# DETERMINANTS AND PERMANENTS OF HESSENBERG MATRICES AND GENERALIZED LUCAS POLYNOMIALS 

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#### Abstract

In this paper, we give some determinantal and permanental representations of generalized Lucas polynomials, which are a general form of generalized bivariate Lucas $p$-polynomials, ordinary Lucas and Perrin sequences etc., by using various Hessenberg matrices. In addition, we show that determinant and permanent of these Hessenberg matrices can be obtained by using combinations. Then we show, the conditions under which the determinants of the Hessenberg matrix become its permanents.


## 1. Introduction

Fibonacci numbers $f_{n}$, Lucas numbers $l_{n}$ and Perrin numbers $r_{n}$ are defined by

$$
\begin{aligned}
f_{n} & =f_{n-1}+f_{n-2} \text { for } n>2 \text { and } f_{1}=f_{2}=1 \\
l_{n} & =l_{n-1}+l_{n-2} \text { for } n>1 \text { and } l_{0}=2, l_{1}=1 \\
r_{n} & =r_{n-2}+r_{n-3} \text { for } n>3 \text { and } r_{0}=3, r_{1}=0, r_{2}=2
\end{aligned}
$$

respectively.

[^0]Generalizations of these sequences have been studied by a number of researchers. For instance; Miles [16] defined generalized order- $k$ Fibonacci numbers, Er [2] defined $k$ sequences of generalized order- $k$ Fibonacci numbers and Kaygısız and Bozkurt [4] defined $k$-generalized order- $k$ Perrin numbers.

MacHenry [12] defined generalized Fibonacci polynomials $F_{k, n}(t)$ and Lucas polynomials $G_{k, n}(t)$ as follows;

$$
\begin{array}{ll}
F_{k, n}(t)=0, n<0, & G_{k, n}(t)=0, n<0, \\
F_{k, 0}(t)=1, & G_{k, 0}(t)=k, \\
F_{k, n}(t)=\sum_{j=1}^{k} t_{j} F_{k, n-j}(t), & G_{k, 1}(t)=t_{1}, \\
& G_{k, n}(t)=G_{k-1, n}(t), 1 \leq n<k, \\
& G_{k, n}(t)=\sum_{j=1}^{k} t_{j} G_{k, n-j}(t), n \geq k
\end{array}
$$

where $t_{i}(1 \leq i \leq k)$ are constant coefficients of the core polynomial

$$
P\left(x ; t_{1}, t_{2}, \ldots, t_{k}\right)=x^{k}-t_{1} x^{k-1}-\cdots-t_{k}
$$

In [13, 14], authors obtained some properties of this polynomials. In addition, in [15], authors obtained ( $n, k \in \mathbb{N}, n \geq 1$ )

$$
\begin{equation*}
G_{k, n}(t)=\sum_{a \vdash n} \frac{n}{|a|}\binom{|a|}{a_{1}, \ldots, a_{k}} t_{1}^{a_{1}} \ldots t_{k}^{a_{k}} \tag{1.1}
\end{equation*}
$$

where $a_{i}$ are nonnegative integers for all $i(1 \leq i \leq k)$, with initial conditions given by

$$
G_{k, 0}(t)=k, G_{k,-1}(t)=0, \cdots, G_{k,-k+1}(t)=0
$$

Throughout this paper, the notations $a \vdash n$ and $|a|$ are used instead of $\sum_{j=1}^{k} j a_{j}=n$ and $\sum_{j=1}^{k} a_{j}$, respectively.

Kaygısız and Şahin [5] defined generalized Perrin polynomials $R_{k, n}(t)$ by using generalized Lucas polynomials.

The generalized bivariate Lucas $p$-polynomials [20] are defined by

$$
L_{p, n}(x, y)=x L_{p, n-1}(x, y)+y L_{p, n-p-1}(x, y)
$$

for $n>p$, with boundary conditions $L_{p, 0}(x, y)=(p+1), L_{p, n}(x, y)=x^{n}$, $n=1,2, \ldots, p$.

Table 1. Cognate polynomial sequences 1.

| $\mathbf{k}$ | $\mathbf{t}_{\mathbf{1}}$ | $\mathbf{t}_{\mathbf{i}}(2 \leq i \leq(k-1))$ | $\mathbf{t}_{\mathbf{k}}$ | $\mathbf{G}_{\mathbf{k}, \mathbf{n}}(\mathbf{t})$ |
| :---: | :---: | :---: | :---: | :---: |
| $k$ | 0 | $t_{i}$ | $t_{k}$ | $R_{k, n}(t)$ |
| $k$ | $x$ | 0 | $y$ | $L_{p, n}(x, y)$ |
| 3 | 0 | 1 | 1 | $r_{n}$ |

TABLE 2. [20]Cognate polynomial sequences 2.

| $\mathbf{x}$ | $\mathbf{y}$ | $\mathbf{p}$ | $\mathbf{L}_{\mathbf{p}, \mathbf{n}}(\mathbf{x}, \mathbf{y})$ |
| :---: | :---: | :---: | :--- |
| $x$ | $y$ | 1 | bivariate Lucas polynomials $L_{n}(x, y)$ |
| $x$ | 1 | $p$ | Lucas p-polynomials $L_{p, n}(x)$ |
| $x$ | 1 | 1 | Lucas polynomials $l_{n}(x)$ |
| 1 | 1 | $p$ | Lucas p-numbers $L_{p}(n)$ |
| 1 | 1 | 1 | Lucas numbers $L_{n}$ |
| $2 x$ | $y$ | $p$ | bivariate Pell-Lucas p-polynomials $L_{p, n}(2 x, y)$ |
| $2 x$ | $y$ | 1 | bivariate Pell-Lucas polynomials $L_{n}(2 x, y)$ |
| $2 x$ | 1 | $p$ | Pell-Lucas p-polynomials $Q_{p, n}(x)$ |
| $2 x$ | 1 | 1 | Pell-Lucas polynomials $Q_{n}(x)$ |
| 2 | 1 | 1 | Pell-Lucas numbers $Q_{n}$ |
| $2 x$ | -1 | 1 | Chebysev polynomials of the first kind $T_{n}(x)$ |
| $x$ | $2 y$ | $p$ | bivariate Jacobsthal-Lucas p-polynomials $L_{p, n}(x, 2 y)$ |
| $x$ | $2 y$ | 1 | bivariate Jacobsthal-Lucas polynomials $L_{n}(x, 2 y)$ |
| 1 | $2 y$ | 1 | Jacobsthal-Lucas polynomials $j_{n}(y)$ |
| 1 | 2 | 1 | Jacobsthal-Lucas numbers $j_{n}$ |

Tables 1 and 2 show that $G_{k, n}(t)$ are general form of many sequences and polynomials. Therefore, any result obtained from the polynomials $G_{k, n}(t)$ is valid for all sequences and polynomials mentioned in these tables.

On the other hand, many researchers studied determinantal and permanental representations of $k$ sequences of generalized order- $k$ Fibonacci and Lucas numbers. For example, Minc [17] defined an $n \times n(0,1)$ matrix $F(n, k)$ and showed that the permanents of $F(n, k)$ is equal to the generalized order- $k$ Fibonacci numbers.

In $[10,11]$, authors defined two ( 0,1 )-matrices and showed that the permanents of these matrices are the generalized Fibonacci and Lucas numbers. Öcal et al. [18] gave some determinantal and permanental representations of $k$-generalized Fibonacci and Lucas numbers and obtained Binet's formulas for these sequences. Kıliç and Stakhov [8] gave
permanent representation of Fibonacci and Lucas $p$-numbers. Kılıç and Taşcı [9] studied permanents and determinants of Hessenberg matrices. Yılmaz and Bozkurt [22] derived some relationships between Pell and Perrin sequences, as well as permanents and determinants of a type of Hessenberg matrices. Kaygısız and Şahin [7] gave some determinantal and permanental representations of Fibonacci type numbers. Kaygisız and Şahin [6] gave some determinantal and permanental representations of generalized bivariate Lucas $p$-polynomials. In [3, 19, 21], authors gave some relations between determinants and permanents.

The main purpose of this paper is to give some determinantal and permanental representations of generalized Lucas polynomials by using various Hessenberg matrices. Then we provide some conditions under which the determinants of the Hessenberg matrix become its permanents.

## 2. The determinantal representations

An $n \times n$ matrix $+A_{n}=\left(a_{i j}\right)$ is called lower Hessenberg matrix if $a_{i j}=0$ when $j-i>1$ i.e.,

$$
{ }_{+} A_{n}=\left[\begin{array}{ccccc}
a_{11} & a_{12} & 0 & \cdots & 0  \tag{2.1}\\
a_{21} & a_{22} & a_{23} & \cdots & 0 \\
a_{31} & a_{32} & a_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1, n} \\
a_{n, 1} & a_{n, 2} & a_{n, 3} & \cdots & a_{n, n}
\end{array}\right]
$$

Lemma 2.1. [1] Let $+A_{n}$ be the $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and define $\operatorname{det}\left({ }_{+} A_{0}\right)=1$. Then, $\operatorname{det}\left({ }_{+} A_{1}\right)=a_{11}$ and for $n \geq 2$
$\operatorname{det}\left(+A_{n}\right)=a_{n, n} \operatorname{det}\left(+A_{n-1}\right)+\sum_{r=1}^{n-1}\left[(-1)^{n-r} a_{n, r}\left(\prod_{j=r}^{n-1} a_{j, j+1}\right) \operatorname{det}\left(+A_{r-1}\right)\right]$.
Now, we define two Hessenberg matrices ${ }_{+} C_{k, m}$ and ${ }_{-} C_{k, m}$ whose determinants give the generalized Lucas polynomials.

Theorem 2.2. Let $k \geq 2$ and $n \geq 1$ be integers, and let $G_{k, n}(t)$ be the generalized Lucas polynomials and ${ }_{+} C_{k, n}=\left(c_{r s}\right)$ an $n \times n$ Hessenberg
matrix, given by

$$
\begin{aligned}
& c_{r s}= \begin{cases}i^{|r-s|} \cdot \frac{t_{r-s+1}}{t_{2}^{(r-s)}}, & \text { if } s \neq 1 \text { and }-1 \leq r-s<k, \\
i^{|r-s|} \cdot \frac{t_{r-s+1}}{t_{2}^{(r-s)}} \cdot(r-s+1), & \text { if } s=1 \text { and }-1 \leq r-s<k, \\
0, & \text { otherwise }\end{cases} \\
& \text { i.e., } \\
& \quad+C_{k, n}=\left[\begin{array}{cccccc}
t_{1} & i t_{2} & 0 & 0 & \cdots & 0 \\
2 i & t_{1} & i t_{2} & 0 & \cdots & 0 \\
3 i^{2} \frac{t_{3}}{t_{2}^{2}} & i & t_{1} & i t_{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
k i^{k-1} \frac{t_{k}}{t_{2}^{k-1}} & i^{k-2} \frac{t_{k-1}}{t_{2}^{k-2}} & i^{k-3} \frac{t_{k-2}}{t_{2}^{k-3}} & i^{k-4} \frac{t_{k-3}}{t_{2}^{k-4}} & \cdots & 0 \\
0 & i^{k-1} \frac{t_{k}}{t_{2}^{k-1}} & i^{k-2} \frac{t_{k-1}^{k-2}}{t_{2}^{k-2}} & i^{k-3} \frac{t_{k-2}}{t_{2}^{k-3}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & i t_{2} \\
0 & 0 & 0 & \cdots & i & t_{1}
\end{array}\right]
\end{aligned}
$$

where $t_{0}=1$ and $i=\sqrt{-1}$. Then,

$$
\begin{equation*}
\operatorname{det}\left({ }_{+} C_{k, n}\right)=G_{k, n}(t) \tag{2.3}
\end{equation*}
$$

Proof. To prove (2.3), we use the mathematical induction on $m$. The result is true for $m=1$ by hypothesis.

Assume that it is true for all positive integers less than or equal to $m$, namely, $\operatorname{det}\left({ }_{+} C_{k, m}\right)=G_{k, m}(t)$. Then, by using Lemma 2.1, we have

$$
\begin{aligned}
\operatorname{det}\left({ }_{+} C_{k, m+1}\right)= & c_{m+1, m+1} \operatorname{det}\left({ }_{+} C_{k, m}\right) \\
& +\sum_{r=1}^{m}\left[(-1)^{m+1-r} c_{m+1, r} \prod_{j=r}^{m} c_{j, j+1} \operatorname{det}\left({ }_{+} C_{k, r-1}\right)\right] \\
= & t_{1} \operatorname{det}\left({ }_{+} C_{k, m}\right)+\sum_{r=1}^{m-k+1}\left[(-1)^{m+1-r} c_{m+1, r} \prod_{j=r}^{m} c_{j, j+1} \operatorname{det}\left({ }_{+} C_{k, r-1}\right)\right] \\
& +\sum_{r=m-k+2}^{m}\left[(-1)^{m+1-r} c_{m+1, r} \prod_{j=r}^{m} c_{j, j+1} \operatorname{det}\left({ }_{+} C_{k, r-1}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= & t_{1} \operatorname{det}\left({ }_{+} C_{k, m}\right) \\
& +\sum_{r=m-k+2}^{m}\left[(-1)^{m+1-r} c_{m+1, r} \prod_{j=r}^{m} c_{j, j+1} \operatorname{det}\left({ }_{+} C_{k, r-1}\right)\right] \\
= & t_{1} \operatorname{det}\left({ }_{+} C_{k, m}\right) \\
& +\sum_{r=m-k+2}^{m}\left[(-1)^{m+1-r} \cdot i^{m+1-r} \frac{t_{m-r+2}}{t_{2}^{m-r+1)}} \prod_{j=r}^{m} i t_{2} \operatorname{det}\left({ }_{+} C_{k, r-1}\right)\right] \\
= & t_{1} \operatorname{det}\left({ }_{+} C_{k, m}\right) \\
& +\sum_{r=m-k+2}^{m}\left[(-i)^{m+1-r} \frac{t_{m-r+2}}{\left.t_{2}^{(m-r+1)} \cdot i^{m+1-r} \cdot t_{2}^{(m-r+1)} \operatorname{det}\left({ }_{+} C_{k, r-1}\right)\right]}\right. \\
= & t_{1} \operatorname{det}\left({ }_{+} C_{k, m}\right)+\sum_{r=m-k+2}^{m} t_{m-r+2} \operatorname{det}\left({ }_{+} C_{k, r-1}\right) \\
= & t_{1} \operatorname{det}\left({ }_{+} C_{k, m}\right)+t_{2} \operatorname{det}\left({ }_{+} C_{k, m-1}\right)+\cdots+t_{k} \operatorname{det}\left({ }_{+} C_{k, m-(k-1)}\right) .
\end{aligned}
$$

From the hypothesis and definition of generalized Lucas polynomials we obtain

$$
\operatorname{det}\left(+C_{k, m+1}\right)=t_{1} G_{k, m}(t)+\cdots+t_{k} G_{k, m-(k-1)}(t)=G_{k, m+1}(t)
$$

Therefore, (2.3) holds for all positive integers.
Example 2.3. We obtain 6 -th generalized Lucas polynomial for $k=5$, by using (2.3).

$$
\begin{aligned}
& \operatorname{det}\left({ }_{+} C_{5,6}\right)= \operatorname{det}\left[\begin{array}{cccccc}
t_{1} & i t_{2} & 0 & 0 & 0 & 0 \\
2 i & t_{1} & i t_{2} & 0 & 0 & 0 \\
3 \frac{-t_{3}}{t_{2}^{2}} & i & t_{1} & i t_{2} & 0 & 0 \\
4 \frac{-i t_{4}}{t_{2}^{3}} & \frac{-t_{3}}{t_{2}^{2}} & i & t_{1} & i t_{2} & 0 \\
5 \frac{t_{5}^{4}}{t_{2}^{4}} & \frac{-t t_{4}}{t_{2}^{3}} & \frac{-t_{3}}{t_{2}^{2}} & i & t_{1} & i t_{2} \\
0 & \frac{t_{5}}{t_{2}^{4}} & \frac{-i t_{4}}{t_{2}^{3}} & \frac{-t_{3}}{t_{2}^{2}} & i & t_{1}
\end{array}\right] \\
&=t_{1}^{6}+6 t_{1}^{4} t_{2}+9 t_{1}^{2} t_{2}^{2}+2 t_{2}^{3}+6 t_{1}^{3} t_{3}+3 t_{3}^{2}+12 t_{1} t_{2} t_{3} \\
&+6 t_{1}^{2} t_{4}+6 t_{2} t_{4}+6 t_{1} t_{5} \\
&= G_{5,6}(t)
\end{aligned}
$$

Theorem 2.4. Let $k \geq 2$ and $n \geq 1$ be integers, $G_{k, n}$ the generalized Lucas polynomial and $C_{k, n}=\left(b_{i j}\right)$ an $n \times n$ lower Hessenberg matrix,
given by

$$
b_{i j}= \begin{cases}-t_{2}, & \text { if } j=i+1, \\ \frac{t_{i-j+1}}{t_{2}^{(i-j)},} & \text { if } j \neq 1 \text { and } 0 \leq i-j<k \\ \frac{t_{i-j+1}}{t_{2}^{(i-j)}} \cdot(i-j+1), & \text { if } j=1 \text { and } 0 \leq i-j<k \\ 0, & \text { otherwise }\end{cases}
$$

i.e.,

$$
{ }_{-} C_{k, n}=\left[\begin{array}{cccccc}
t_{1} & -t_{2} & 0 & 0 & \cdots & 0 \\
2 & t_{1} & -t_{2} & 0 & \cdots & 0 \\
3 \frac{t_{3}}{t_{2}^{2}} & 1 & t_{1} & -t_{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
k \frac{t_{k}}{t_{2}^{k-1}} & \frac{t_{k-1}}{t_{2}^{k-2}} & \frac{t_{k-2}}{t_{2}^{k-3}} & \ddots & \cdots & 0 \\
0 & \frac{t_{k}}{t_{2}^{k-1}} & \frac{t_{k-1}}{t_{2}^{k-2}} & \frac{t_{k-2}}{t_{2}^{k-3}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & -t_{2} \\
0 & 0 & 0 & \cdots & \cdots & t_{1}
\end{array}\right]
$$

where $t_{0}=1$. Then,

$$
\begin{equation*}
\operatorname{det}\left({ }_{-} C_{k, n}\right)=G_{k, n}(t) \tag{2.4}
\end{equation*}
$$

Proof. The proof is similar to the proof of Theorem 2.2, by using Lemma 2.1.

Example 2.5. We obtain 5 -th generalized Lucas polynomial for $k=4$, by using (2.4).

$$
\begin{aligned}
\operatorname{det}\left({ }_{-} C_{4,5}\right) & =\operatorname{det}\left[\begin{array}{ccccc}
t_{1} & -t_{2} & 0 & 0 & 0 \\
2 & t_{1} & -t_{2} & 0 & 0 \\
3 \frac{t_{3}}{t_{2}^{2}} & 1 & t_{1} & -t_{2} & 0 \\
4 \frac{t_{4}}{t_{2}^{3}} & \frac{t_{3}}{t_{2}^{2}} & 1 & t_{1} & -t_{2} \\
0 & \frac{t_{4}}{t_{2}^{3}} & \frac{t_{3}}{t_{2}^{2}} & 1 & t_{1}
\end{array}\right] \\
& =t_{1}^{5}+5 t_{1}^{3} t_{2}+5 t_{1} t_{2}^{2}+5 t_{1}^{2} t_{3}+5 t_{2} t_{3}+5 t_{1} t_{4} \\
& =G_{4,5}(t)
\end{aligned}
$$

Corollary 2.6. If we rewrite equalities (2.3) and (2.4) for $t_{i}=1$ and $k=2$, then we obtain

$$
\operatorname{det}\left({ }_{+} C_{k, n}\right)=\operatorname{det}\left({ }_{-} C_{k, n}\right)=l_{n}
$$

where $l_{n}$ are the ordinary Lucas numbers.

Corollary 2.7. If we rewrite equalities (2.3) and (2.4) for $t_{1}=0$, then we obtain

$$
\operatorname{det}\left({ }_{+} C_{k, n}\right)=\operatorname{det}\left({ }_{-} C_{k, n}\right)=R_{k, n}(t)
$$

where $R_{k, n}(t)$ are the generalized Perrin polynomials.
Corollary 2.8. If we rewrite the right hand side of equalities (2.3) and (2.4) for $t_{1}=x, t_{k}=y, t_{i}=0(2 \leq i \leq k-1)$ and $k=(p+1)$, then we obtain

$$
\operatorname{det}\left(+C_{k, n}\right)=\operatorname{det}\left({ }_{-} C_{k, n}\right)=L_{p, n}(x, y)
$$

where $L_{p, n}(x, y)$ are the generalized bivariate Lucas p-polynomials.
Corollary 2.9. If we rewrite equalities (2.3) and (2.4) for $t_{1}=0$ and $t_{i}=1(2 \leq i \leq k)$ and $k=3$, then we obtain

$$
\operatorname{det}\left({ }_{+} C_{k, n}\right)=\operatorname{det}\left({ }_{-} C_{k, n}\right)=r_{n}
$$

where $r_{n}$ are the ordinary Perrin numbers.
Now we show that determinants of Hessenberg matrices $C_{k, n}$ and $+C_{k, n}$ can be obtained by using combinations.

## Corollary 2.10 .

$$
\operatorname{det}\left({ }_{+} C_{k, n}\right)=\operatorname{det}\left(C_{k, n}\right)=\sum_{a \vdash n} \frac{n}{|a|}\binom{|a|}{a_{1, \ldots,}, a_{k}} t_{1}^{a_{1}} \ldots t_{k}^{a_{k}}
$$

Proof. It is obvious from Theorem 2.2, Theorem 2.4 and (1.1).

## 3. The permanent representations

Let $A=\left(a_{i, j}\right)$ be an $n \times n$ square matrix over a ring R . Then, it is well known that, the permanent of $A$ is defined by

$$
\operatorname{per}(A)=\sum_{\sigma \in S_{n}} \prod_{i=1}^{n} a_{i, \sigma(i)}
$$

where $S_{n}$ denotes the symmetric group on $n$ letters.
Lemma 3.1. [18] Let $A_{n}$ be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and define $\operatorname{per}\left(A_{0}\right)=1$. Then, $\operatorname{per}\left(A_{1}\right)=a_{11}$ and for $n \geq 2$

$$
\begin{equation*}
\operatorname{per}\left(A_{n}\right)=a_{n, n} \operatorname{per}\left(A_{n-1}\right)+\sum_{r=1}^{n-1}\left(a_{n, r} \prod_{j=r}^{n-1} a_{j, j+1} \operatorname{per}\left(A_{r-1}\right)\right) \tag{3.1}
\end{equation*}
$$

We define two Hessenberg matrices $H_{k, n}$ and ${ }_{+} H_{k, n}$ whose permanents give the generalized Lucas polynomials.

Theorem 3.2. Let $k \geq 2$ and $n \geq 1$ be integers, $G_{k, n}(t)$ the generalized Lucas polynomials and $H_{k, n}=\left(h_{r s}\right)$ an $n \times n$ lower Hessenberg matrix, given by

$$
\begin{aligned}
& h_{r s}= \begin{cases}i^{(r-s)} \cdot \frac{t_{r-s+1}}{t_{2}^{(r-s)}}, & \text { if } s \neq 1 \text { and }-1 \leq r-s<k, \\
i^{(r-s)} \cdot \frac{r-s+1}{t_{2}^{(r-s)}} \cdot(r-s+1), & \text { if } s=1 \text { and }-1 \leq r-s<k, \\
0, & \text { otherwise }\end{cases} \\
& \text { i.e., } \\
& { }_{-} H_{k, n}=\left[\begin{array}{cccccc}
t_{1} & -i t_{2} & 0 & 0 & \cdots & 0 \\
2 i & t_{1} & -i t_{2} & 0 & \cdots & 0 \\
3 i^{2} \frac{t_{3}}{t_{2}^{2}} & i & t_{1} & -i t_{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
k i^{k-1} \frac{t_{k}}{t_{2}^{k-1}} & i^{k-2} \frac{t_{k-1}}{t_{k}^{k-2}} & i^{k-3} \frac{t_{k-2}}{t_{k}^{k-3}} & i^{k-4} \frac{t_{k-3}}{t_{k}^{k-4}} & \cdots & 0 \\
0 & i^{k-1} \frac{t_{2}}{t_{2}^{k-1}} & i^{k-2} \frac{2 t_{k-1}}{t_{2}^{k-2}} & i^{k-3} \frac{t_{k-2}}{t_{k}^{k-3}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & \cdots & \cdots & t_{1}
\end{array}\right]
\end{aligned}
$$

where $t_{0}=1$ and $i=\sqrt{-1}$. Then,

$$
\begin{equation*}
\operatorname{per}\left(-H_{k, n}\right)=G_{k, n}(t) . \tag{3.2}
\end{equation*}
$$

Proof. The proof is similar to the proof of Theorem 2.2, by using Lemma 3.1.

Example 3.3. We obtain the 3 -rd generalized Lucas polynomial for $k=$ 4, by using (3.2)

$$
\begin{aligned}
\operatorname{per}\left(-H_{4,3}\right) & =\operatorname{per}\left[\begin{array}{ccc}
t_{1} & -i t_{2} & 0 \\
2 i & t_{1} & -i t_{2} \\
3 \frac{-t_{3}}{t_{2}} & i & t_{1}
\end{array}\right] \\
& =t_{1}^{3}+3 t_{1} t_{2}+3 t_{3} .
\end{aligned}
$$

Theorem 3.4. Let $k \geq 2$ and $n \geq 1$ be integers, $G_{k, n}(t)$ the generalized Lucas polynomials and $+H_{k, n}=\left(p_{i j}\right)$ an $n \times n$ lower Hessenberg matrix
given by

$$
p_{i j}= \begin{cases}\frac{t_{i-j+1}}{t_{2}^{(i-j)}}, & \text { if } j \neq 1 \text { and }-1 \leq i-j<k \\ \frac{t_{i-j+1}}{t_{2}^{(i-j)}} \cdot(i-j+1), & \text { if } j=1 \text { and } 0 \leq i-j<k \\ 0, & \text { otherwise }\end{cases}
$$

i.e.,

$$
{ }_{+} H_{k, n}=\left[\begin{array}{cccccc}
t_{1} & t_{2} & 0 & 0 & \cdots & 0 \\
2 & t_{1} & t_{2} & 0 & \cdots & 0 \\
3 \frac{t_{3}}{t_{2}^{2}} & 1 & t_{1} & t_{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
k \frac{t_{k}}{t_{2}^{k-1}} & \frac{t_{k-1}}{t_{2}^{k-2}} & \frac{t_{k-2}}{t_{2}^{k-3}} & \frac{t_{k-3}}{t_{2}^{k-4}} & \cdots & 0 \\
0 & \frac{t_{k}}{t_{2}^{k-1}} & \frac{t_{k-1}}{t_{2}^{k-2}} & \frac{t_{k-2}}{t_{2}^{k-3}} & \cdots & 0 \\
& \vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & \cdots & \cdots & t_{1}
\end{array}\right]
$$

where $t_{0}=1$. Then,

$$
\begin{equation*}
\operatorname{per}\left({ }_{+} H_{k, n}\right)=G_{k, n}(t) \tag{3.3}
\end{equation*}
$$

Proof. The proof is similar to the proof of Theorem 2.2, by using Lemma 3.1.

Corollary 3.5. If we rewrite equalities (3.2) and (3.3) for $t_{i}=1(1 \leq$ $i \leq k$ ) and $k=2$, then we obtain

$$
\operatorname{per}\left({ }_{-} H_{k, n}\right)=\operatorname{per}\left({ }_{+} H_{k, n}\right)=l_{n}
$$

where $l_{n}$ are the ordinary Lucas numbers.
Corollary 3.6. If we rewrite equalities (3.2) and (3.3) for $t_{1}=0$ and $t_{i}=1(2 \leq i \leq k)$, then we obtain

$$
\operatorname{per}\left(-H_{k, n}\right)=\operatorname{per}\left(+H_{k, n}\right)=R_{k, n}(t)
$$

where $R_{k, n}(t)$ are the generalized Perrin polynomials.
Corollary 3.7. If we rewrite the right hand side of equalities (3.2) and (3.3) for $t_{1}=x, t_{k}=y, t_{i}=0 \quad(2 \leq i \leq k-1)$ and $k=(p+1)$, then we obtain

$$
\operatorname{per}\left(-H_{k, n}\right)=\operatorname{per}\left(+H_{k, n}\right)=L_{p, n}(x, y)
$$

where $L_{p, n}(x, y)$ are the generalized bivariate Lucas p-polynomials.

Corollary 3.8. If we rewrite equalities (3.2) and (3.3) for $t_{1}=0, t_{i}=1$ $(2 \leq i \leq k)$ and $k=3$, then we obtain

$$
\operatorname{per}\left(-H_{k, n}\right)=\operatorname{per}\left(+H_{k, n}\right)=r_{n}
$$

where $r_{n}$ are the ordinary Perrin numbers.
Now we show that permanent of Hessenberg matrices $H_{k, n}$ and $+H_{k, n}$ can be obtained by using combinations.

## Corollary 3.9.

$$
\operatorname{per}\left(-H_{k, n}\right)=\operatorname{per}\left({ }_{+} H_{k, n}\right)=\sum_{a \vdash n} \frac{n}{|a|}\binom{|a|}{a_{1}, \ldots, a_{k}} t_{1}^{a_{1}} \ldots t_{k}^{a_{k}}
$$

Proof. It is obvious from Theorem 3.2, Theorem 3.4 and (1.1).

## 4. Determinant and permanent of a Hessenberg matrix

Gibson [3] gave an identity between permanent and determinant of a semitriangular matrix. We give a different proof of this identity for Hessenberg matrices.

Theorem 4.1. Let $+A_{n}$ be the Hessenberg matrix in (2.1) and
_ $A_{n}=\left(b_{i j}\right)$ an $n \times n$ Hessenberg matrix, given by

$$
b_{i j}= \begin{cases}0, & \text { if } j-i>1, \\ -a_{i j}, & \text { if } j-i=1, \\ a_{i j}, & \text { otherwise }\end{cases}
$$

i.e.,

$$
-A_{n}=\left[\begin{array}{ccccc}
a_{11} & -a_{12} & 0 & \cdots & 0 \\
a_{21} & a_{22} & -a_{23} & \cdots & 0 \\
a_{31} & a_{32} & a_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & -a_{n-1, n} \\
a_{n, 1} & a_{n, 2} & a_{n, 3} & \cdots & a_{n, n}
\end{array}\right] .
$$

Then,

$$
\begin{equation*}
\operatorname{det}\left(-A_{n}\right)=\operatorname{per}\left(+A_{n}\right) \text { and } \operatorname{det}\left(+A_{n}\right)=\operatorname{per}\left(-A_{n}\right) \tag{4.1}
\end{equation*}
$$

Proof. To prove (4.1), we use the mathematical induction on $m$. The result is true for $m=1$ by hypothesis.

Assume that it is true for all positive integers less than or equal to $m$, namely $\operatorname{det}\left(A_{m}\right)=\operatorname{per}\left(+A_{m}\right)$. Then, by using (2.2) and (3.1), we have

$$
\begin{aligned}
\operatorname{det}\left(-A_{m+1}\right)= & a_{m+1, m+1} \operatorname{det}\left(-A_{m}\right) \\
& +\sum_{r=1}^{m}\left[(-1)^{m+1-r} a_{m+1, r} \prod_{j=r}^{m} b_{j, j+1} \operatorname{det}\left(-A_{r-1}\right)\right] \\
= & a_{m+1, m+1} \operatorname{per}\left({ }_{+} A_{m}\right) \\
& +\sum_{r=1}^{m}\left[(-1)^{m+1-r} a_{m+1, r} \prod_{j=r}^{m}\left(-a_{j, j+1}\right) \operatorname{per}\left({ }_{+} A_{r-1}\right)\right] \\
= & a_{m+1, m+1} \operatorname{per}\left(+A_{m}\right) \\
& +\sum_{r=1}^{m}\left[(-1)^{m+1-r} a_{m+1, r}(-1)^{m+1-r} \prod_{j=r}^{m} a_{j, j+1} \operatorname{per}\left(+A_{r-1}\right)\right] \\
= & a_{m+1, m+1} \operatorname{per}\left({ }_{+} A_{m}\right)+\sum_{r=1}^{m}\left[a_{m+1, r} \prod_{j=r}^{m+1} a_{j, j+1} \operatorname{per}\left({ }_{+} A_{r-1}\right)\right] \\
= & \operatorname{per}\left({ }_{+} A_{m+1}\right) .
\end{aligned}
$$

Therefore, the result is true for all positive integers.

## Conclusion

Generalized Lucas polynomials are a general form of several polynomials and number sequences. Therefore any result obtained from these polynomials is applicable to the others. In addition, the relation between determinant and permanent of a Hessenberg matrix make it possible to transfer any result between them.

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