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RICCI TENSOR FOR GCR-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

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ABSTRACT. We obtain the expression of Ricci tensor for a GCR-lightlike submanifold of indefinite complex space form and discuss its properties on a totally geodesic GCR-lightlike submanifold of an indefinite complex space form. Moreover, we prove that every proper totally umbilical GCR-lightlike submanifold of an indefinite Kaehler manifold is a totally geodesic GCR-lightlike submanifold.

1. Introduction

The geometry of CR-submanifolds of Kaehler manifolds was initiated by Bejancu [2], as a generalization of complex and totally real submanifolds of Kaehler manifolds and has been further developed by many others [3–6, 12, 13]. Due to growing importance of lightlike geometry in mathematical physics and relativity, Duggal and Bejancu [7], introduced the notion of CR-lightlike submanifolds of indefinite Kaehler manifolds, which does not include complex and totally real subcases. Then Duggal and Sahin [9], obtained a new class of submanifolds, namely, SCR-lightlike submanifolds of indefinite Kaehler manifolds but there was no inclusion relation between CR and SCR-cases. Later on, Duggal and Sahin [10], introduced GCR-lightlike submanifolds of indefinite

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Kaehler manifolds. This class of submanifolds is an umbrella over real hypersurfaces, invariant, screen real and CR-lightlike submanifolds. In this paper, we obtain the expression of Ricci tensor for a GCR-lightlike submanifold of indefinite complex space form and discuss its properties for a totally geodesic GCR-lightlike submanifold of an indefinite complex space form. We also prove that every proper totally umbilical GCR-lightlike submanifold of an indefinite Kaehler manifold is a totally geodesic GCR-lightlike submanifold.

2. Lightlike submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If g is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$, (for detail see [7]). For a degenerate metric g on M

$$T_xM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\}, \tag{2.1}$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$, which is known as radical (null) subspace. If the mapping

$$RadTM : x \in M \longrightarrow RadT_xM, \tag{2.2}$$

defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M.

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is

$$TM = RadTM \oplus S(TM), \tag{2.3}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M} \mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.4}$$

$$T\bar{M} \mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.5}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\}$, where $\{\xi_1, ..., \xi_r\}$, $\{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(RadTM\mid_u)$, $\Gamma(ltr(TM)\mid_u)$, respectively and $\{W_{r+1}, ..., W_n\}, \{X_{r+1}, ..., X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$, $\Gamma(S(TM)\mid_u)$, respectively. For this quasi-orthonormal fields of frames, one has

Theorem 2.1. *([7]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathrm{u}}$, where* u *is a coordinate neighborhood of M such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \textit{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.6}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ then, using the decomposition of $T\bar{M}\mid_M$ in (2.5), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall \quad X, Y \in \Gamma(TM), \tag{2.7}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \quad \forall \quad X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{2.8}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$, which is called second fundamental form, A_U is a linear operator on M, known as shape operator.

Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then (2.7) and (2.8) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.9}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.10}$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.12}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2.9)-(2.12) we obtain

$$\bar{g}(h^s(X,Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.13}$$

$$\bar{g}(h^l(X,Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.14}$$

$$\bar{g}(D^s(X,N), W) = \bar{g}(N, A_W X), \tag{2.15}$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0, \tag{2.16}$$

for any $\xi \in \Gamma(RadTM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(ltr(TM))$.

Let $\bar{P}$ be the projection morphism of TM on $S(TM)$, then using (2.3), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{2.17}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.18}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X \bar{P}Y, A^*_\xi X\}$ and $\{h^*(X, \bar{P}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. h^* and A^* are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms, known as the second fundamental forms of distributions $S(TM)$ and $RadTM$, respectively.

Using (2.9), (2.10), (2.17) and (2.18), the following relations hold good.

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \tag{2.19}$$

$$\bar{g}(h^*(X, \bar{P}Y), N) = \bar{g}(A_N X, \bar{P}Y), \tag{2.20}$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇, respectively, then by straightforward calculations ([7]), we have

$$\begin{aligned}\bar{R}(X,Y)Z \;&= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ &\quad -A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ &\quad +D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ &\quad -(\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)),\end{aligned} \tag{2.21}$$

where

$$
\begin{aligned}
(\nabla_X h^s)(Y,Z) &= \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y, Z) \\
&\quad - h^s(Y, \nabla_X Z),
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
(\nabla_X h^l)(Y,Z) &= \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y, Z) \\
&\quad - h^l(Y, \nabla_X Z).
\end{aligned} \tag{2.23}
$$

Let $\bar{M}(c)$ be a complex space form of constant holomorphic curvature c, then the curvature tensor $\bar{R}$ is given by

$$
\begin{aligned}
\bar{R}(X,Y)Z &= \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX \\
&\quad - \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\},
\end{aligned} \tag{2.24}
$$

for X, Y, Z vector fields on $\bar{M}$.

Using (2.24) and (2.21), we obtain

$$
\begin{aligned}
g(R(X,Y)Z,W) &= \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\
&\quad + g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) \\
&\quad + 2g(X,JY)g(JZ,W)\} - g(A_{h^l(X,Z)}Y,W) \\
&\quad + g(A_{h^l(Y,Z)}X,W) - g(A_{h^s(X,Z)}Y,W) \\
&\quad + g(A_{h^s(Y,Z)}X,W) - \bar{g}((\nabla_X h^l)(Y,Z),W) \\
&\quad + \bar{g}((\nabla_Y h^l)(X,Z),W) - \bar{g}(D^l(X,h^s(Y,Z)),W) \\
&\quad + \bar{g}(D^l(Y,h^s(X,Z)),W).
\end{aligned} \tag{2.25}
$$

Definition 2.2. *([1]): Let $(\bar{M}, J, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold if J is parallel with respect to $\bar{\nabla}$, that is*

$$
(\bar{\nabla}_X J)Y = 0, \quad \forall \quad X, Y \in \Gamma(T\bar{M}). \tag{2.26}
$$

3. **Generalized Cauchy-Riemann lightlike submanifolds**

Definition 3.1. *([10]). Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, J)$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied*

(A) *There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that*

$$
Rad(TM) = D_1 \oplus D_2, \quad J(D_1) = D_1, \quad J(D_2) \subset S(TM). \tag{3.1}
$$

(B) *There exist two subbundles* D_0 *and* D' *of* $S(TM)$ *such that*

$$S(TM) = \{JD_2 \oplus D'\} \perp D_0, \quad J(D_0) = D_0, \quad J(D') = L_1 \perp L_2, \tag{3.2}$$

where D_0 *is a non-degenerate distribution on* M, L_1 *and* L_2 *are vector subbundle of* $ltr(TM)$ *and* $S(TM)^{\perp}$, *respectively.*

Then the tangent bundle TM of M is decomposed as

$$TM = D \oplus D', \quad \text{where} \quad D = Rad(TM) \oplus D_0 \oplus JD_2. \tag{3.3}$$

M is called a proper GCR-lightlike submanifold if $D_1 \neq \{0\}, D_2 \neq \{0\}, D_0 \neq \{0\}$ and $L_2 \neq \{0\}$.

Let Q, P_1 and P_2 be the projections on D, $J(L_1) = M_1 \subset D'$ and $J(L_2) = M_2 \subset D'$, respectively. Then for any $X \in \Gamma(TM)$, we have

$$X = QX + P_1X + P_2X = QX + PX, \tag{3.4}$$

applying J to (3.4), we obtain

$$JX = TX + wP_1X + wP_2X = TX + wX, \tag{3.5}$$

where TX and wX are the tangential and transversal components of JX, respectively.
Similarly

$$JV = BV + CV, \tag{3.6}$$

for any $V \in \Gamma(tr(TM))$, where BV and CV are the sections of TM and $tr(TM)$, respectively. Differentiating (3.5) and using (2.9)-(2.12) and (3.6), we have

(3.7)
$$D^s(X, wP_1Y) = -\nabla^s_X wP_2Y + wP_2\nabla_X Y - h^s(X, TY) + Ch^s(X, Y),$$

$$D^l(X, wP_2Y) = -\nabla^l_X wP_1Y + wP_1\nabla_X Y - h^l(X, TY) + Ch^l(X, Y). \tag{3.8}$$

Using Kaehlerian property of $\bar{\nabla}$ with (2.11) and (2.12), we have the following lemmas.

Lemma 3.2. *Let* M *be a GCR-lightlike submanifold of an indefinite Kaehler manifold* $\bar{M}$. *Then we have*

$$(\nabla_X T)Y = A_{wY}X + Bh(X, Y) \tag{3.9}$$

and

$$(\nabla^t_X w)Y = Ch(X, Y) - h(X, TY), \tag{3.10}$$

where $X, Y \in \Gamma(TM)$ *and*

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{3.11}$$

$$(\nabla^t_X w)Y = \nabla^t_X wY - w\nabla_X Y. \tag{3.12}$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then we have*

$$(\nabla_X B)V = A_{CV}X - TA_V X \tag{3.13}$$

and

$$(\nabla^t_X C)V = -wA_V X - h(X, BV), \tag{3.14}$$

where $X \in \Gamma(TM)$, $V \in \Gamma(tr(TM))$ and

$$(\nabla_X B)V = \nabla_X BV - B\nabla^t_X V, \tag{3.15}$$

$$(\nabla^t_X C)V = \nabla^t_X CV - C\nabla^t_X V. \tag{3.16}$$

4. Ricci tensor of GCR-lightlike Submanifold of an indefinite complex space form

Let $\{E_1, E_2, ..., E_m\}$ be a local orthonormal frame field on M such that $\{E_1, E_2, ..., E_p, E_{p+1} = JE_1, E_{p+2} = JE_2, ..., E_{2p} = JE_p\}$, $\{\xi_1, \xi_2, ..., \xi_s, \xi_{s+1} = J\xi_1, \xi_{s+2} = J\xi_2, ..., \xi_{2s} = J\xi_s\}$, $\{\xi_{2s+1}, \xi_{2s+2}, ..., \xi_r\}$ and $\{J\xi_{2s+1}, J\xi_{2s+2}, ..., J\xi_r\}$ be local frame fields on D_0, D_1, D_2 and JD_2, respectively and $\{F_1, F_2, ..., F_q\}$ be a local frame field on D', then by direct computation, we have

$$\sum_{i=1}^{m} g(U, E_i)g(E_i, V) = g(U, V), \tag{4.1}$$

$$\sum_{i=r+1}^{m} g(U, E_i)g(E_i, V) = g(\bar{P}U, \bar{P}V), \tag{4.2}$$

$$\sum_{i=1}^{m-q} g(U, E_i)g(E_i, V) = g(QU, QV) \tag{4.3}$$

and

$$\sum_{i=1}^{q} g(U, E_i)g(E_i, V) = g(PU, PV), \tag{4.4}$$

for any $U, V \in \Gamma(TM)$ and the Ricci tensor is given by

$$Ric(U, V) = \sum_{a=1}^{r} g(R(U, \xi_a)V, N_a) + \sum_{b=r+1}^{m} g(R(U, U_b)V, U_b). \tag{4.5}$$

Using (2.25), we obtain

$$
\begin{aligned}
\sum_{a=1}^{r} g(R(U,\xi_a)V,N_a) &= -\frac{c}{4}\sum_{a=1}^{r} g(JV,\xi_a)g(JU,N_a) - \frac{cr}{4}g(U,V)\\
&-\frac{c}{4}\sum_{a=1}^{r} g(JU,V)g(J\xi_a,N_a)\\
&-\frac{c}{2}\sum_{a=1}^{r} g(JU,\xi_a)g(JV,N_a)\\
&+\sum_{a=1}^{r} g(A_{h^l(\xi_a,V)}U,N_a) - \sum_{a=1}^{r} g(A_{h^l(U,V)}\xi_a,N_a)\\
&+\sum_{a=1}^{r} g(A_{h^s(\xi_a,V)}U,N_a) - \sum_{a=1}^{r} g(A_{h^s(U,V)}\xi_a,N_a). \qquad (4.6)
\end{aligned}
$$

Now, using equation (2.30) of [7] at page 158, for any $U \in \Gamma(T(M))$, define a differential 1-form as

$$\eta_a(U) = \bar{g}(U,N_a), \forall a \in \{1,2,...,r\},$$

then any vector field U on M can be expressed as

$$U = \bar{P}U + \sum_{a=1}^{r} \eta_a(U)\xi_a, \qquad (4.7)$$

where $\bar{P}$ is the projection morphism of TM on $S(TM)$. Therefore, we have

$$g(U,JV) = g(\bar{P}U,JV) + \sum_{a=1}^{r} \bar{g}(U,N_a)g(\xi_a,JV). \qquad (4.8)$$

Also, using (2.15), (2.16) and (4.8) in (4.6), we obtain

$$\begin{aligned}\sum_{a=1}^{r} g(R(U,\xi_a)V,N_a) \quad = \quad & -\frac{c(r+3)}{4}g(U,V)+\frac{3c}{4}g(TU,TV) \\ & -\frac{c}{4}\sum_{a=1}^{r} g(JU,V)g(J\xi_a,N_a) \\ & -\sum_{a=1}^{r} g(A_{N_a}U,h^l(\xi_a,V)) \\ & +\sum_{a=1}^{r} g(A_{N_a}\xi_a,h^l(U,V)) \\ & +\sum_{a=1}^{r} g(D^s(U,N_a),h^s(\xi_a,V)) \\ & -\sum_{a=1}^{r} g(D^s(\xi_a,N_a),h^s(U,V)). \end{aligned} \tag{4.9}$$

Using (2.13), (2.20), (2.25) and (4.2), we obtain

$$\begin{aligned}\sum_{b=r+1}^{m} g(R(U,U_b)V,U_b) \quad = \quad & -\frac{c}{2}g(\bar{P}U,\bar{P}V)-\frac{(m-r)c}{4}g(U,V) \\ & +\sum_{b=r+1}^{m}\{g(h^l(U_b,V),h^*(U,U_b)) \\ & -g(h^l(U,V),h^*(U_b,U_b)) \\ & +g(h^s(U_b,V),h^s(U,U_b)) \\ & -g(h^s(U,V),h^s(U_b,U_b))\}. \end{aligned} \tag{4.10}$$

Thus subsituting (4.9) and (4.10) in (4.5), we obtain the expression of Ricci tensor of a GCR-lightlike submanifold as

$$\begin{aligned} Ric(U,V) &= -\frac{(m+3)c}{4}g(U,V) + \frac{3c}{4}g(TU,TV) - \frac{c}{2}g(\bar{P}U,\bar{P}V) \\ &-\frac{c}{4}\sum_{a=1}^{r} g(JU,V)g(J\xi_a,N_a) + \sum_{a=1}^{r} g(A_{N_a}\xi_a, h^l(U,V)) \\ &-\sum_{a=1}^{r} g(A_{N_a}U, h^l(\xi_a,V)) + \sum_{a=1}^{r} g(D^s(U,N_a), h^s(\xi_a,V)) \\ &-\sum_{a=1}^{r} g(D^s(\xi_a,N_a), h^s(U,V)) - \sum_{b=r+1}^{m} g(h^l(U,V), h^*(U_b,U_b)) \\ &+\sum_{b=r+1}^{m} g(h^l(U_b,V), h^*(U,U_b)) - \sum_{b=r+1}^{m} g(h^s(U,V), h^s(U_b,U_b)) \\ &+\sum_{b=r+1}^{m} g(h^s(U_b,V), h^s(U,U_b)). \end{aligned} \tag{4.11}$$

Next, by using an orthonormal frame fields on D', D_0, JD_2 and $Rad(TM)$, we can also express the Ricci tensors as

$$\begin{aligned} Ric(U,V) &= \sum_{i=1}^{q} g(R(U,F_i)V,F_i) + \sum_{k=1}^{2p} g(R(U,E_k)V,E_k) \\ &+ \sum_{l=2s+1}^{r} g(R(U,J\xi_l)V,JN_l) + \sum_{a=1}^{r} g(R(U,\xi_a)V,N_a), \end{aligned} \tag{4.12}$$

therefore, we have

$$\begin{aligned} Ric_D(U,V) &= \sum_{k=1}^{2p} g(R(U,E_k)V,E_k) + \sum_{l=2s+1}^{r} g(R(U,J\xi_l)V,JN_l) \\ &+ \sum_{a=1}^{r} g(R(U,\xi_a)V,N_a), \end{aligned} \tag{4.13}$$

$$Ric_{D'}(U,V) = \sum_{i=1}^{q} g(R(U,F_i)V,F_i). \tag{4.14}$$

Using (2.15), (2.20), (2.25) and (4.3), we obtain

$$\begin{aligned}\sum_{k=1}^{2p} g(R(U,E_k)V,E_k) &= -\frac{c}{2}g(QU,QV)-\frac{pc}{2}g(U,V)\\ &+\sum_{k=1}^{2p} g(h^l(E_k,V),h^*(U,E_k))\\ &-\sum_{k=1}^{2p} g(h^l(U,V),h^*(E_k,E_k))\\ &+\sum_{k=1}^{2p} g(h^s(E_k,V),h^s(U,E_k))\\ &-\sum_{k=1}^{2p} g(h^s(U,V),h^s(E_k,E_k)) \qquad (4.15)\end{aligned}$$

and

$$\begin{aligned}\sum_{l=2s+1}^{r} g(R(U,J\xi_l)V,JN_l) &= -\frac{c(r-2s)}{4}g(U,V)\\ &+\frac{c}{4}\sum_{l=2s+1}^{r} g(J\xi_l,V)g(U,JN_l)\\ &-\sum_{l=2s+1}^{r} g(A_{h^l(U,V)}J\xi_l,JN_l)\\ &+\sum_{l=2s+1}^{r} g(A_{h^l(J\xi_l,V)}U,JN_l)\\ &-\sum_{l=2s+1}^{r} g(A_{h^s(U,V)}J\xi_l,JN_l)\\ &+\sum_{l=2s+1}^{r} g(A_{h^s(J\xi_l,V)}U,JN_l). \qquad (4.16)\end{aligned}$$

Using (4.9), (4.15) and (4.16) in (4.13), we obtain

$$
\begin{aligned}
Ric_D(U,V)=&\frac{3c}{4}g(TU,TV)-\frac{c(2p+2r-2s+3)}{4}g(U,V)-\frac{c}{2}g(QU,QV)\\
&-\frac{c}{4}\sum_{a=1}^{r}g(JU,V)g(J\xi_a,N_a)+\frac{c}{4}\sum_{l=2s+1}^{r}g(J\xi_l,V)g(U,JN_l)\\
&+\sum_{a=1}^{r}g(A_{N_a}\xi_a,h^l(U,V))-\sum_{a=1}^{r}g(A_{N_a}U,h^l(\xi_a,V))\\
&-\sum_{a=1}^{r}g(D^s(\xi_a,N_a),h^s(U,V))+\sum_{a=1}^{r}g(D^s(U,N_a),h^s(\xi_a,V))\\
&+\sum_{k=1}^{2p}g(h^l(E_k,V),h^*(U,E_k))-\sum_{k=1}^{2p}g(h^l(U,V),h^*(E_k,E_k))\\
&+\sum_{k=1}^{2p}g(h^s(E_k,V),h^s(U,E_k))-\sum_{k=1}^{2p}g(h^s(U,V),h^s(E_k,E_k))\\
&-\sum_{l=2s+1}^{r}g(A_{h^l(U,V)}J\xi_l,JN_l)+\sum_{l=2s+1}^{r}g(A_{h^l(J\xi_l,V)}U,JN_l)\\
&-\sum_{l=2s+1}^{r}g(A_{h^s(U,V)}J\xi_l,JN_l)+\sum_{l=2s+1}^{r}g(A_{h^s(J\xi_l,V)}U,JN_l). \qquad (4.17)
\end{aligned}
$$

Also using (2.13), (2.20), (2.25) and (4.4), we obtain

$$
\begin{aligned}
Ric_{D'}(U,V)=&-\frac{c}{2}g(PU,PV)-\frac{qc}{4}g(U,V)-\sum_{i=1}^{q}g(h^l(U,V),h^*(F_i,F_i))\\
&+\sum_{i=1}^{q}g(h^l(F_i,V),h^*(U,F_i))-\sum_{i=1}^{q}g(h^s(F_i,F_i),h^s(U,V))\\
&+\sum_{i=1}^{q}g(h^s(F_i,V),h^s(U,F_i)). \qquad (4.18)
\end{aligned}
$$

Let $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D')$, then particularly, we have

$$
\begin{aligned}
Ric_{D'}(X,Y) = & -\frac{qc}{4} g(X,Y) - \sum_{i=1}^{q} g(h^l(X,Y), h^*(F_i,F_i)) \\
& + \sum_{i=1}^{q} g(h^l(F_i,Y), h^*(X,F_i)) - \sum_{i=1}^{q} g(h^s(F_i,F_i), h^s(X,Y)) \\
& + \sum_{i=1}^{q} g(h^s(F_i,Y), h^s(X,F_i)), \qquad (4.19)
\end{aligned}
$$

$$
\begin{aligned}
Ric_D(X,Y) = & \frac{c(s-r-p-1)}{2} g(X,Y) - \frac{c}{4} \sum_{a=1}^{r} g(JX,Y) g(J\xi_a, N_a) \\
& + \sum_{a=1}^{r} g(A_{N_a}\xi_a, h^l(X,Y)) - \sum_{a=1}^{r} g(A_{N_a}X, h^l(\xi_a,Y)) \\
& - \sum_{a=1}^{r} g(D^s(\xi_a,N_a), h^s(X,Y)) + \sum_{a=1}^{r} g(D^s(X,N_a), h^s(\xi_a,Y)) \\
& + \sum_{k=1}^{2p} g(h^l(E_k,Y), h^*(X,E_k)) - \sum_{k=1}^{2p} g(h^l(X,Y), h^*(E_k,E_k)) \\
& + \sum_{k=1}^{2p} g(h^s(E_k,Y), h^s(X,E_k)) - \sum_{k=1}^{2p} g(h^s(X,Y), h^s(E_k,E_k)) \\
& - \sum_{l=2s+1}^{r} g(A_{h^l(X,Y)}J\xi_l, JN_l) + \sum_{l=2s+1}^{r} g(A_{h^l(J\xi_l,Y)}X, JN_l) \\
& - \sum_{l=2s+1}^{r} g(A_{h^s(X,Y)}J\xi_l, JN_l) + \sum_{l=2s+1}^{r} g(A_{h^s(J\xi_l,Y)}X, JN_l), \qquad (4.20)
\end{aligned}
$$

$$
\begin{aligned}
Ric_{D'}(X,Z) = & \sum_{i=1}^{q} g(h^l(X,Z), h^*(F_i,F_i)) + \sum_{i=1}^{q} g(h^l(F_i,Z), h^*(X,F_i)) \\
& - \sum_{i=1}^{q} g(h^s(F_i,F_i), h^s(X,Z)) + \sum_{i=1}^{q} g(h^s(F_i,Z), h^s(X,F_i)), \qquad (4.21)
\end{aligned}
$$

$$\begin{aligned}
Ric_D(X,Z)=&\frac{c}{4}\sum_{l=2s+1}^{r} g(J\xi_l,Z)g(X,JN_l)+\sum_{a=1}^{r} g(A_{N_a}\xi_a,h^l(X,Z))\\
&-\sum_{a=1}^{r} g(A_{N_a}X,h^l(\xi_a,Z))-\sum_{a=1}^{r} g(D^s(\xi_a,N_a),h^s(X,Z))\\
&+\sum_{a=1}^{r} g(D^s(X,N_a),h^s(\xi_a,Z))+\sum_{k=1}^{2p} g(h^l(E_k,Z),h^*(X,E_k))\\
&-\sum_{k=1}^{2p} g(h^l(X,Z),h^*(E_k,E_k))+\sum_{k=1}^{2p} g(h^s(E_k,Z),h^s(X,E_k))\\
&-\sum_{k=1}^{2p} g(h^s(X,Z),h^s(E_k,E_k))-\sum_{l=2s+1}^{r} g(A_{h^l(X,Z)}J\xi_l,JN_l)\\
&+\sum_{l=2s+1}^{r} g(A_{h^l(J\xi_l,Z)}X,JN_l)-\sum_{l=2s+1}^{r} g(A_{h^s(X,Z)}J\xi_l,JN_l)\\
&+\sum_{l=2s+1}^{r} g(A_{h^s(J\xi_l,Z)}X,JN_l) \qquad (4.22)
\end{aligned}$$

and

$$\begin{aligned}
Ric_{D'}(Z,W)=&-\frac{(q+2)c}{4}g(Z,W)-\sum_{i=1}^{q} g(h^l(Z,W),h^*(F_i,F_i))\\
&+\sum_{i=1}^{q} g(h^l(F_i,W),h^*(Z,F_i))-\sum_{i=1}^{q} g(h^s(F_i,F_i),h^s(Z,W))\\
&+\sum_{i=1}^{q} g(h^s(F_i,W),h^s(Z,F_i)). \qquad (4.23)
\end{aligned}$$

$$\begin{aligned}
Ric_D(Z,W) =& -\frac{c(2p+2r-2s+3)}{4}g(Z,W) \\
&+\frac{c}{4}\sum_{l=2s+1}^{r} g(J\xi_l,W)g(Z,JN_l)+\sum_{a=1}^{r} g(A_{N_a}\xi_a,h^l(Z,W)) \\
&-\sum_{a=1}^{r} g(A_{N_a}Z,h^l(\xi_a,W))-\sum_{a=1}^{r} g(D^s(\xi_a,N_a),h^s(Z,W)) \\
&+\sum_{a=1}^{r} g(D^s(Z,N_a),h^s(\xi_a,W))+\sum_{k=1}^{2p} g(h^l(E_k,W),h^*(Z,E_k)) \\
&-\sum_{k=1}^{2p} g(h^l(Z,W),h^*(E_k,E_k)) \\
&+\sum_{k=1}^{2p} g(h^s(E_k,W),h^s(Z,E_k))-\sum_{k=1}^{2p} g(h^s(Z,W),h^s(E_k,E_k)) \\
&-\sum_{l=2s+1}^{r} g(A_{h^l(Z,W)}J\xi_l,JN_l)+\sum_{l=2s+1}^{r} g(A_{h^l(J\xi_l,W)}Z,JN_l) \\
&-\sum_{l=2s+1}^{r} g(A_{h^s(Z,W)}J\xi_l,JN_l)+\sum_{l=2s+1}^{r} g(A_{h^s(J\xi_l,W)}Z,JN_l), \qquad (4.24)
\end{aligned}$$

Definition 4.1. *A GCR-lightlike submanifold of an indefinite Kaehler manifold is called:*

(i) *Totally geodesic GCR-lightlike submanifold if its second fundamental form h vanishes, that is, $h(X,Y)=0$, for any $X,Y\in\Gamma(TM)$.*
(ii) *D-geodesic GCR-lightlike submanifold if $h(X,Y)=0$, for any $X,Y\in\Gamma(D)$.*
(iii) *D'-geodesic GCR-lightlike submanifold $h(X,Y)=0$, for any $X,Y\in\Gamma(D')$.*
(iv) *Mixed-geodesic GCR-lightlike submanifold if $h(X,Y)=0$, for any $X\in\Gamma(D)$ and $Y\in\Gamma(D')$.*

Thus from (4.19) to (4.24), we have the following results.

Theorem 4.2. *Let M be a totally geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $X,Y\in\Gamma(D)$*

and $Z, W \in \Gamma(D')$

$$Ric_D(X,Y) = \frac{c(s-r-p-1)}{2}g(X,Y) - \frac{c}{4}\sum_{a=1}^{r} g(JX,Y)g(J\xi_a, N_a),$$

$$Ric_D(X,Z) = \frac{c}{4}\sum_{l=2s+1}^{r} g(J\xi_l, Z)g(X, JN_l),$$

$$\begin{aligned} Ric_D(Z,W) &= -\frac{c(2p+2r-2s+3)}{4}g(Z,W) \\ &\quad + \frac{c}{4}\sum_{l=2s+1}^{r} g(J\xi_l, W)g(Z, JN_l) \end{aligned}$$

and

$$Ric_{D'}(X,Y) = -\frac{qc}{4}g(X,Y),$$

$$Ric_{D'}(X,Z) = 0,$$

$$Ric_{D'}(Z,W) = -\frac{(q+2)c}{4}g(Z,W).$$

Theorem 4.3. *Let M be a D-geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $X, Y \in \Gamma(D)$*

$$Ric_D(X,Y) = \frac{c(s-r-p-1)}{2}g(X,Y) - \frac{c}{4}\sum_{a=1}^{r} g(JX,Y)g(J\xi_a, N_a)$$

and

$$\begin{aligned} Ric_{D'}(X,Y) &= -\frac{qc}{4}g(X,Y) + \sum_{i=1}^{q} g(h^l(F_i,Y), h^*(X,F_i)) \\ &\quad + \sum_{i=1}^{q} g(h^s(F_i,Y), h^s(X,F_i)). \end{aligned}$$

Theorem 4.4. *Let M be a D'-geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $Z, W \in \Gamma(D')$*

$$Ric_D(Z,W) = -\frac{c(2p+2r-2s+3)}{4}g(Z,W) + \frac{c}{4}\sum_{l=2s+1}^{r} g(J\xi_l, W)g(Z, JN_l)$$
$$- \sum_{a=1}^{r} g(A_{N_a}Z, h^l(\xi_a, W)) + \sum_{a=1}^{r} g(D^s(Z, N_a), h^s(\xi_a, W))$$
$$+ \sum_{k=1}^{2p} g(h^l(E_k, W), h^*(Z, E_k)) + \sum_{k=1}^{2p} g(h^s(E_k, W), h^s(Z, E_k))$$
$$+ \sum_{l=2s+1}^{r} g(A_{h^l(J\xi_l, W)}Z, JN_l) + \sum_{l=2s+1}^{r} g(A_{h^s(J\xi_l, W)}Z, JN_l)$$

and

$$Ric_{D'}(Z,W) = -\frac{(q+2)c}{4}g(Z,W) + \sum_{i=1}^{q} g(h^l(F_i, W), h^*(Z, F_i)) + \sum_{i=1}^{q} g(h^s(F_i, W), h^s(Z, F_i)).$$

Theorem 4.5. *Let M be a mixed-geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $X \in \Gamma(D)$ and $Z \in \Gamma(D')$*

$$Ric_D(X,Z) = \frac{c}{4}\sum_{l=2s+1}^{r} g(J\xi_l, Z)g(X, JN_l)$$

and

$$Ric_{D'}(X,Z) = \sum_{i=1}^{q} g(h^l(F_i, Z), h^*(X, F_i)) + \sum_{i=1}^{q} g(h^s(F_i, Z), h^s(X, F_i)).$$

Definition 4.6. *([8]). A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a totally umbilical in $\bar{M}$, if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that, for $X, Y \in \Gamma(TM)$*

$$h(X,Y) = H\bar{g}(X,Y). \tag{4.25}$$

Using (2.9), it is clear that M is totally umbilic, if and only if, on each coordinate neighborhood u there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that

$$h^l(X,Y) = H^l \quad g(X,Y), \quad h^s(X,Y) = H^s g(X,Y),$$
$$D^l(X,W) = 0, \tag{4.26}$$

for $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$.

Theorem 4.7. *Let M be a proper totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $H^l = 0$.*

Proof. Using (2.9), (2.10) and (2.26) with the hypothesis that M is a totally umbilic GCR-lightlike submanifold and then taking tangential parts of the resulting equation, we obtain

$$A_{wZ}Z + T\nabla_Z Z + Bh^l(Z,Z) + Bh^s(Z,Z) = 0, \tag{4.27}$$

where $Z \in \Gamma(JL_2)$. Taking inner product with $\xi \in \Gamma(D_2)$, we obtain $g(A_{wZ}Z, J\xi) + g(h^l(Z,Z),\xi) = 0$ and further using (2.13), we have $g(h^s(Z,J\xi), wZ) + g(h^l(Z,Z),\xi) = 0$. Hence, using (4.26), we get $g(Z,Z)g(H^l,\xi) = 0$, then the non-degeneracy of JL_2 implies that $H^l = 0$, which completes the proof. □

Lemma 4.8. *Let M be a totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $\nabla_X X \in \Gamma(D)$, for any $X \in \Gamma(D)$.*

Proof. Since $D' = J(L_1 \perp L_2)$, therefore $\nabla_X X \in \Gamma(D)$, if and only if, $g(\nabla_X X, J\xi) = 0$ and $g(\nabla_X X, JW) = 0$, for any $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$. Since M is a totally umbilic GCR-lightlike submanifold, we obtain

$$\begin{aligned} g(\nabla_X X, J\xi) &= -\bar{g}(\bar{\nabla}_X JX, \xi) = -\bar{g}(h^l(X,JX),\xi) \\ &= -\bar{g}(H^l,\xi)g(X,JX) = 0 \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} g(\nabla_X X, JW) &= -\bar{g}(\bar{\nabla}_X JX, W) = -\bar{g}(h^s(X,JX),W) \\ &= -\bar{g}(H^s,W)g(X,JX) = 0. \end{aligned} \tag{4.29}$$

Thus from (4.28) and (4.29), the result follows. □

Theorem 4.9. *Let M be a proper totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $H^s \in \Gamma(L_2)$.*

Proof. Since M is a totally umbilic GCR-lightlike submanifold, therefore using (3.7), we get $g(X, JX)H^s = wP_2\nabla_X X + g(X, X)CH^s$, for any $X \in \Gamma(D_0)$. Then, using the Lemma (4.8), we have $g(X, X)CH^s = 0$. As D_0 is non-degenerate, we get $CH^s = 0$. Hence $H^s \in \Gamma(L_2)$. □

Theorem 4.10. *Let M be a totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $\nabla_X JX = J\nabla_X X$, for any $X \in \Gamma(D_0)$.*

Proof. Using (3.10) and (3.12), we have $w\nabla_X X = h(X, JX) - Ch(X, X)$, for any $X \in \Gamma(D_0)$. Since M is totally umbilic, using (4.25), we have $w\nabla_X X = Hg(X, JX) - CH^l g(X, X) - CH^s g(X, X)$. Consequently, using the Theorems (4.7) and (4.9), we get $w\nabla_X X = 0$. Hence

$$\nabla_X X \in \Gamma(D). \tag{4.30}$$

Moreover, using (2.26) and the fact that D_0 is non-degenerate, we obtain that $\nabla_X JX = J\nabla_X X$, for any $X \in \Gamma(D_0)$. □

Theorem 4.11. *Let M be a proper totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $H^s = 0$.*

Proof. For any $W \in \Gamma(S(TM^\perp))$ and $X \in \Gamma(D_0)$, using (2.26), (4.25) and the Theorem (4.10), we have

$$\begin{aligned} g(J\bar{\nabla}_X X, JW) &= g(\bar{\nabla}_X JX, JW) = g(\nabla_X JX, JW) + g(h(X, JX), JW) \\ &= g(\nabla_X JX, JW) + g(X, JX)g(H, JW) \\ &= g(J\nabla_X X, JW) = g(\nabla_X X, W) = 0. \end{aligned} \tag{4.31}$$

Also using (4.26), we have

$$\begin{aligned} g(J\bar{\nabla}_X X, JW) &= g(\bar{\nabla}_X X, W) = g(h^s(X, X), W) \\ &= g(X, X)g(H^s, W). \end{aligned} \tag{4.32}$$

Comparing (4.31) and (4.32), we have $g(X, X)g(H^s, W) = 0$. Thus, the non-degeneracy of D_0 and $S(TM^\perp)$ implies that $H^s = 0$. Hence the result follows. □

As a consequence of Theorems (4.7) and (4.11), we obtain the following result.

Theorem 4.12. *Let M be a proper totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then M is a totally geodesic GCR-lightlike submanifold.*

In [10], Duggal and Sahin proved a theorem for non-existence of totally umbilic proper GCR-lightlike submanifolds in a complex space form as

Theorem 4.13. *([10]). There exist no totally umbilic proper GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, such that $c \neq 0$.*

Finally, using the theorems (4.12) and (4.13) in (4.11), we obtain

Theorem 4.14. *Let M be a totally umbilic GCR-lightlike submanifold of an complex space form $\bar{M}(c)$, then $Ric(U, V) = 0$, for any $U, V \in \Gamma(TM)$.*

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