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DISTRIBUTION OF RATIONAL POINTS: A SURVEY

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To Joseph Shalika, on the occasion of his retirement

Communicated by Samad Hedayat

Abstract. In this survey we discuss Manin's conjectures about the distribution of rational points on certain classes of algebraic varieties.

1. **Introduction**

One of the most important areas of investigation in number theory is the study of the distribution of rational or integral points on algebraic varieties. It is not a small historical miracle that an area of investigation in mathematics that started some 2000 years ago would still be of great interest. This field now has profound connections to various areas of modern mathematics including algebraic geometry, complex analysis, logic, and more recently harmonic analysis and automorphic representation theory. One of the central themes in the theory is to explore the relationship between geometric and arithmetic properties of algebraic varieties. One of the guiding principles is the idea that the rough geometric classification of algebraic varieties according to the ampleness of the canonical (respectively anticanonical) line bundle should directly influence the arithmetic properties. For example, a conjecture due to

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Bombieri, Lang and Vojta asserts that on a variety with ample canonical class all of its rational points are contained in some Zariski closed subset. This conjecture is valid for curves and subvarieties of abelian varieties by the work of Faltings. It is largely open in general. Next there is the class of varieties where neither the canonical nor the anticanonical line bundle is ample. There has been some research to investigate the behavior of rational points on these varieties, but the theory still lacks a general picture of what might be true. On the opposite end of the spectrum we have the class of varieties with ample anticanonical line bundle, called Fano varieties. The class of Fano varieties is very rich and includes many varieties defined by classical diophantine equations (for example, cubic hypersurfaces). The study of the arithmetic properties of Fano varieties motivates many interesting and concrete problems in number theory, automorphic forms, and algebraic geometry for which I refer the reader to the book [19] and the papers listed in the bibliography. Here, I would also like to mention the paper [13] where the authors have considered the distribution of integral points on homogeneous varieties. In fact, their analysis involves understanding the full automorphic spectrum, and is very close to our approach in spirit. The developments up to 1990 along the lines of [13] are explained in Sarnak's ICM talk [25].

In this survey we discuss the distribution of rational points on algebraic varieties with many rational points, and in particular Manin's conjectures for Fano varieties. The paper is organized as follows. In section 2 we review basic properties of height functions on algebraic varieties and set up the basic framework. Section 3 includes a survey of Manin's conjectures and a resume of known results, as well as an introductory subsection on counting zeta functions and tauberian theorems. The last section discusses two explicit one dimensional examples in detail. Here I should point out that there is nothing strictly new in this article, except possibly for the argument presented in 4.2, which is an adaptation of a classical argument due to Deuring. Both of the examples of the fourth Section are easy consequences of much more general results due to Batyrev, Chambert-Loir, and Tschinkel (see Remark 6 and Remark 8 for exact references).

The bulk of these notes are based on a series of lectures delivered by the author [35] at the Sharif University of Technology and the Institute for Studies in Theoretical Physics and Mathematics (IPM) in Tehran, during the summer of 2006. These lectures were largely based on the book of Hindry and Silverman [19]. It would be obvious to the reader that this book has greatly influenced our presentation of the material, although the author has also been influenced by [20] and [27]. Our reference for the algebraic geometry material has been [17] and [28], although here too the book of Hindry and Silverman has been quite useful. Some of these materials were also discussed during colloquium lectures delivered at IPM, Institute of Basic Sciences in Zanjan, and Ahwaz University all during July of 2006, and CUNY Graduate Center and the New York Number Theory Seminar in the Academic Year 2006-2007.

On a personal note, it is a great pleasure to dedicate this modest work to my PhD adviser and mentor Professor Joseph A. Shalika of Johns Hopkins on the occasion of his retirement. I will forever be indebted to him for his support and guidance during my graduate study at Hopkins. It was under his supervision that I first started thinking about the distribution of rational points on homogeneous varieties. Professor Shalika is an amazing human being in the truest sense of the word. He is an influential teacher and an extraordinary mathematician whose teachings, mathematical and otherwise, have changed the lives and careers of those around him. I wish him best of luck in his retirement.

2. **Height functions and metrized line bundles**

2.1. **Weil Height.**

2.1.1. *Height functions on* $\mathbb{P}^n(\overline{\mathbb{Q}})$. The field of rational numbers $\mathbb{Q}$ has a canonical family $\mathcal{M}_{\mathbb{Q}}$ of valuations. We will denote a typical element of $\mathcal{M}_{\mathbb{Q}}$ by v. If $v = \infty$, the corresponding valuation denoted by $\|.\|_\infty$ is the ordinary absolute value. If $v = p$, a prime number, then the corresponding valuation is the p-adic absolute value. It is well-known that if $r \in \mathbb{Q}^\times$, then

$$\prod_{v \in \mathcal{M}_{\mathbb{Q}}} \|r\|_v = 1.$$

If k is a finite extension of $\mathbb{Q}$ it is possible to construct a canonical family of valuations $\mathcal{M}_k$ in such a way that for all $\alpha \in k^\times$ we have

$$\prod_{w \in \mathcal{M}_k} \|\alpha\|_w = 1. \tag{2.1}$$

For details see [27], Section 2.1. We now define a height function on $\mathbb{P}^n(\overline{\mathbb{Q}})$. Let $x \in \mathbb{P}^n(\overline{\mathbb{Q}})$. Then x has a representative of the form $(x_0, x_1, \ldots, x_n)$, with $x_i \in \overline{\mathbb{Q}}$, not all zero, for $0 \leq i \leq n$. Then there is a finite extension of $\mathbb{Q}$, say K, such that $x_i \in K$ for all i. We set

$$H(x) = \left(\prod_{w \in \mathcal{M}_K} \sup_{0 \leq i \leq n} \|x_i\|_w \right)^{\frac{1}{[K:\mathbb{Q}]}}.$$

If for example $x_0 \neq 0$, we have $H(x) \geq \prod_{w \in \mathcal{M}_K} \|x_0\|_w = 1$. It follows from the product formula (2.1) that $H(x)$ is independent of the choice of the representative $(x_0, \ldots, x_n)$ and the finite extension K. As a result we obtain a function $H : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}$ such that $H(x) \geq 1$ for all x. It should be noted that the above construction also works for function fields. In the situation where $x_0, \ldots, x_n$ are rational numbers, we may clear denominators to obtain a representative of x of the form $(y_0, \ldots, y_n)$ with $y_i \in \mathbb{Z}$, $0 \leq i \leq n$, and $\gcd(y_0, \ldots, y_n) = 1$. Such an $(n+1)$-tuple is called *primitive*. One can show that in this case

$$H(x) = \max(|y_0|, \ldots, |y_n|).$$

This identity shows that this definition of the height function agrees with that considered in 3.4.1.

2.1.2. *The pull-back of the height function.* Here we follow [19]. Let V be a projective variety defined over $\overline{\mathbb{Q}}$, and suppose $\phi : V \to \mathbb{P}^n$ is a morphism. We define the *height with respect to* ϕ to be the function $H_\phi : V(\overline{\mathbb{Q}}) \to [1, \infty)$ given by $H_\phi(P) = H(\phi(P))$. Given a variety V, we define an equivalence relation $\sim_V$ on the collection of functions on $V(\overline{\mathbb{Q}})$ with values in the positive real numbers: We say $f \sim_V g$ if $\log f(P) = \log g(P) + O(1)$ for all $P \in V(\overline{\mathbb{Q}})$.

Theorem 2.1. *Let V be a projective variety defined over $\overline{\mathbb{Q}}$, and let $\phi : V \to \mathbb{P}^n$ and $\psi : V \to \mathbb{P}^m$ be morphisms again defined over $\overline{\mathbb{Q}}$. Suppose $\phi^* L_0$ and $\psi^* L_0'$, L_0 and L_0' hyperplane sections in $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, are linearly equivalent. Then $H_\phi \sim_V H_\psi$.*

2.1.3. *Weil's height machine.* We have the following theorem:

Theorem 2.2. *Let k be a number field. For every smooth projective variety V/k there exists a map*

$$H_V : \mathrm{Div}(V) \to \{ \text{ real valued functions on } V(\overline{k})\},$$

$\mathrm{Div}(V)$ *the divisor group of V, with the following properties:*

1. *Let $L_0 \subset \mathbb{P}^n$ be a hyperplane section. Then $H_{\mathbb{P}^n,L_0} \sim_{\mathbb{P}^n} H$.*
2. *Let $\phi : V \to W$ be a morphism, and $D \in \mathrm{Div}(W)$. Then $H_{V,\phi^*D} \sim_V H_{W,D} \circ \phi$.*
3. *Let $D, E \in \mathrm{Div}(V)$. Then $H_{V,D+E} \sim_V H_{V,D}.H_{V,E}$.*
4. *If $D \in \mathrm{Div}(V)$ is principal, then $H_{V,D} \sim_V 1$.*
5. *Let $D \in \mathrm{Div}(V)$ be an effective divisor, and let B be the base locus of the linear system $|D|$. Then there is $c > 0$ such that $H_{V,D}(P) \geq c$ for all $P \in (V\backslash B)(\overline{\mathbb{Q}})$.*
6. *Let $D, E \in \mathrm{Div}(V)$. Suppose D is ample and E is algebraically equivalent to 0. Then*
$$\lim_{H_{V,D}(P)\to\infty} \frac{\log H_{V,E}(P)}{\log H_{V,D}(P)} = 0.$$
7. *Let $D \in \mathrm{Div}(V)$ be ample. then for every finite extension k'/k and every constant B, the set*
$$\{P \in V(k'); H_{V,D}(P) \leq B\}$$
is finite.
8. *The function H_V is unique up to $\sim_V$.*

The function H_V descends to a function on the Picard group of the variety V, which we will denote by $\mathrm{Pic}(V)$, *to $\{$real functions on $V(\overline{\mathbb{Q}})\}/\sim_V$.*

For the notion of *algebraic equivalence* mentioned above see [28], Book 1, Chapter III, 4.4. Briefly, a family of divisors on a variety X with base T, is any map $f : T \to \mathrm{Div}X$. We say the family f is *algebraic* if there exists a divisor $C \in \mathrm{Div}(X \times T)$, T an algebraic variety, such that for each $t \in T$, $(X \times \{t\}) \cap C = f(t) \times \{t\}$. Divisors D_1, D_2 are said to be *algebraically equivalent* if there is an algebraic family $f : T \to \mathrm{Div}X$ and $t_1, t_2 \in T$ such that $D_1 = f(t_1)$ and $D_2 = f(t_2)$.

The construction of the function H_V is as follows. If D is very ample, we choose a morphism $\phi_D : V \to \mathbb{P}^n$ associated to D and we define

$$H_{V,D}(P) = H(\phi_D(P))$$

for all $P \in V(\overline{\mathbb{Q}})$. Next if D is an arbitrary divisor, we write D as $D_1 - D_2$, with D_1, D_2 very ample. We then define

$$H_{V,D}(P) = \frac{H_{V,D_1}(P)}{H_{V,D_2}(P)}.$$

If $[D] \in \mathrm{Pic}(V)$ is a class, we obtain a class of functions $H_{V,[D]} : V(\overline{\mathbb{Q}}) \to \mathbb{R}$, determined up to $\sim_V$. We typically work with a specific function $f \in H_{V,[D]}$. We will explain a more general way to define height functions in 2.2.

2.1.4. *Counting points.* The finiteness statement of the last theorem leads us to the following question. Let k be a number field and let $[D] \in \mathrm{Pic}(V)$ be such that for all B and all $f \in H_{V,[D]}$ we have

$$N_f(B) = |\{P \in V(k); f(P) \le B\}|$$

is finite. We often suppress the subscript f. We know that every ample class satisfies this requirement. The question that we are going to consider in this paper is how the number $N(B)$ changes as $B \to \infty$. The general feeling is that the geometry of V should determine the growth of $N(B)$ if one desensitizes the arithmetic by passing to a sufficiently large ground field (via finite extensions of k), and perhaps deleting certain exceptional subvarieties the growth of the number of rational points of which dominates the behavior of the whole set $V(k)$. We will see instances of these exceptional subvarieties in 3.4.3 and 3.4.4.

We believe that the distribution of rational points on an algebraic variety is tightly related to its geometry. An invariant of a smooth variety V that is expected to play an important role is its anti-canonical class $-K_V$; here K_V is the canonical class of the variety V defined by top degree differential forms. In order to illustrate this expectation we examine a heuristic for hypersurfaces [20]. Suppose F is a homogeneous form with integral coefficients of degree d on $\mathbb{P}^n$. We have already seen that there are about B^{n+1} rational points of heights less than B in $\mathbb{P}^n(\mathbb{Q})$; we will think of these points as primitive $(n+1)$-tuples of integers. Since F is of degree d, it takes about B^d values on the set of primitive integral points of height less than B. Let us assume that the values taken by F are equally likely so that the probability of getting 0 is about B^{-d}. This means that $F(X_0, \dots, X_n)$ should be zero for about B^{n+1-d} of the rational points of height less than B. Recall that if $V = (F = 0)$, then $-K_V = (n+1-d)H$. The basic principle is that the occurrence of $n+1-d$ is more than a mere accident! When $n+1>d$ one expects an abundance of rational points, and otherwise scarcity. In geometric terms, when $n+1>d$ the resulting hypersurface is *Fano*; a non-singular projective variety the negative of whose canonical class is ample is called *Fano*. Manin [14] has formulated certain conjectures describing the behavior of $N(B)$ for Fano varieties; for non-Fano varieties one expects that rational points should be rare; see however Remark 2.3. We survey Manin's original conjectures in 3.3 below. For curves this is completely worked out, and the results can be described in terms of the genus of the curve. So let C be a non-singular projective curve of genus g defined over a number field k. Then

- When $g=0$, the curve C is Fano. Such a curve will have a Zariski dense set of rational points if we go over a finite extension of k; over such extension, the number of rational points of height less than B grows at least like a positive power of B. The Pythagorean and Pellean curves considered below are in fact examples of this situation.
- When $g=1$, then neither the canonical class, nor its negative is ample; this is the so-called *intermediate type*. In this case, if the curve has a rational point, it will be an elliptic curve. For any number field k, by the celebrated Mordell-Weil theorem, the set $C(k)$ is a finitely-generated abelian group. The logarithm of any height function is equivalent, in the sense of $\sim_C$, to a quadratic form on $C(k)$. It is then easily seen that $N(B)$ grows at most like a power of $\log B$. This is the scarcity alluded to above.
- When $g>1$, then the curve is of *general type*. In this case, by Mordell's conjecture, proved by Faltings and Vojta, for any number field k, the set $C(k)$ is finite. These are clearly quite rare!

So clearly even for curves the problem of understanding the distribution of rational points is not trivial! Next, following [20], we make some comments on surfaces. In this discussion we assume familiarity with the conjectures of 3.3. Let V be a non-singular projective surface defined over a number field k. Then

- $-K_V$ ample. Such a surface is called a *del Pezzo* surface. The study of such surfaces poses some serious arithmetical problems. There is a classification of del Pezzo surfaces over $\mathbb{C}$; there is a $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$, and the remaining classes all can be obtained as blow-ups of $\mathbb{P}^2$ at at most eight points in general position. This

classification is not valid over a non-algebraically closed field as one has to account for non-trivial $\mathrm{Gal}(\bar{k}/k)$ actions. At any rate, such surfaces can have subvarieties isomorphic to $\mathbb{P}^1$ embedded in such a way that $(-K_V)|_{\mathbb{P}^1} = \frac{1}{2}(-K_{\mathbb{P}^1})$, so that if we do the counting with respect to the height function given by the anti-canonical class, then there will be B^2 points of height less than B on the $\mathbb{P}^1$ subvariety alone (see 3.4.1 for the computation of the number of rational points on $\mathbb{P}^n$ for $n \geq 2$ – the case of $\mathbb{P}^1$ is similar). After throwing out such an embedded subvariety one would only have $B^{1+\epsilon}$ rational points of bounded height for any $\epsilon > 0$. These are the exceptional subvarieties mentioned at the end of the first paragraph of 2.1.4.
- Intermediate cases: There are various interesting class of such surfaces, e.g. abelian surfaces, $K3$ surfaces, etc. Some of these are easy to study and some others are hard.
- General type: There are the conjectures of Bombieri and Lang which are higher dimensional analogues of the Mordell's conjecture, i.e. one expects that over any number field k, the collection of rational points $V(k)$ should be contained in a proper Zariski closed set. In general, there is very little known about this. See [19], F.5.2 for more details.

Remark 2.3. While this is in principle expected to be valid, there are varieties which are "almost Fano" (e.g. toric varieties that are not Fano) for which Manin's conjectures hold and therefore rational points on them are not "rare"; see [5, 3].

2.2. **Heights for metrized line bundles.** The definition of $N_f(B)$ in 2.1.4 depended on the choice of a function $f \in H_{V,[D]}$. In this paragraph, we make the choice of the function f geometric. Our reference here is [20].

2.2.1. *Metrizations.* Let us start with a very general definition. For a moment, Let L be a one-dimensional space over a number field k. An *adèlic metrization of* L is a choice at each place v of k of a norm $\|.\|_v : L \to \mathbb{R}$ with the following properties

(1) $\|a\lambda\|_v = |a|_v.\|\lambda\|_v$ for all $a \in k$ and $\lambda \in L$; and
(2) For each $\lambda \in L\backslash\{0\}$, $\|\lambda\|_v = 1$ for all except finitely many v.

It is clear that $(k, \{|.|_v\}_v)$ is an adèlic metrization.

Now let V be an algebraic variety, and L an invertible sheaf on V. We proceed to define the notion of an *L-height on $V(k)$*. First let L be a very ample sheaf, and let $\underline{s} = (s_0, \ldots, s_n)$ be a basis for $\Gamma(L)$. For every $x \in V(k)$, there is j such that $s_j(x) \neq 0$. Set

$$H_{L,\underline{s}}(x) = \prod_v \max_i \left\{ \left| \frac{s_i(x)}{s_j(x)} \right|_v \right\}.$$

This is clearly independent of the chosen j. If L is not very ample, we fix an isomorphism $\sigma : L \to L_1 \otimes L_2^{-1}$ and bases $\underline{s}$ and $\underline{t}$ of $\Gamma(L_1)$ and $\Gamma(L_2)$, respectively, and put

$$H_{L,\sigma,s}(x) = H_{L_1,\underline{s}}(x) H_{L_2,\underline{t}}(x)^{-1}.$$

Example 1. For $k = \mathbb{Q}$, $V = \mathbb{P}^n$, $L = \mathcal{O}(1)$, and $\underline{s}$ a homogeneous coordinate system on V, then

$$H_{\mathcal{O}(1),\underline{s}}(x) = \max_i (|x_i|)$$

where $(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ is a primitive representative for x.

Now we explain the relation between these height functions and adèlizations. Given an invertible sheaf L on V, we call a family $\|.\|_{v,x} : L_x \to \mathbb{R}$ of v-adic norms, one for each $x \in V(k_v)$, a *v-adic metric on L* if for every Zariski open $U \subset V$ and every section $s \in \Gamma(U, L)$ the map $x \mapsto \|s(x)\|_{x,v}$ is a continuous function $U(k_v) \to \mathbb{R}$. What we did earlier is in fact a v-adic metric in disguise: If L is very ample, it is generated by global sections, and we can choose a basis $\underline{s} = (s_0, \ldots, s_n)$ of $\Gamma(L)$ defined over k. If s is a section such that $s(x) \neq 0$, we set

$$\|s(x)\|_{x,v} := \max_i \left\{ \left| \frac{s_i(x)}{s(x)} \right|_v^{-1} \right\},$$

otherwise $\|s(x)\|_{x,v} = 0$. It is easy to see that this depends only on the point $s(x) \in L_x$ and not on the choice of the section s, and thus we obtain a function $L_x \to \mathbb{R}$. We call this the *metrization defined by means of the basis $\underline{s} = (s_0, \ldots, s_n)$*.

We can now define what we mean by an arbitrary adèlic metric on an invertible sheaf. Again we assume that L is very ample. An adèlic metric on L is a collection of v-adic metrics such that for all but finitely many v the v-adic metric on L is defined by means of some fixed basis $\underline{s} = (s_0, \ldots, s_n)$ of $\Gamma(L)$. We call the data $\mathcal{L} = (L, \{\|.\|_v\}_v)$ an *adèlically metrized line bundle*. To extend the construction to arbitrary invertible sheaves, as before we represent an arbitrary sheaf L as $L_1 \otimes L_2^{-1}$ with L_1, L_2 very ample. If L_1 and L_2 are adèlically metrized, then their v-adic metrizations naturally extend to the tensor product. An adèlic metrizations of L is any metrization which for all but finitely many v is induced from the metrizations of L_1, L_2.

Given an adèlically metrized line bundle $\mathcal{L} = (L, \{\|.\|_v\}_v)$, we define local and global height functions as follows. Let s be a local section of L. Let $U \subset X$ be the maximal Zariski open subset of X where s is defined and is non-zero. For $x = (x_v)_v \in U(\mathbb{A})$, $\mathbb{A}$ the ring of adèles, we define the local height

$$H_{\mathcal{L},s,v}(x_v) := \|s(x_v)\|_{x_v,v}^{-1}$$

and the global height function

$$H_{\mathcal{L},s}(x) := \prod_v H_{\mathcal{L},s,v}(x_v).$$

It is important to note that by the *product formula* the restriction of global height to $U(k)$ does not depend on the choice of s. We then obtain a function

$$H_{\mathcal{L}} : V(k) \to \mathbb{R}.$$

To complete the circle, suppose we are given a class $[D] \in \mathrm{Pic}(V)$, V a smooth variety over a number field k. There is a standard procedure to associate to $[D]$ a line bundle $L_{[D]}$. Let $\mathcal{L}$ be any adèlic metrization of $L_{[D]}$. Then $H_{\mathcal{L}} \in H_{V,[D]}$.

3. **Conjectures**

In this section, we state Manin's conjectures on the distribution of rational points on Fano varieties. First we need some preparation on zeta functions.

3.1. **Zeta functions and counting.**

Let S be any countable, possibly finite, set, and let $H : S \to \mathbb{R}_+$ be a function such that

$$N_S(H,B) = |\{x \in S; H(x) \leq B\}|$$

is finite for all B. Put for $s \in \mathbb{C}$, at first formally,

$$\mathcal{Z}_S(H;s) = \sum_{x \in S} H(x)^{-s},$$

and set $\beta_H = \inf\{\sigma \in \mathbb{R}; \mathcal{Z}_S(H;s) \text{ converges for } \Re s > \sigma\}$. It is important to note that if H, H' are two functions such that $\log H = \log H' + O(1)$, then $\beta_H = \beta_{H'}$. Also, β_H is non-negative unless S is finite, in which case it is $-\infty$. When $\beta_H \geq 0$, we have

$$\beta_H = \limsup_{B\to\infty} \frac{\log N_S(H,B)}{\log B}.$$

This implies that for $\beta_H \geq 0$ we have $N_S(H,B) = O(B^{\beta_H+\epsilon})$, and $N_S(H,B) = \Omega(B^{\beta_H-\epsilon})$ for all $\epsilon > 0$. Finer growth properties of $N_S(H,B)$ can be obtained from the analytic properties of $\mathcal{Z}_S$ as the following theorem shows:

Theorem 3.1 (Ikehara's Tauberian theorem)**.** *Suppose for some $t > 0$ we have $\mathcal{Z}_S(H,s) = (s-\beta_H)^{-t}G(s)$ where $G(\beta_H) \neq 0$ and $G(s)$ is holomorphic for $\Re s \geq \beta_H$. Then if $\beta_H > 0$*

$$N_S(H,B) = \frac{G(\beta_H)}{\beta_H \Gamma(t)} B^{\beta_H} (\log B)^{t-1}(1+o(1))$$

as $B \to \infty$.

Better understanding of the analytic properties of G, e.g. holomorphic continuation to a larger domain with growth conditions, usually leads to better error estimate in the asymptotic formula for N_S.

Remark 3.2. It was pointed out that if H, H' satisfy $\log H = \log H' + O(1)$ then $\beta_H = \beta_{H'}$. The same is not true of the analytic properties of $\mathcal{Z}_S$. Here is an example taken from [20]. Choose a sequence of positive integers d_i such that $d_i \to \infty$ and $\frac{d_{i+1}}{d_i} \to \infty$. Set $S = \mathbb{N}$ and let H be any function satisfying the finiteness condition for $N_S(B,H)$ for all B. Set

$$H'(x) = \begin{cases} 2H(x) & d_{2i} \leq H(x) < d_{2i+1}; \\ \frac{H(x)}{2} & d_{2i+1} \leq H(x) < d_{2i+2}. \end{cases}$$

Then one can show that $N_S(H';B)/B^{\beta_{H'}}(\log B)^{t-1}$ does not tend to a limit as $B \to \infty$, so that $\mathcal{Z}(H',s)(s-\beta_{H'})^t$ cannot be holomorphic for $\Re s \geq \beta_{H'}$.

3.2. **Height zeta function.**

3.2.1. *Néron-Severi group.* Let V be an algebraic variety defined over a number field k. We define the Néron-Severi group $NS(V)$ to be the quotient of $\mathrm{Pic}(V)$ by algebraic equivalence. For any field F we set $NS(V)_F = NS(V) \otimes F$. Given a divisor D in $\mathrm{Div}(V)$, we denote by $\overline{L}$ its class in any of the Néron-Severi groups. Ample classes form a semi-group in $\mathrm{Pic}(V)$ (Think about the Veronese map!); this semi-group will generate a convex open cone $N_+^\circ(V)$ in $NS(V)_{\mathbb{R}}$. The closure of $N_+^\circ(V)$, denoted by $N_+(V)$, is the collection of *nef* (*numerically effective*) classes. Now let L be an ample sheaf on V, and $\mathcal{L}$ an adèlic metrization. Let $H_{\mathcal{L}}$ be the associated height function. Let U be a subset in $V(k)$, and let $\beta_U(\mathcal{L})$ be the abscissa of convergence of $\mathcal{Z}_U(H_{\mathcal{L}}, s)$. One can show that $\beta_U(\mathcal{L})$ depends only on $\overline{L}$ in $NS(V)_{\mathbb{Q}}$. We will then denote $\beta_U(\mathcal{L})$ by $\beta_U(\overline{L})$. Also if $\beta_U(\overline{L}) \geq 0$ for one ample class L, then the same holds for all ample classes. In general, β_U extends uniquely to a continuous function on $N_+^\circ(V)$ which is inverse linear on each half-line. For a subset $U \subset V(k)$, we let $N_U(B, \mathcal{L})$, or when convenient $N_U(B, H_{\mathcal{L}})$, denote the number of points $P \in U$ such that $H_{\mathcal{L}}(P) \leq B$.

3.2.2. *Accumulating subvarieties.* A subset $X \subset U$ ($U \subset V(k)$) is called *accumulating* if

$$\beta_U(\overline{L}) = \beta_X(\overline{L}) > \beta_{U \setminus X}(\overline{L}),$$

and *weakly accumulating* if

$$\beta_U(\overline{L}) = \beta_X(\overline{L}) = \beta_{U \setminus X}(\overline{L}).$$

Example 2. Let V be $\mathbb{P}^2$ blown up at a point. For any height function associated to the anti-canonical class, the exceptional divisor is accumulating.

3.3. **Manin's Conjectures.** In the following conjectures V is a Fano variety defined over a number field k.

Conjecture 1. *There is a finite extension k' of k such that $V(k')$ is Zariski dense in V.*

Conjecture 2. *Let k' be as in the previous conjecture. Then for sufficiently small Zariski open $U \subset V$ we have $\beta_{U(k')}(\overline{-K_V}) = 1$.*

Conjecture 3. *Let U, k' be as above. Then if $-\mathcal{K}_V$ is a metrization of $-K_V$, there is a constant $C > 0$ such that*

$$N_{U(k')}(B, -\mathcal{K}_V) = CB(\log B)^{t-1}(1 + o(1))$$

as $B \to \infty$, where $t = \operatorname{rank} NS(V)$.

There is a conjectural description of the constant C in the last conjecture due to Peyre [24].

Remark 3.3. If a variety V has a rational curve on it, then clearly it will have an infinite number of rational points over some extension. The *Rational Curve Conjecture* says that if for a quasi-projective variety V and an open subset $U \subset V$ we have $\beta_{U(k)}(\overline{L}) > 0$, then U contains a rational curve.

Remark 3.4. Batyrev and Tschinkel [3] have introduced conjectures for arbitrary metrized line bundles refining the conjectures of [1]. These conjectures aim to describe the behavior of $N_U(B, \mathcal{L})$ for any metrized line bundle $\mathcal{L} = (L, \|.\|)$ whose underlying invertible sheaf L is in the interior of the cone of effective divisors. They have introduced notions of $\mathcal{L}$-saturation, $\mathcal{L}$-primitiveness, etc. (See [3], sections 2.3 and 3.2). In the situation where V is a strongly $\mathcal{L}$-saturated (and $\mathcal{L}$-primitive) smooth quasi-projective variety according to Step 4 of section 3.4 of [3] the expectation is that

$$N_U(B, \mathcal{L}) \sim CB^a(\log B)^{b-1}$$

with $C \geq 0$, $a \in \mathbb{N}$, $b \in \frac{1}{2}\mathbb{N}$. Indeed one expects the following to hold. First off, $N_+(V)$ is believed to be a polyhedral cone, so for a point in the boundary of $N_+(V)$ it would make sense to talk about the *codimension of the face containing* it. Given $\mathcal{L}$ as above, we set

$$a = \inf\{a \in \mathbb{R}; a[L] + [K_V] \in N_+(V)\}$$

and b will be the codimension of the face containing $a(L)[L] + [K_V]$. We note that a and b are independent of the metrization and depend only on the underlying geometric data. The constant C however depends on the metrization. In the situation where V is not $\mathcal{L}$-primitive or strongly $\mathcal{L}$-saturated one needs to use a fiber-by-fiber analysis of $\mathcal{L}$-primitive fibrations (See section 3.5 of the [3] for some details).

3.4. **Examples.** In this paragraph we collect several explicit examples of various degrees of difficulty. Two more examples are worked out in 4.1 and 4.2.

3.4.1. *Projective spaces.* We start by defining the standard height function on $\mathbb{P}^n(\mathbb{Q})$. If $x \in \mathbb{P}^n(\mathbb{Q})$, there is a representative of x of the form $(x_0, \ldots, x_n)$ with $x_i \in \mathbb{Z}$, and $\gcd(x_0, \ldots, x_n) = 1$. We then define

$$H(x) = \max_i |x_i|.$$

The choice of the max norm is fairly arbitrary, and we could have instead used any other Banach norm on $\mathbb{R}^{n+1}$ restricted to $\mathbb{Z}^{n+1}$. Here we will consider the following quantity:

$$N_{\mathbb{P}^n}(B) = \Big|\{x \in \mathbb{P}^n(\mathbb{Q}) \big| H(x) \leq B\}\Big|.$$

Problem 1. *Determine the asymptotic behavior of $N(B)$ as $B \to \infty$.*

For simplicity we assume $n \geq 2$. Let us start by setting

$$\begin{aligned}
\mathcal{Z}(s) &= \sum_{\gamma \in \mathbb{P}^n(\mathbb{Q})} \frac{1}{H(\gamma)^s} \\
&= \sum_{(x_0,\ldots,x_n) \in \mathbb{Z}^{n+1}_{\text{prim}} \setminus \{(0,\ldots,0)\}} \frac{1}{\max(|x_0|, \ldots, |x_n|)^s}.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta(s)\mathcal{Z}(s) &= \sum_{(x_0,\ldots,x_n) \in \mathbb{Z}^{n+1} \setminus \{(0,\ldots,0)\}} \frac{1}{\max(|x_0|, \ldots, |x_n|)^s} \\
&= \sum_{k=1}^{\infty} \frac{(2k+1)^{n+1} - (2k-1)^{n+1}}{k^s}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\mathcal{Z}(s) &= \left(\sum_{r=1}^{\infty} \frac{\mu(r)}{r^s}\right) \cdot \left(\sum_{k=1}^{\infty} \frac{(2k+1)^{n+1} - (2k-1)^{n+1}}{k^s}\right) \\
&= \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{rk=m} \mu(r) \left\{(2k+1)^{n+1} - (2k-1)^{n+1}\right\}.
\end{aligned}$$

So we need to determine the asymptotic behavior of

$$\sum_{m\leq B}\sum_{rk=m}\mu(r)\left\{(2k+1)^{n+1}-(2k-1)^{n+1}\right\}$$

as $B\to\infty$. After applying the binomial theorem to the inner expression, this is seen to be equal to

$$\begin{aligned}&\sum_{j=0}^{n+1}\binom{n+1}{j}2^j\left[1-(-1)^{n+1-j}\right]\sum_{m\leq B}\sum_{rk=m}\mu(r)k^j\\ =&\frac{2^n}{\zeta(n+1)}B^{n+1}+O(B^n).\end{aligned}$$

Remark 3.5. There is a theorem of Schanuel [26] that generalizes this to $\mathbb{P}^n(k)$ for any number field k.

3.4.2. *Products of varieties.* If $V = V_1\times V_2$ and $H_i : V_i(k)\to\mathbb{R}_+$, $U_i\subset V_i$ open subsets, then we have a function $H : V\to\mathbb{R}_+$ defined by $H(x_1,x_2) = H_1(x_1)H_2(x_2)$ corresponding to the Segre embedding. Assume that

$$N_{U_i}(B,H_i) = C_iB(\log B)^{t_i-1}+O(B(\log B)^{t_i-2}).$$

Then by a proposition in §1 of [14] (also see [23])

$$N_{U_1\times U_2}(B,H) = \frac{(t_1-1)!(t_2-1)!}{(t_1+t_2-1)!}C_1C_2B(\log B)^{t_1+t_2-1}(1+o(1)).$$

3.4.3. *Blowup of* $\mathbb{P}^2$ *at one point.* This example is [19], F.5.4.3 (originally [27] 2.12; also [20] for generalizations). Let $X\to\mathbb{P}^2$ be the blowup at one point. Let L be the pullback to X of a generic line on $\mathbb{P}^2$, and E the exceptional divisor. Then one shows [27] that $\mathrm{Pic}(X)$ is freely generated by L,E. The canonical divisor is $K_X = -3L+E$. It is seen using the Nakai-Moishezon Criterion ([17], Chapter V, Theorem 1.10) that a divisor $D = aL-bE$ is ample if $a > b > 0$. For an ample divisor $D = aL-bE$ set

$$\alpha(D) = \max\left\{\frac{3}{a},\frac{2}{a-b}\right\}.$$

Let $U = X \backslash E$. Then if $\mathcal{L}$ is a metrization of a line bundle associated to D

$$N_{U(k)}(B, \mathcal{L}) = \begin{cases} cB^{\alpha(D)} & \text{if } D \text{ is not proportional to } -K_X \\ cB^{\alpha(D)} \log B & \text{if } D \text{ is proportional to } -K_X. \end{cases}$$

3.4.4. *Blow-up of $\mathbb{P}^2$ at three points.* This is example 2.4 of [23]. Let $V \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, and $P_3 = (0 : 0 : 1)$. Then V is a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by the equation $x_1 x_2 x_3 = y_1 y_2 y_3$. Let's set

$$H(P_1, P_2, P_3) = H_{\mathbb{P}^1}(P_1) H_{\mathbb{P}^1}(P_2) H_{\mathbb{P}^1}(P_3)$$

where $H_{\mathbb{P}^1}$ is the standard height on $\mathbb{P}^1$. This will define a height function on $V(\mathbb{Q})$. There are six exceptional lines on V defined by $E_{ij} = (x_i = 0, y_j = 0)$ for $i \neq j$. Let $U = V \backslash \bigcup_{i \neq j} E_{ij}$. We have

$$N_{E_{ij}}(B, H) \sim CB^2$$

and

$$N_U(B, H) \sim \frac{1}{6} \left(\prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right) B (\log B)^3.$$

3.5. **Results and methods.**

3.5.1. *Some bad news.* Before we get our hopes too high let us point out that of these conjectures at least the Conjecture 3 is wrong! Let us explain a family of counter-examples due to Batyrev and Tschinkel [2]. Let X_{n+2} be a hypersurface in $\mathbb{P}^n \times \mathbb{P}^3$ $(n \geq 1)$ defined by the equation

$$\sum_{i=0}^{3} l_i(\underline{x}) y_i^3 = 0$$

where $\underline{x} = (x_0, \ldots, x_n)$, and $l_0(\underline{x}), \ldots, l_3(\underline{x})$ are homogeneous linear forms in $x_0, \ldots, x_n$. We will assume that any $\min(n+1, 4)$ forms among the l_i's are linearly independent. Then one can check that X_{n+2} is a smooth Fano variety containing a Zariski open subset U_{n+2} which is isomorphic to $\mathbb{A}^{n+2}$, and that the Picard group of X_{n+2} over an arbitrary field containing $\mathbb{Q}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Then the blow comes from the fact that for any open subset $U \subset X_{n+2}$ there exists a number field

k_0 containing $\mathbb{Q}(\sqrt{-3})$ which may depend on U such that for any field k containing k_0 one has

$$N_{U(k_0)}(B, -\mathcal{K}_{X_{n+2}}) \geq cB(\log B)^3$$

for all $B > 0$ and some positive constant c. This clearly contradicts the conjecture.

3.5.2. *But it's not all bad news...* There are many cases where the conjectures of Manin and their refinements and generalizations are proved. Here is an incomplete list:

- smooth complete intersections of small degree in $\mathbb{P}^n$ (circle method, e.g. [6]);
- generalized flag varieties [14];
- toric varieties [4], [5];
- horospherical varieties [33];
- equivariant compactifications of $\mathbb{G}_a^n$ [8];
- bi-equivariant compactifications of unipotent groups [32];
- wonderful compactifications of semi-simple groups of adjoint type [15, 31].

We expect that Manin's conjecture (and its refinements) should hold for equivariant compactifications of *all* linear algebraic groups G and their homogeneous spaces G/H.

3.5.3. *Methods.* There are a number of methods that have been employed to treat special cases of the conjectures 1, 2, and 3. This is a summary of the basic ideas:

Elementary methods: Sometimes elementary tricks can be used to treat some special examples, e.g. our treatment of $\mathbb{P}^n(\mathbb{Q})$.

Circle method: This method, invented by Hardy and Ramanujan, first appeared in connection with classical problems in number theory [36, 21, 6]: Waring's problems, Vinogradov's work on writing numbers as sums of three primes, partitions, etc. The idea is indeed quite simple: Suppose we have a finite set $\mathcal{A} = \{a_m\}$ of positive integers and we want to know whether or in how many ways a given integer n can be written as a sum of s integers from the set $\mathcal{A}$. In order to do this one forms the generating function $f(x) = \sum_m e^{2\pi i a_m x}$, and considers

$$f(x)^s = \sum_n R_s(n) e^{2\pi i n x}$$

where $R_s(n)$ is the number of ways of writing n as a sum of s elements of $\mathcal{A}$. It is then clear that

$$R_s(n) = \int_0^1 f(x)^s e^{-2\pi i n x}\, dx.$$

This is called the circle method as $e^{2\pi i x}$ traces a circle in $\mathbb{C}$ when x is in $[0,1]$. One then writes $[0,1]$ as a union $\mathcal{M} \cup \mathfrak{m}$. $\mathcal{M}$ is where the function f is supposed to be large, i.e. small neighborhoods of rational numbers, and $\mathfrak{m}$ is the left-over. In a successful application of the circle method one expects the main term of $R_s(n)$ to come from $\int_{\mathcal{M}}$ and the rest $\int_{\mathfrak{m}}$ to be error term.

Harmonic analysis: This method is most successful when the variety in question has a large group of automorphisms, and in this situation the harmonic analysis of the group of automorphisms enters the picture. This has been the case in applications to toric varieties, additive group compactifications, wonderful compactifications, and flag varieties stated above. For example in [31] the idea is to consider the height zeta function $\mathcal{Z}_{\mathcal{L}}(s)$ for a given metrized line bundle $\mathcal{L}$ and interpret it as a special value of an appropriately defined automorphic form. Then one uses the spectral theory of automorphic forms to derive the desired results.

Dynamics and ergodic theory: This is one of the emerging ideologies of the subject and number theory in general. The method has already yielded a major theorem [15] and is expected to produce more results. The basic idea in [15] is to first prove the equidistribution of rational points of bounded height using mixing techniques of ergodic theory. Then the asymptotics of the number of rational points follow from volume computations. See [34] for a motivated description of the method, and [16] for a spectacular new development.

Universal torsors: The idea is to relate the rational points on a projective variety to the integral points of the *universal torsor* of the variety. For example, the points of $\mathbb{P}^n(\mathbb{Q})$ are in one to one correspondence with primitive $(n+1)$-tuples of integers. This is clearly an oversimplified example! See [18, 10] for some computations and [7, 11, 23] and the references therein for a list of applications to various counting problems.

4. **Two more examples**

4.1. **The Pythagorean Equation.** In this section, we will study the Pythagorean equation

$$x^2 + y^2 = z^2$$

with $x, y, z \in \mathbb{Z}$. After dividing by any common factors that x, y, and z might have we can assume that $\gcd(x, y, z) = 1$. We call such a triple *Pythagorean*. Note that if (x, y, z) is a Pythagorean triple, then $\max(|x|, |y|, |z|) = |z|$. We define

$$N_{pyt}(B) = \big|\{(x, y, z) \in \mathbb{Z}^3; (x, y, z) \text{ Pythagorean}, |z| \leq B\}\big|.$$

Here we will solve the following problem:

Problem 2 (Exercise 4.3.5 of [21]). *Determine the asymptotic behavior of $N_{pyt}(B)$ as $B \to \infty$.*

This is nothing but counting the number of points of bounded height in $\mathbb{P}^2$ lying on the particular curve. We set

$$\mathcal{Z}_{pyt}(s) = \sum_{(x,y,z) \text{ Pythagorean}} \frac{1}{|z|^s}.$$

We now proceed to find an expression of this zeta function in terms of the standard zeta functions of arithmetic. The desired analytic properties of $\mathcal{Z}_H$ will then follow from standard results in analytic number theory. If $a, b \in \mathbb{Z}$ with $ab \neq 0$ and $\gcd(a, b) = 1$, we define

$$\delta(a, b) = \begin{cases} 2 & a \equiv b \equiv 1 \mod 2; \\ 1 & \text{otherwise.} \end{cases}$$

Then it is well-known that a parametrization of the rational points on the Pythagorean conic $x^2 + y^2 = z^2$ in $\mathbb{P}^2$ is given by

$$\begin{cases} x = \frac{a^2 - b^2}{\delta(a,b)} \\ y = \frac{2ab}{\delta(a,b)} \\ z = \frac{a^2 + b^2}{\delta(a,b)} \end{cases}$$

where (a, b) runs over co-prime pairs of integers with $ab \neq 0$. Here the pairs (a, b) and $(-a, -b)$ give the same points on the conic. This parametrization misses two points on the conic: $(-1:0:1)$ and $(1:0:1)$. We equip $\mathbb{P}^2$ with the maximum height, i.e. the height of a primitive triple of integers (A, B, C) is defined to be $\max(|A|, |B|, |C|)$. The height zeta function is

$$\mathcal{Z}_{pyt}(s) = 2 + \frac{1}{2} \sum_{\substack{ab \neq 0, \\ \gcd(a,b)=1}} \frac{1}{((a^2+b^2)/\delta(a,b))^s} = 2 + \frac{1}{2} S_1 + \frac{1}{2} S_2.$$

Here S_1 is the sum over those (a, b) with different parity, and S_2 is the sum over those (a, b) which are both odd. I claim that $S_1 = S_2$. This follows from the following trivial lemma:

Lemma 4.1. *If* (a, b) *is such that* $\gcd(a, b) = 1$ *and* $a \equiv b \equiv 1 \mod 2$, *then*

$$\frac{a+bi}{1+i} = \frac{b+a}{2} + i\frac{b-a}{2} \in \mathbb{Z}[i].$$

Furthermore, $\gcd(\frac{b+a}{2}, \frac{b-a}{2}) = 1$, *and the two numbers have different parity.*

It is also directly seen that

$$\frac{a^2+b^2}{2} = \left(\frac{b-a}{2}\right)^2 + \left(\frac{b+a}{2}\right)^2.$$

Consequently

$$\mathcal{Z}_{pyt}(s) = 2 + S_1.$$

Now we observe that the lemma also implies the following identity

$$(1 + 2^{-s}) S_1 = \sum_{\substack{ab \neq 0, \\ \gcd(a,b)=1}} \frac{1}{(a^2+b^2)^s}.$$

This means

$$\begin{aligned} \mathcal{Z}_{pyt}(s) &= 2 + \frac{1}{1+2^{-s}} \sum_{\substack{ab \neq 0, \\ \gcd(a,b)=1}} \frac{1}{(a^2+b^2)^s} \\ &= 2 + \frac{1}{(1+2^{-s})\zeta(2s)} \sum_{ab \neq 0} \frac{1}{(a^2+b^2)^s}. \end{aligned}$$

We now relate the sum on the right hand side of this last expression to the Dedekind zeta function of the quadratic extension $\mathbb{Q}(i)$. Fortunately, $\mathbb{Q}(i)$ has class number one, and every ideal in $\mathbb{Z}[i]$ is principal. Also

$(a+bi)=(c+di)$ if and only if $a+bi=\epsilon(c+di)$ with $\epsilon\in\{\pm1,\pm i\}$. It is then seen that

$$\zeta_{\mathbb{Q}(i)}(s)=\zeta(2s)+\frac{1}{4}\sum_{ab\neq 0}\frac{1}{(a^2+b^2)^s}.$$

Combining everything we obtain

$$\mathcal{Z}_{pyt}(s)=\frac{4}{(1+2^{-s})\zeta(2s)}\zeta_{\mathbb{Q}(i)}(s)-2\frac{1-2^{-s}}{1+2^{-s}}.$$

This has a simple pole at $s=1$ with computable residue. Also we observe that

$$\frac{1}{(1+2^{-s})\zeta(2s)}\zeta_{\mathbb{Q}(i)}(s)=\frac{(1-2^{-s})\zeta_{\mathbb{Q}(i)}(s)}{(1-2^{-2s})\zeta(2s)}$$

is the ratio of "2-adically corrected" zeta functions. We leave it to the reader to determine the exact asymptotic behavior of $N_{pyt}(B)$.

Remark 4.2. We note that the above computations remain valid for height function

$$H'(x,y,z)=\sqrt{x^2+y^2+z^2}$$

whenever (x,y,z) is a primitive integral triple. In fact if (x,y,z) is Pythagorean, then

$$H'(x,y,z)=\sqrt{2}H(x,y,z).$$

For other choices of the height function it is not obvious to me how one can adapt the above method to obtain the asymptotic behavior of rational points. For these more general height functions the desired result follows from [4, 5] as well as [8].

4.2. **Homogeneous Pell's equation.** Let D be a positive odd square-free integer, and consider the equation $X^2-DY^2=1$. Here we will consider the rational solutions of the homogenization of this equation:

$$x^2+Dy^2=z^2.$$

Again we will be interested in integral solutions (x,y,z), not all zero, satisfying $\gcd(x,y,z)=1$. We will call a triple of this type *Pellean.* As before, we define

$$N_{pel}(B)=\left|\{(x,y,z)\in\mathbb{Z}^3;(x,y,z)\text{ Pellean},|z|\leq B\}\right|,$$

and we pose the following problem:

Problem 3. *Determine the asymptotic behavior of $N_{pel}(B)$ as $B \to \infty$.*

A parametrization for primitive solutions of this equations is given by the following:

$$\begin{cases} x = (u^2 - Dv^2)/\delta(u,v) \\ y = 2uv/\delta(u,v) \\ z = (u^2 + Dv^2)/\delta(u,v). \end{cases} \tag{4.1}$$

Here we may take $\gcd(u,v) = 1$, and $\delta(u,v) = \gcd(u^2 - Dv^2, 2uv, u^2 + Dv^2)$. It is easy to see that whenever $\gcd(u,v) = 1$ and $uv \neq 0$ we have

$$\delta(u,v) = \begin{cases} 2\gcd(u,D) & u \equiv v \equiv 1 \mod 2; \\ \gcd(u,D) & \text{otherwise.} \end{cases}$$

As in the last section we use the Tauberian theorem. The height zeta function obtained is given by

$$\begin{aligned} \mathcal{Z}_D(s) =& 2 + \sum_{\substack{\gcd(u,v)=1 \\ uv\neq 0}} \frac{1}{\left(\frac{u^2+Dv^2}{\delta(u,v)}\right)^s} \\ =& 2 + 2^s \sum_{d|D} d^s \sum_{\substack{\gcd(u,v)=1, uv\neq 0, \\ \gcd(u,D)=d, u\equiv v(2)}} \frac{1}{(u^2+Dv^2)^s} \\ &+ \sum_{d|D} d^s \sum_{\substack{\gcd(u,v)=1, uv\neq 0, \\ \gcd(u,D)=d, u\not\equiv v(2)}} \frac{1}{(u^2+Dv^2)^s}. \end{aligned}$$

We will treat this expression by modifying Deuring's argument to treat the Epstein zeta function from [12].

4.2.1. *First series.* Set

$$\mathcal{Z}_1(s) = \sum_{\substack{\gcd(u,v)=1, uv\neq 0, \\ \gcd(u,D)=d, u\equiv v(2)}} \frac{1}{(u^2+Dv^2)^s}.$$

Then

$$
\begin{aligned}
(1-2^{-2s})\prod_{p|\frac{D}{d}}(1-p^{-2s})\zeta(2s)\mathcal{Z}_1(s)& \\
&= \sum_{\substack{uv\neq 0, \gcd(u,D)=d,\\ u\equiv v\equiv 1(2)}} \frac{1}{(u^2+Dv^2)^s} \\
&= \sum_{\substack{u\neq 0, \gcd(u,D)=d,\\ u\equiv 1(2)}} \sum_{l\in\mathbb{Z}} \frac{1}{(4Dl^2+4Dl+D+u^2)^s}.
\end{aligned}
$$

For reasons that will soon become apparent treating this modified expression is simpler. We now follow Deuring. We start with the following identity

$$
\sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \sum_{\nu=1}^{\infty}\int_a^b f(x)(e^{2\pi i\nu x}+e^{-2\pi i\nu x})\,dx
$$

valid for f with continuous second derivative. As in [12], this formula implies

$$
\begin{aligned}
\sum_{l\in\mathbb{Z}} \frac{1}{(4Dl^2+4Dl+D+u^2)^s} &= \int_{-\infty}^{\infty} \frac{1}{(4Dx^2+4Dx+D+u^2)^s}\,dx \\
&\quad + \sum_{\nu=1}^{\infty}\int_{-\infty}^{\infty} \frac{(e^{2\pi i\nu x}+e^{-2\pi i\nu x})}{(4Dx^2+4Dx+D+u^2)^s}\,dx.
\end{aligned}
$$

Now we have the following integral formula

$$
\int_{-\infty}^{\infty} \frac{dx}{(ax^2+bx+c)^s} = \pi^{\frac{1}{2}}\left(\frac{\Delta}{4}\right)^{-s+\frac{1}{2}} a^{s-1}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}
$$

with $\Delta = 4ac-b^2>0$ and $a>0$. We consequently get

$$
\begin{aligned}
\sum_{l\in\mathbb{Z}} \frac{1}{(4Dl^2+4Dl+D+u^2)^s} &= \Big(\frac{\pi}{4D}\Big)^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{1}{|u|^{2s-1}} \\
&\quad + \sum_{\nu=1}^{\infty}[\omega(s,\nu,u)+\omega(s,-\nu,u)]
\end{aligned}
$$

where $\omega(s,\nu,u)$ is defined by

$$
\omega(s,\nu,u) = \int_{-\infty}^{\infty} \frac{e^{2\pi i\nu x}}{(4Dx^2+4Dx+D+u^2)^s}\,dx, \qquad \Re s>0.
$$

By section 4 of [12], the series

$$R_1(s) = \sum_{\substack{u\neq 0, \gcd(u,D)=d,\\ u\equiv 1(2)}} \sum_{\nu=1}^{\infty} [\omega(s,\nu,u) + \omega(s,-\nu,u)]$$

is absolutely and uniformly convergent in any bounded s-region, and therefore represents an entire function. Consequently,

$$(1-2^{-2s})\prod_{p|\frac{D}{d}}(1-p^{-2s})\zeta(2s)\mathcal{Z}_1(s)$$
$$= \left(\frac{\pi}{4D}\right)^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{\substack{u\neq 0, \gcd(u,D)=d,\\ u\equiv 1(2)}} \frac{1}{|u|^{2s-1}} + R_1(s).$$

We now observe that since D is square-free, $\gcd(u,D)=d$ means $u=dw$ with $\gcd(w,\frac{D}{d})=1$. This then implies

$$\sum_{\substack{u\neq 0, \gcd(u,D)=d,\\ u\equiv 1(2)}} \frac{1}{|u|^{\sigma}} = 2(1-2^{-\sigma})\prod_{q|\frac{D}{d}}(1-q^{-\sigma})\frac{\zeta(\sigma)}{d^{\sigma}}$$

for $\Re\sigma > 1$. We get

$$\mathcal{Z}_1(s) = \left(\frac{\pi}{D}\right)^{\frac{1}{2}} d^{1-2s}\left(\frac{1-2^{1-2s}}{1-2^{-2s}}\right)\prod_{p|\frac{D}{d}}\left(\frac{1-p^{1-2s}}{1-p^{-2s}}\right)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}$$
$$+ \frac{R_1(s)}{(1-2^{-2s})\prod_{p|\frac{D}{d}}(1-p^{-2s})\zeta(2s)}.$$

4.2.2. *Second series.* We now consider

$$\mathcal{Z}_2(s) = \sum_{\substack{\gcd(u,v)=1, uv\neq 0,\\ \gcd(u,D)=d, 2|u, v \text{ odd}}} \frac{1}{(u^2+Dv^2)^s}.$$

We have

$$(1-2^{-2s})\prod_{p|\frac{D}{d}}(1-p^{-2s})\zeta(2s)\mathcal{Z}_2(s)$$

$$= \sum_{\substack{\gcd(u,v)=1,uv\neq 0,\\ \gcd(u,D)=d,2|u,v\text{ odd}}} \frac{1}{(u^2+Dv^2)^s}$$

$$= \sum_{\substack{u\neq 0,\\ \gcd(u,D)=d,2|u}} \sum_{l} \frac{1}{(4Dl^2+4Dl+D+u^2)^s}.$$

The same argument as before gives

$$(1-2^{-2s})\prod_{p|\frac{D}{d}}(1-p^{-2s})\zeta(2s)\mathcal{Z}_2(s)$$

$$= \left(\frac{\pi}{4D}\right)^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{\substack{u\neq 0,\gcd(u,D)=d,\\ 2|u}} \frac{1}{|u|^{2s-1}} + R_2(s)$$

where now

$$R_2(s) = \sum_{\substack{u\neq 0,\gcd(u,D)=d,\\ 2|u}} \sum_{\nu=1}^{\infty}[\omega(s,\nu,u)+\omega(s,-\nu,u)].$$

This gives

$$\mathcal{Z}_2(s) = \left(\frac{\pi}{D}\right)^{\frac{1}{2}} 2^{1-2s}d^{1-2s}\left(\frac{1}{1-2^{-2s}}\right)\prod_{p|\frac{D}{d}}\left(\frac{1-p^{1-2s}}{1-p^{-2s}}\right)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}$$
$$+ \frac{R_2(s)}{(1-2^{-2s})\prod_{p|\frac{D}{d}}(1-p^{-2s})\zeta(2s)}.$$

4.2.3. *Third series.* Finally we consider

$$\mathcal{Z}_3(s) = \sum_{\substack{\gcd(u,v)=1,uv\neq 0,\\ \gcd(u,D)=d,u\text{ odd},2|v}} \frac{1}{(u^2+Dv^2)^s}.$$

We have

$$(1-2^{-2s})\prod_{p|\frac{D}{d}}(1-p^{-2s})\zeta(2s)\mathcal{Z}_3(s) = \sum_{\substack{u\neq 0,\\ \gcd(u,D)=d,u\text{ odd}}} \sum_{l} \frac{1}{(4Dl^2+u^2)^s}.$$

An identical argument gives

$$(1-2^{-2s})\prod_{p|\frac{D}{d}}(1-p^{-2s})\zeta(2s)\mathcal{Z}_3(s)$$
$$=\left(\frac{\pi}{4D}\right)^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{\substack{u\neq 0,\gcd(u,D)=d,\\ u\equiv 1(2)}}\frac{1}{|u|^{2s-1}}+R_3(s).$$

Here

$$R_3(s)=\sum_{\nu}[\omega_1(s,\nu,u)+\omega_1(s,-\nu,u)]$$

with

$$\omega_1(s,\nu,u)=\int_{-\infty}^{\infty}\frac{e^{2\pi i\nu x}}{(4Dx^2+u^2)^s}dx.$$

This gives

$$\mathcal{Z}_3(s)=\left(\frac{\pi}{D}\right)^{\frac{1}{2}}d^{1-2s}\left(\frac{1-2^{1-2s}}{1-2^{-2s}}\right)\prod_{p|\frac{D}{d}}\left(\frac{1-p^{1-2s}}{1-p^{-2s}}\right)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}$$
$$+\frac{R_3(s)}{(1-2^{-2s})\prod_{p|\frac{D}{d}}(1-p^{-2s})\zeta(2s)}.$$

4.2.4. *Going back to $\mathcal{Z}_D(s)$.* Recall that

$$\mathcal{Z}_D(s)=2+2^s\sum_{d|D}d^s\sum_{\substack{\gcd(u,v)=1,uv\neq 0,\\ \gcd(u,D)=d,u\equiv v(2)}}\frac{1}{(u^2+Dv^2)^s}$$
$$+\sum_{d|D}d^s\sum_{\substack{\gcd(u,v)=1,uv\neq 0,\\ \gcd(u,D)=d,u\not\equiv v(2)}}\frac{1}{(u^2+Dv^2)^s}.$$

What we have established so far says

$$\mathcal{Z}_D(s)=\left(\frac{\pi}{D}\right)^{\frac{1}{2}}\left(\frac{2^s-2^{1-s}+1}{1-2^{-2s}}\right)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}$$
$$\times\sum_{d|D}d^{1-s}\prod_{p|\frac{D}{d}}\left(\frac{1-p^{1-2s}}{1-p^{-2s}}\right)+R(s)$$

with $R(s)$ analytic in $\Re s \geq \frac{1}{2}$. The rest of the expression has a simple pole at $s=1$ with residue equal to

$$\begin{aligned}\frac{8}{\pi\sqrt{D}}\sum_{d|D}\prod_{p|\frac{D}{d}}\frac{1}{1+p^{-1}} &= \frac{8}{\pi\sqrt{D}}\sum_{d|D}\prod_{p|d}\frac{1}{1+p^{-1}}\\ &= \frac{8}{\pi\sqrt{D}}\prod_{p|D}(1+\frac{1}{1+p^{-1}})\\ &= \frac{8}{\pi\sqrt{D}}\prod_{p|D}\frac{2+p^{-1}}{1+p^{-1}}.\end{aligned}$$

We have then established the following theorem:

Theorem 4.3. *Let D be a positive odd square-free integer. Then*

$$N_{pel}(B) = \left(\frac{8}{\pi\sqrt{D}}\prod_{p|D}\frac{2+p^{-1}}{1+p^{-1}}\right)B + o(B)$$

as $B \to \infty$.

Remark 4.4. By using the estimates obtained in Deuring's paper one can also give pretty sharp error estimates for the asymptotic formula obtained above.

Remark 4.5. It is not obvious to me how one can use the above method to deduce the asymptotic behavior of rational points for other choices of the height function. Again Theorem 4.3 and its variants for other choices of the height function follow from [4, 5] and [8].

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