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SOME CLASSES OF STRONGLY CLEAN RINGS

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Abstract. A ring R is a strongly clean ring if every element in R is the sum of an idempotent and a unit that commute. We construct some classes of strongly clean rings which have stable range one. It is shown that such cleanness of 2×2 matrices over commutative local rings is completely determined in terms of solvability of quadratic equations.

1. **Introduction**

Throughout, all rings are associative rings with identity. A ring R is strongly clean provided that for any $x \in R$ there exists an idempotent $e \in R$ such that $x - e \in U(R)$ and $ex = xe$. Many authors investigated strong cleanness of 2×2 matrices over local rings [1–5,7] and [10]. Recall that a ring R has stable range one provided that $aR + bR = R$ implies that there exists $y \in R$ such that $a + by \in U(R)$. A long standing question asks whether strongly clean rings have stable range one. So to check that strongly clean rings have stable range one, some subclasses of such rings are introduced. A ring R is strongly rad clean in case for any $x \in R$, there is an idempotent $e \in R$ such that $x - e \in U(R), ex = xe$ and $ex \in J(R)$, where $J(R)$ is the Jacobson radical of R. A ring R is strongly π-rad clean in case for any $x \in R$, there exist an idempotent $e \in R$ and an $n \in \mathbb{N}$ such that $x - e \in U(R), ex = xe$ and $(ex)^n \in J(R)$. For a fixed natural number n, we say that $x \in R$ is strongly J_n-clean provided

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that there exists an idempotent $e \in R$ such that $x - e \in U(R), ex = xe$ and $(ex)^n \in J(R)$. A ring R is strongly J_n-clean in case each element in R is strongly J_n-clean. Clearly, a ring R is strongly rad clean if and only if R is strongly J_1-clean. For elementary properties of these rings, we refer the reader to [6] and [12].

The motivation of this article is to give a characterization of strongly J_n-clean rings by virtue of strongly π-regularity, and then construct many of such rings. Finally, we determine strong J_n-cleanness for 2×2 matrices over commutative local rings in terms of solvability of quadratic equations. We will show that strong J_n-cleanness $(n \geq 2)$, strong J_2-cleanness and strongly π-rad cleanness coincide for 2×2 matrices over local rings.

Throughout, $J(R)$ and $U(R)$ will denote, respectively, the Jacobson radical and the group of units in R. Furthermore $M_n(R)$ stands for the ring of all $n \times n$ matrices over a ring R. $tr(A)$ denotes the trace of $A \in M_n(R)$, $t_A = tr^2(A) - 4det(A)$, and $\mathbb{N}$ is the set of all natural numbers.

2. Strong π-regularity

A ring R is strongly π-regular in case for any $x \in R$ there exists some $n \in \mathbb{N}$ such that $x^n \in x^{n+1}R$. The aim of this section is to characterize the strongly J_n-cleanness by means of the strong π-regularity.

Theorem 2.1. *Let R be a ring, and let n be a fixed natural number. Then the following are equivalent:*

(1) *R is strongly J_n-clean.*
(2) *For any $x \in R$, there exist an idempotent $e \in R$ and a unit $u \in R$ such that $x^n \equiv eu \equiv ue\big(\ mod\ J(R)\big), ex = xe$.*

Proof. (1) $\Rightarrow$ (2) For any $x \in R$, there exist an idempotent $e \in R$ and a unit $u \in R$ such that $x = e + u, ex = xe$ and $ex^n \in J(R)$. Clearly, $x = ex + (1-e)x$. This implies that $x^n = ex^n + (1-e)x^n$. It is easy to verify that $(1-e)x = (1-e)u$, and so $(1-e)x^n = (1-e)u^n$. Let $f = 1-e$ and $v = u^n$. Then $x^n \equiv fv \equiv vf\big(\ mod\ J(R)\big)$ and $fx = xf$, as required.

(2) $\Rightarrow$ (1) Let $x \in R$. Then there exist an idempotent $e \in R$ and a unit $u \in R$ such that $x^n \equiv eu \equiv ue\big(\ mod\ J(R)\big), ex = xe$. Let $f = 1-e$

and $w=x-f$. Choose $v=x^{n-1}u^{-1}e-(1+x+\cdots+x^{n-1})f$. Then

$$\begin{array}{rcl} wv & = & (x-f)\big(x^{n-1}u^{-1}e-(1+x+\cdots+x^{n-1})f\big)\\ & = & x^nu^{-1}e+(1-x)(1+x+\cdots+x^{n-1})f-fx^{n-1}u^{-1}e\\ & \equiv & e+(1-x^n)f+fx^{n-1}u^{-1}x^nu^{-1}\big(mod\ J(R)\big)\\ & \equiv & e+(1-x^n)f+fx^{n-1}eu^{-1}\big(mod\ J(R)\big)\\ & = & e+f\\ & = & 1, \end{array}$$

and so $w\in R$ is right invertible. Likewise, we get $vw=1$, and so $w\in R$ is left invertible. Hence, $w\in U(R)$. Therefore, $x=f+w$ is a sum of an idempotent $f\in R$ and a unit $w\in R$, and that $fw=wf$. One easily checks that $fx^n\in J(R)$, as required. □

For a fixed natural number n, a ring R is said to be n-regular provided that for any $x\in R$ there exists a $y\in R$ such that $x^n=x^nyx^n$. Clearly, $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2\\ 0 & \mathbb{Z}_2\end{pmatrix}$ is 2-regular, while it is not regular. Let $S=T_n(\mathbb{Z}_2)$, the ring of all $n\times n$ upper triangular matrix over $\mathbb{Z}_2$. Then S is n-regular. Let $R=\{\begin{pmatrix} A_i & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}\mid A_i$ is an $i\times i$ upper triangular matrix over $\mathbb{Z}_2, i=1,2,3\cdots\}$. Then R is π-regular, while it is not n-regular for all $n\in\mathbb{N}$. Thus,

$$\{\text{ regular rings }\}\subsetneqq\{n\text{ -regular rings }\}\subsetneqq\{\pi\text{ -regular rings }\}$$

for all $n\geq 2$.

Corollary 2.2. *Let R be commutative, and let n be a fixed natural number. Then the following are equivalent:*

(1) *R is strongly J_n-clean.*
(2) *$R/J(R)$ is n-regular and every idempotent lifts modulo $J(R)$.*

Proof. (1) $\Rightarrow$ (2) Since R is strongly J_n-clean, it is clean. In view of [11, Theorem 30.2], R is an exchange ring. Hence, every idempotent lifts modulo $J(R)$, by [11, Theorem 29.2]. Let $x\in R$. According to Theorem 2.1, there exist an idempotent $e\in R$ and a unit $u\in R$ such that $x^n\equiv eu\equiv ue\big(\ mod\ J(R)\big), ex=xe$. This implies that $x^n\equiv ex^n\equiv u^{-1}x^{2n}\big(\ mod\ J(R)\big)$. Therefore, $x^nu^{-1}x^n\equiv x^ne\equiv(ue)e=ue\equiv x^n\big(\ mod\ J(R)\big)$, and so $R/J(R)$ is n-regular, as required.

(2) $\Rightarrow$ (1) As every idempotent lifts modulo $J(R)$, $R/J(R)$ is commutative. For any $x \in R$, there exists a $y \in R$ such that $x^n \equiv x^n y x^n \big(mod\ J(R)\big)$. Set $z = yx^n y$. Then $x^n \equiv x^n z x^n \big(mod\ J(R)\big)$ and $z \equiv z x^n z \big(mod\ J(R)\big)$. Set $u = z + 1 - x^n z$. Since R is commutative, it is easy to verify that

$$(x^n + 1 - x^n z)u \equiv u(x^n + 1 - x^n z) \equiv 1\big(mod\ J(R)\big).$$

Hence, $u \in U(R)$. Let $e = x^n z$. Then we can find an idempotent $f \in R$ such that $e \equiv f\big(mod\ J(R)\big)$. In addition, $x^n \equiv fu \equiv uf\big(mod\ J(R)\big)$. Further, $fx = xf$. According to Theorem 2.1, R is strongly J_n-clean. $\square$

In view of [11, Remark 29.7], all commutative n-regular rings are exchange rings; hence, it follows from [11, Theorem 29.2] and Corollary 2.2 that every commutative n-regular ring is strongly J_n-clean. Obviously, $\mathbb{Z}_4$ is 2-regular, while it is not regular. Also we see that $\mathbb{Z}_8$ is 3-regular, while it is not 2-regular. In light of Proposition 2.4, $\mathbb{Z}_4$ is strongly J_2-clean and $\mathbb{Z}_8$ is strongly J_3-clean. Let $R = \mathbb{F}[x]/(x^2) = \{a + bt \mid a, b \in \mathbb{F}, t^2 = 0\}$, where $\mathbb{F}$ is a field of characteristic 2. Then for any $a \in R$, there exists a $b \in R$ such that $a^2 = a^3 b$. Hence, $a^2 = a^2 b^2 a^2$. That is, R is commutative 2-regular. Thus, R is strongly J_2-clean. In this case, R is not commutative regular.

Recall that a ring R is of bounded index n provided that $x^n = 0$ for any nilpotent $x \in R$.

Lemma 2.3. *Let R be a strongly π-regular ring of bounded index n. Then for any $x \in R$, there exists $\mathbb{N}$ such that $x^n R = x^{n+1} R$.*

Proof. Let $x \in R$. In light of [4, Proposition 13.1.18], there exist $e = e^2 \in R, u \in U(R)$ and a nilpotent $w \in R$ such that $x = eu + w$ and e, u, w commute. It is easy to verify that $x - x^2 u^{-1} = (w + eu) - (w + eu)^2 u^{-1} = w - w(2e + wu^{-1})$. By hypothesis, $w^n = 0$, and so $\big(x - x^2 u^{-1}\big)^n = 0$. This implies that $x^n \in x^{n+1} R$, as required. $\square$

As is well known, a ring R is strongly π-regular if and only if for any $x \in R$ there exist $y \in R$ and $n \in R$ such that $x^n = x^{2n} y, y = y^2 x$ and $xy = yx$. By a similar method of the proof of this fact, we now derive the following.

Theorem 2.4. *Every strongly π-regular ring of bounded index n is strongly J_n-clean.*

Proof. Let R be a strongly π-regular ring of bounded index n, and let $x \in R$. According to Lemma 2.3, we have some $y \in R$ such that $x^n = x^{n+1}y$, and then $x^n = x^{n+i}y^i$ for any $i \in \mathbb{N}$. Also we have some $z \in R$ such that $y^n = y^{n+1}z$; hence, $y^n = y^{n+j}z^j$ for any $j \in \mathbb{N}$. Furthermore, we get $x^n = x^{2n}y^n$. As in the proof of [4, Proposition 13.15], we get $x^n \in Rx^{n+1}$.

Write $x^n = x^{n+1}y = rx^{n+1}$. Then $x^ny = rx^{n+1}y = rx^n$. By induction, we get $x^ny^k = r^kx^n$ for any $k \in \mathbb{N}$. Let $t = x^ny^{n+1}$. Then $t = r^{n+1}x^n$. It is easy to verify that $xt = x^{n+1}y^{n+1} = (x^{n+1}y)y^n = x^ny^n = r^nx^n = r^n(rx^{n+1}) = (r^{n+1}x^n)x = tx; x^{n+1}t = x^{2n+1}y^{n+1} = x^n(x^{n+1}y)y^n = x^{2n}y^n = \cdots = x^{n+1}y = x^n; t^2x = t(xt) = t(x^{n+1}y^{n+1}) = x^{n+1}ty^{n+1} = x^ny^{n+1} = t$. Let $s = t^n$. Then $x^n = x^{2n}s, xs = sx, s = s^2x^n$. Set $u = s + 1 - x^ns$. Then $x^nu = x^ns + x^n(1 - x^ns) = x^ns$ is an idempotent. In addition, $u^{-1} = x^n + 1 - x^ns$. Let $e = x^ns$. Then $x^n = eu^{-1} = u^{-1}e$ and $ex = xe$. Accordingly, the result follows by Theorem 2.1. □

As a consequence, we deduce that every finite ring R is strongly J_n-clean for all $n \geq |R|$. In view of [8, Theorem 7.15] and Theorem 2.4, every regular ring of bounded index n is strongly J_n-clean. Further, we show that every right P-exchange ring of bounded index n is strongly J_n-clean [4, Corollary 15.3.8].

By the decompositions of 2×2 matrices, Ying considered the strong J_2-cleanness of 2×2 matrix ring over a division ring [12, Theorem 3.3.10]. As a consequence of Theorem 2.4, we extend Ying's result to the general case.

Example 2.5. *Let D be a division ring. Then $M_n(D)$ is strongly J_n-clean, while it is not strongly J_{n-1}-clean.*

Proof. In view of [8, Theorem 7.2], $M_n(D)$ is a strongly π-regular ring of bounded index n. Therefore, $M_n(D)$ is strongly J_n-clean from Theorem 2.4. By a similar method [12, Proposition 3.3.9], one can check the next assertion. Choose $A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(D)$. Then A is not strongly J_{n-1}-clean. Otherwise, we have some $E = E^2 \in M_n(D)$ and $U \in GL_n(D)$ such that $EA = AE$ and $(EA)^{n-1} \in J\big(M_n(R)\big)$. Write $E = (e_{ij})$. As $EA^{n-1} = A^{n-1}E$, we get $e_{ni} = 0$ $(i = 1, \cdots, n-1)$.

As $(EA)^{n-1} \in J\big(M_n(R)\big)$, we get $e_{nn} = 0$. So every entry at the last row of U are zero, a contradiction. Accordingly, $M_n(D)$ is not strongly J_{n-1}-clean. □

Example 2.6. *Let D be a division ring, and let $R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D\}$. Then R is strongly J_n-clean for all $n \in \mathbb{N}$.*

Proof. Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in R$. If $a \neq 0$, then $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in R$ is invertible. If $a = 0$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$. Accordingly, R is strongly J_n-clean for all $n \in \mathbb{N}$. □

3. **Examples**

Let R be a ring, and let $a \in R$. Let $\ell(a) = \{r \in R \mid ra = 0\}$ and $r(a) = \{r \in R \mid ar = 0\}$. We begin this section with considering the strong J_n-cleanness in corner rings.

Lemma 3.1. *Let R be a ring, and let $a = e + w$ be a strongly J_n-clean decomposition of a in R. Then $\ell(a) \subseteq \ell(e)$ and $r(a) \subseteq r(e)$.*

Proof. Let $r \in \ell(a)$. Then $ra = 0$. Write $a = e + w, e = e^2, w \in U(R), ew = we$ and $ew^n \in J(R)$. Hence, $ew^{2n+1} \in J(R)$. We observe that $1 + ew \in U(R)$. Thus, $re = -rw$, and so $re = -rwe = -rew$. It follows that $re(1 + ew) = 0$, and then $re = 0$. That is, $r \in \ell(e)$. Therefore, $\ell(a) \subseteq \ell(e)$. A similar argument shows that $r(a) \subseteq r(e)$. □

Theorem 3.2. *Let R be a ring, let n be a fixed natural number, and let $f \in R$ be an idempotent. Then $a \in fRf$ is strongly J_n-clean in R if and only if a is strongly J_n-clean in fRf.*

Proof. Suppose $a = e - w, e = e^2 \in fRf, w \in U(fRf), ew = we$ and $ea^n \in J(fRf)$. Hence, $a = (e+1-f)-(1-f+w) \in fRf$ is strongly clean in R. In addition, $(e + 1 - f)a^n = ea^n \in J(R)$. Clearly, $(e + 1 - f)a = ea = ae = a(e + 1 - f)$, and thus, $a \in fRf$ is strongly J_n-clean in R.

Conversely, suppose that $a = e + w, e = e^2 \in R, w \in U(R), ew = we$ and $ew^n \in J(R)$. As $a \in fRf$, it follows from Lemma 3.1 that

$$1-f \in \ell(a)\bigcap r(a) \subseteq \ell(e)\bigcap r(e) = R(1-e)\bigcap(1-e)R = (1-e)R(1-e).$$

Hence, $ef = e = fe$. We observe that $a = fef + fwf$, $(fef)^2 = fef$, $fwf = a - fef = a - e = w \in U(fRf)$. Furthermore, $fef \cdot$

$fwf = fewf = fwef = fwf \cdot fef$. In addition, $fef(fwf)^n = few^nf \in fJ(R)f \subseteq J(fRf)$, as required. □

As a consequence, every corner of a strongly J_n-clean ring is strongly J_n-clean.

Example 3.3. *Let D be a division ring, and let $R = \bigoplus_{i=1}^{\infty} M_i(D)$. Then R is strongly π-rad clean, while R is not strongly J_n-clean for all $n \in \mathbb{N}$.*

Proof. Let $x \in R$. Write $x = (x_1, x_2, \cdots, x_m, 0, 0, \cdots) \in R$. In view of Corollary 2.2, all $x_i \in M_i(D)(i = 1, \cdots, m)$ are strongly J_m-clean. Therefore we conclude that $x \in R$ is strongly J_m-clean; hence, it is strongly π-rad clean. If R is strongly J_n-clean, then so is $M_{n+1}(D)$ by Theorem 3.2. This contradicts the result of Corollary 2.2, and we are done. □

For a commutative ring R, R is strongly J_n-clean if and only if R is strongly π-rad clean. But such rings are very different in the nonocommutative case. From Example 3.3, we see that

$$\begin{aligned}&\{\text{strongly rad clean }\} = \{\text{ strongly } J_1 \text{ -clean rings }\}\\ &\subsetneqq \{\text{ strongly } J_2 \text{ -clean rings }\} \subsetneqq \{\text{ strongly } J_3 \text{ -clean rings }\}\\ &\subsetneqq \cdots \subsetneqq \{\text{ strongly } \pi \text{ -rad clean rings }\}.\end{aligned}$$

We say that B is a subring of a ring A provided that B is a non-empty subset of A such that for any $x, y \in B$, $x - y, xy \in B$ and that $1_A \in B$. Let B be a subring of A. Set $R[A, B] = \{(a_1, a_2, \cdots, a_n, b, b, \cdots) \mid a_i \in A, b \in B, n \geq 1\}$. Then $R[A, B]$ is a ring under the general addition and multiplication.

Theorem 3.4. *Let B be a subring of a ring A, and let n be a fixed natural number. Then $R[A, B]$ is strongly J_n-clean if and only if*

(1) *A is strongly J_n-clean.*
(2) *For any $x \in B$, there exists an idempotent $e \in B$ such that $x - e \in U(B), ex = xe$ and $(ex)^n \in J(A) \bigcap J(B)$.*

Proof. Suppose $R[A, B]$ is strongly J_n-clean. Then A is strongly J_n-clean by Theorem 3.2. For any $x \in B$, we see that $(0, x, x, \cdots) \in R[A, B]$. By hypothesis, there exist an idempotent $(f_1, \cdots, f_s, f, f, \cdots) \in R[A, B]$ and a unit $(v_1, \cdots, v_t, v, v, \cdots) \in R[A, B]$ such that

$$(0, x, x, \cdots) = (f_1, \cdots, f_s, f, f, \cdots) + (v_1, \cdots, v_t, v, v, \cdots)$$

is a strongly clean expression and $(f_1, \cdots, f_s, f, f, \cdots)(0, x, x, \cdots)^n \in J\big(R[A,B]\big)$. This implies that $x = f + v$ is a strongly clean expression in B. One easily checks that $J\big(R[A,B]\big) = \big[J(A), J(A)\bigcap J(B)\big]$. As a result, we see that $fx^n \in J(A)\bigcap J(B)$, as required.

Conversely, assume that (1) and (2) hold. Let $(a_1, \cdots, a_m, a, a, \cdots) \in R[A,B]$. Then we have strongly clean expressions $a_1 = e_1 + u_1, \cdots, a_m = e_m + u_m, a = e + u$, and that $e_1 a_1^n, \cdots, e_m a_m^n \in J(A), ea^n \in J(A)\bigcap J(B)$. Thus, we have a strongly clean decomposition $(a_1, \cdots, a_m, a, a, \cdots) = (e_1, \cdots, e_m, e, e, \cdots) + (u_1, \cdots, u_m, u, u, \cdots)$. In addition,
$(e_1, \cdots, e_m, e, e, \cdots)(a_1, \cdots, a_m, a, a, \cdots)^n \in R\big[J(A), J(A)\bigcap J(B)\big]$.
Therefore $R[A,B]$ is strongly J_n-clean. □

Example 3.5. *Let D be a division ring, and let $A = M_2(D)$, $B = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D\}$. Then $R[A,B]$ is strongly J_2-clean.*

Proof. In view of Example 2.5, A is strongly J_2-clean. Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in B$. If $a \neq 0$, then $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is a strongly clean expression. Obviously, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in J(A)\bigcap J(B)$. If $a = 0$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ is a strongly clean expression. In addition, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^2 \in J(A)\bigcap J(B)$. According to Theorem 3.4, $R[A,B]$ is strongly J_2-clean. In this case, $R[A,B]$ is not strongly J_1-clean. □

Example 3.6. *Let D be a division ring, and let $A = M_3(D)$, $B = \{\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in D\}$. Then $R[A,B]$ is strongly J_3-clean.*

Proof. In view of Example 2.5, A is strongly J_3-clean. Let $\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in B$. If $a \neq 0$, then $\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ is a strongly clean expression. Obviously, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in$ $J(A) \bigcap J(B)$. If $a = 0$,

then $\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & b & c \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is a strongly clean expression. In addition, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^3 \in J(A) \bigcap J(B)$. By virtue of Theorem 3.4, $R[A, B]$ is strongly J_3-clean. In this case, $R[A, B]$ is not strongly J_2-clean. □

4. **Matrices over local rings**

Let $a \in R$. Let $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$. Following Diesl, a local ring R is bleached provided that for any $a \in U(R), b \in J(R)$, $l_a - r_b, l_b - r_a$ are both surjective. Let $T_2(R)$ denote the ring of all upper triangular 2×2 matrices over R. For J_n-cleanness of $T_2(R)$, we can derive the following.

Proposition 4.1. *Let R be a local ring, and let n be a fixed natural number. Then the following are equivalent:*

(1) *$T_2(R)$ is strongly J_n-clean.*
(2) *$T_2(R)$ is strongly J_1-clean.*
(3) *$T_2(R)$ is strongly π-rad clean.*
(4) *R is bleached.*

Proof. (1) ⇒ (3) is trivial.

(3) ⇒ (4) Since $T_2(R)$ is strongly π-rad clean, it follows from [6, Theorem 4.2.4] that R is bleached, as desired.

(4) ⇒ (2) As R is bleached, it follows from [6, Theorem 4.2.4] that $T_2(R)$ is strongly rad clean, and so R is strongly J_1-clean.

(2) ⇒ (1) is trivial. □

Let R be a commutative local ring. Then $T_2(R)$ is strongly J_n-clean for all $n \in \mathbb{N}$. But it is hard to determine the strongly J_n-cleanness for general case. In the next, we discuss the strongly J_n-cleanness for full matrices.

Lemma 4.2. *Let R be a local ring, and let $E \in M_2(R)$ be idempotent. Then $E \in M_2(R)$ is similar to an idempotent matrix.*

Proof. In view of [4, Lemma 16.4.11], $E \in GL_2(R)$ or $I_2 - E \in GL_2(R)$ or there exists some $P \in GL_2(R)$ such that $P^{-1}EP = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$. If $E \in GL_2(R)$, then $E = I_2$. If $I_2 - E \in GL_2(R)$, then $E = 0$. If there exists some $P \in GL_2(R)$ such that $P^{-1}EP = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, then $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} \in M_2(R)$ is idempotent; hence, $\lambda = 0$ and $\mu = 1$. One easily checks that

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, E is similar to an idempotent matrix. □

Lemma 4.3. *Let $a \in R, u \in U(R)$, and let n be a fixed natural number. Then the following are equivalent:*

(1) *$a \in R$ is strongly J_n-clean.*
(2) *$uau^{-1} \in R$ is strongly J_n-clean.*

Proof. (1) ⇒ (2) Write $a = e + w, e = e^2, w \in U(R), ew = we$ and $ea^n \in J(R)$. Then $uau^{-1} = ueu^{-1} + uwu^{-1}$. Clearly, $ueu^{-1} \in R$ is an idempotent and $uwu^{-1} \in U(R)$. In addition, $\big(ueu^{-1}\big)\big(uwu^{-1}\big) = \big(uwu^{-1}\big)\big(ueu^{-1}\big)$. Further, $(ueu^{-1})(uau^{-1})^n = uea^nu^{-1} \in J(R)$. Therefore, $uau^{-1} \in R$ is strongly J_n-clean.

(2) ⇒ (1) is proved in the same manner. □

Theorem 4.4. *Let R be a local ring, and let n be a fixed natural number. Then $A \in M_2(R)$ is strongly J_n-clean if and only if $A \in GL_2(R)$ or $A^n \in J\big(M_2(R)\big)$ or A admits a diagonal reduction.*

Proof. If $A \in GL_2(R)$, then $A = 0 + A$ is a strongly J_n-clean decomposition. If $A^n \in J\big(M_2(R)\big)$, then $A = I_2 + (A - I_2)$ is a strongly J_n-clean decomposition. Suppose A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. If

$\alpha \in J(R), \beta \in U(R)$, then

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix}$$

is a strongly J_1-clean decomposition. In view of Lemma 4.3, A is strongly J_1-clean. If $\alpha \in U(R), \beta \in J(R)$, similarly, A is strongly J_1-clean. If $\alpha, \beta \in U(R)$, then $A \in GL_2(R)$. If $\alpha, \beta \in J(R)$, then $A \in J\big(M_2(R)\big)$, and so A is strongly J_1-clean. Thus one direction is proved.

Conversely, assume that $A \in M_2(R)$ is strongly J_n-clean. Then there exists an idempotent $E \in M_2(R)$ and a $W \in GL_2(R)$ such that $A = E + W, EW = WE$ and $EA^n \in J\big(M_2(R)\big)$. According to Lemma 4.2, there exists $K \in GL_2(R)$ such that $KEK^{-1} = diag(f_1, f_2)$, where $f_1, f_2 \in R$ are idempotents. As R is local, every idempotent in R is 0 or 1. If $f_1 = f_2 = 0$, then $A \in GL_2(R)$. If $f_1 = f_2 = 1$, then $E = I_2$; hence, $A^n \in J\big(M_2(R)\big)$. So we may assume that $f_1 = 1, f_2 = 0$ or $f_1 = 0, f_2 = 1$. Thus, there exists some $H \in GL_2(R)$ such that $HEH^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Hence

$$HAH^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + HWH^{-1}.$$

Set $V = (v_{ij}) := HWH^{-1}$. It follows from $EW = WE$ that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and so $v_{12} = v_{21} = 0$ and $v_{11}, v_{22} \in U(R)$. We conclude that $HAH^{-1} = \begin{pmatrix} 1 + v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$, and therefore A admits a diagonal reduction. □

Corollary 4.5. *Let R be a local ring, and let $n \geq 2$ be a fixed natural number. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly J_n-clean.*
(2) *$A \in M_2(R)$ is strongly J_2-clean.*
(3) *$A \in M_2(R)$ is strongly π-rad clean.*

Proof. (1) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2) Suppose $A \in M_2(R)$ is strongly π-rad clean. Then there exists some $m \in \mathbb{N}$ such that $A \in M_2(R)$ is strongly J_m-clean. By virtue of Theorem 4.4, $A \in GL_2(R)$ or $A^n \in J\big(M_2(R)\big)$ or A admits a diagonal reduction. If $A^n \in J\big(M_2(R)\big)$, then $\overline{A}^n = \overline{0}$ in $M_2\big(R/J(R)\big)$. As $R/J(R)$ is a division ring, it follows by [8, Theorem 7.2] that $M_2\big(R/J(R)\big)$ is a

regular ring of bounded index 2. Hence, $\overline{A}^2 = \overline{0}$, and so $A^2 \in J\big(M_2(R)\big)$. By using Theorem 4.4 again, $A \in M_2(R)$ is strongly J_2-clean.

(2) ⇒ (1) is obvious. □

Corollary 4.6. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *$M_2(R)$ is strongly clean.*
(2) *$M_2(R)$ is strongly J_2-clean.*
(3) *$M_2(R)$ is strongly π-rad clean.*

Proof. (1) ⇒ (3) is clear by [6, Corollary 4.3.7].

(3) ⇒ (2) is obvious from Corollary 4.5.

(2) ⇒ (1) is trivial. □

We naturally ask whether $M_2(R)$ is strongly clean if and only if $M_2(R)$ is strongly J_1-clean? The answer is negative. For instance, letting $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2\big(\mathbb{Z}_2\big)$, one easily checks that A is not strongly J_1-clean, and so $M_2\big(\mathbb{Z}_2\big)$ is not strongly J_1-clean, but $M_2\big(\mathbb{Z}_2\big)$ is strongly clean.

Corollary 4.7. *Let R be a commutative local ring. Then $A \in M_2(R)$ is strongly clean if and only if $A \in GL_2(R)$ or $I_2 - A \in GL_2(R)$ or $A \in M_2(R)$ is strongly J_1-clean.*

Proof. One direction is obvious. Conversely, assume that $A \in M_2(R)$ is strongly clean. If $A \notin GL_2(R)$ or $I_2 - A \notin GL_2(R)$, it follows by [4, Corollary 16.4.7] that there exists some $U \in GL_2(R)$ such that $UAU^{-1} = diag(1+\alpha, \beta)$, where $\alpha, \beta \in U(R)$. Thus, $A \in M_2(R)$ is strongly J_1-clean by Theorem 4.4. □

Recall that $A \in M_2(R)$ is purely singular provided that $A, I_2 - A \notin GL_2(R)$.

Corollary 4.8. *Let R be a commutative local ring, and let $A \in M_2(R)$ be purely singular. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly J_1-clean.*
(2) *$A \in M_2(R)$ is strongly J_2-clean.*
(3) *$A \in M_2(R)$ is strongly clean.*

Proof. (1) ⇒ (2) ⇒ (3) are trivial.

(3) ⇒ (2) Since $A \in M_2(R)$ is purely singular, we complete the proof by Corollary 4.7. □

Let $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m, n \in \mathbb{N}, 2 \nmid n\}$. Then $\left(\begin{array}{cc} 1 & 1 \\ -\frac{2}{9} & 0 \end{array}\right) \in M_2\big(\mathbb{Z}_{(2)}\big)$ is strongly J_1-clean for all $n \in \mathbb{N}$, by Corollary 4.8.

Let $\chi(A)$ denote the characteristic polynomial of a matrix $A \in M_2(R)$, i.e., $\chi(A) = x^2 - tr(A)x + det(A)$. Furthermore, we can derive the following.

Corollary 4.9. *Let R be a commutative local ring. Then $A \in M_2(R)$ is strongly J_2-clean if and only if $det(A) \in U(R)$, or $\chi(A) \equiv t^2\big(mod\ J(R)\big)$, or A admits a diagonal reduction.*

Proof. Suppose $A \in M_2(R)$ is strongly J_2-clean. In light of Theorem 4.4, $A \in GL_2(R)$ or $A^2 \in J\big(M_2(R)\big)$ or A admits a diagonal reduction. If $A^2 \in J\big(M_2(R)\big)$, then $\overline{A} \in M_2\big(R/J(R)\big)$ is nilpotent. Therefore, $\chi(\overline{A}) \equiv t^2\big(mod\ N(R/J(R))\big)$. Write $\chi(A) = t^2 + at + b$. Then there exists some $m \in \mathbb{N}$ such that $a^m \in J(R)$. As $J(R)$ is maximal, it is prime. Hence, $a \in J(R)$. Likewise, $b \in J(R)$. This shows that $\chi(A) \equiv t^2\big(mod\ J(R)\big)$, as required.

Conversely, assume that $det(A) \in U(R)$, or $\chi(A) \equiv t^2\big(mod\ J(R)\big)$, or A admits a diagonal reduction. If $\chi(A) \equiv t^2\big(mod\ J(R)\big)$, then $\chi(\overline{A}) \equiv t^2\big(mod\ R/J(R)\big)$. This implies that $\overline{A}$ is nilpotent in $M_2\big(mod\ R/J(R)\big)$. Therefore, $A^m \in J\big(M_2(R)\big)$. In light of Theorem 4.4, $A \in M_2(R)$ is strongly J_m-clean, and so it is strongly π-rad clean. Accordingly, $A \in M_2(R)$ is strongly J_2-clean, by Corollary 4.6. □

5. **Criteria via quadratic equations**

Many authors characterized strong cleanness by means of quadratic equations. The purpose of this section is to study the strongly J_n-cleanness of a single 2×2 matrix over commutative local rings. It is shown that such property can be characterized by a kind of quadratic equations. We begin this section with two elementary lemmas.

Lemma 5.1. *Let R be a commutative local ring. Then $A \in M_2(R)$ is an idempotent if and only if $A = 0$ or $A = I_2$ or $A = \left(\begin{array}{cc} a & b \\ c & 1-a \end{array}\right)$ where $bc = a - a^2$ in R.*

Proof. See [4, Lemma 16.4.10]. □

Lemma 5.2. *Let R be a commutative ring, and let $A = (a_{ij}) \in M_2(R)$. If $a_{21} \in U(R)$ and $\chi(A)$ has two roots $x_1, x_2 \in R$ such that $x_1 - x_2 \in U(R)$, then A is similar to $diag(x_1, x_2)$.*

Proof. See [4, Lemma 16.4.30]. □

Lemma 5.3. *Let R be a commutative local ring, let n be a fixed natural number, and let $A = (a_{ij}) \in M_2(R)$ be strongly J_n-clean. Then $A \in GL_2(R)$ or $A^n \in J\big(M_2(R)\big)$, or A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$.*

Proof. If $A \notin GL_2(R)$ and $A^n \notin J\big(M_2(R)\big)$. It follows from Theorem 4.4 that there exists a $P \in GL_2(R)$ such that $P^{-1}AP = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in J(R), \beta \in U(R)$. Thus, $B_{21}(1)P^{-1}APB_{21}(-1) = \begin{pmatrix} \alpha & 0 \\ \alpha-\beta & \beta \end{pmatrix}$. As $\alpha - \beta \in U(R)$, it is easy to check that

$$\begin{aligned} & B_{12}\big(-\alpha(\alpha-\beta)^{-1}\big)B_{21}(1)P^{-1}APB_{21}(-1)B_{12}\big(\alpha(\alpha-\beta)^{-1}\big) \\ = & \begin{pmatrix} 0 & -\alpha(\alpha-\beta)^{-1}\beta \\ \alpha-\beta & (\alpha-\beta)\alpha(\alpha-\beta)^{-1}+\beta \end{pmatrix}. \end{aligned}$$

This implies that

$$\begin{aligned} & [\alpha-\beta, 1]B_{12}\big(-\alpha(\alpha-\beta)^{-1}\big)B_{21}(1)P^{-1}APB_{21}(-1) \\ & B_{12}\big(\alpha(\alpha-\beta)^{-1}\big)[(\alpha-\beta)^{-1}, 1] \\ = & \begin{pmatrix} 0 & -(\alpha-\beta)\alpha(\alpha-\beta)^{-1}\beta \\ 1 & (\alpha-\beta)\alpha(\alpha-\beta)^{-1}+\beta \end{pmatrix}. \end{aligned}$$

Let $\lambda = -(\alpha-\beta)\alpha(\alpha-\beta)^{-1}\beta$ and $\mu = (\alpha-\beta)\alpha(\alpha-\beta)^{-1}+\beta$.

As $\alpha \in J(R)$, we see that $\beta - (\beta-\alpha)\alpha(\alpha-\beta)^{-1} \in U(R)$, i.e., $\mu \in U(R)$. Therefore A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. □

Theorem 5.4. *Let R be a commutative local ring, and let $n \in \mathbb{N}$ be a fixed natural number. Then the following are equivalent:*

1. *$A \in M_2(R)$ is strongly J_n-clean.*
2. *$A \in GL_2(R)$ or $A^n \in J\big(M_2(R)\big)$, or $\chi(A)$ has a root in $J(R)$ and a root in $U(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $A \in M_2(R)$ be strongly J_n-clean. Assume that $A \notin GL_2(R)$ and $A^n \notin J\big(M_2(R)\big)$. In view of Theorem 4.4, A is similar to the matrix $B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M_2(R)$, where $\lambda^n \in J(R), \mu \in U(R)$. Thus,

$$\chi(A) = det(xI_2 - A) = det(xI_2 - B) = (x - \alpha)(x - \beta),$$

and so $\chi(A) = 0$ has a root α and a root β. As $J(R)$ is maximal, it is prime. Hence, $\alpha \in J(R)$, as required.

(2) $\Rightarrow$ (1) Let $A \in M_2(R)$. If $A \in GL_2(R)$ or $A^n \in J\big(M_2(R)\big)$, then $A \in M_2(R)$ is strongly J_n-clean. Otherwise, it follows by the hypothesis that the equation $\chi(A) = 0$ has a root $x_1 \in J(R)$ and a root $x_2 \in U(R)$. Clearly, $x_1 - x_2 \in U(R)$.

Case I. $a_{21} \in U(R)$. It follows from Lemma 5.2 that there exists a $P \in GL_2(R)$ such that $P^{-1}AP = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ for some $\alpha_1 \in J(R), \alpha_2 \in U(R)$. Thus,

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_1 - 1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

is a strongly J_n-clean expression. Therefore, $A \in M_2(R)$ is strongly J_n-clean by Lemma 4.3.

Case II. $a_{12} \in U(R)$. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix} \in M_2(R).$$

Applying Case I and Lemma 4.3, we see that $A \in M_2(R)$ is strongly J_1-clean.

Case III. $a_{12}, a_{21}, a_{11} \in J(R)$. Clearly, $tr(A) = x_1 + x_2$ and $det(A) = x_1x_2$. As $tr(A) \in U(R)$, $a_{22} \in U(R)$. Obviously,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12} & a_{12} \\ a_{21} + a_{22} - a_{11} - a_{12} & a_{22} - a_{12} \end{pmatrix} \in M_2(R),$$

where $a_{21} + a_{22} - a_{11} - a_{12} \in U(R)$. Applying Case I and Lemma 4.3, $A \in M_2(R)$ is strongly J_1-clean.

Case IV. $a_{12}, a_{21} \in J(R), a_{11} \in U(R)$. Clearly, $det(A) = x_1x_2 \in J(R)$. This implies that $a_{22} \notin U(R)$, and so $a_{22} \in J(R)$. Obviously,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12} & a_{12} \\ a_{21} + a_{22} - a_{11} - a_{12} & a_{22} - a_{12} \end{pmatrix} \in M_2(R),$$

where $a_{21} + a_{22} - a_{11} - a_{12} \in U(R)$. Applying Case I and Lemma 4.3, $A \in M_2(R)$ is strongly J_1-clean.

In any case, $A \in M_2(R)$ is strongly J_n-clean, as asserted. □

Let $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \in M_2\big(\mathbb{Z}_{(2)}\big)$. In light of Theorem 5.4, we claim that $A \in M_2\big(\mathbb{Z}_{(2)}\big)$ is strongly J_2-clean, but it is not strongly J_1-clean. Clearly, $A^2 \in J\big(M_2(\mathbb{Z}_{(2)})\big)$, and so $A \in M_2\big(\mathbb{Z}_{(2)}\big)$ is strongly J_2-clean. But $A \notin GL_2\big(\mathbb{Z}_{(2)}\big)$ and $A \notin J\big(M_2(\mathbb{Z}_{(2)})\big)$. On the other hand, χ_A has no root in $J\big(\mathbb{Z}_{(2)}\big)$. Accordingly, $A \in M_2\big(\mathbb{Z}_{(2)}\big)$ is not strongly J_1-clean, by Theorem 5.4. Furthermore, we can derive the following.

Corollary 5.5. *Let R be a commutative local ring, let $n \in \mathbb{N}$ be a fixed natural number, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly J_n-clean.*
(2) *$A \in GL_2(R)$ or $A^n \in J\big(M_2(R)\big)$, or the equation $x^2 - x = t_A^{-1}det(A)$ is a root in $J(R)$.*

Corollary 5.6. *Let R be a commutative local ring, let $n \in \mathbb{N}$ be a fixed natural number, and let $A \in M_2(R)$. If $\frac{1}{2} \in R$, then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly J_n-clean.*
(2) *$A \in GL_2(R)$ or $A^n \in J\big(M_2(R)\big)$, or $tr^2(A) - 4det(A) = u^2$ for some $u \in U(R)$.*

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