible and

\[
TX = [A_1 X b_1 | \ldots | A_m X b_m].
\]

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