*Bulletin of the Iranian Mathematical Society Vol.* 39 *No.* 6 (*2013*), *pp 1117-1123.*

# CHARACTERISTIC FUNCTION OF A MEROMORPHIC FUNCTION AND ITS DERIVATIVES

J. WU AND Z. WU *

Communicated by Javad Mashreghi

Abstract. In this paper, some results of Singh, Gopalakrishna and Kulkarni (1970s) have been extended to higher order derivatives. It has been shown that, if $\sum_a \Theta(a,f)=2$ holds for a meromorphic function $f(z)$ of finite order, then for any positive integer $k$, $T(r,f) \sim T(r,f^{(k)}), r\to\infty$ if $\Theta(\infty,f)=1$ and $T(r,f^{(k)}) \sim (k+1)T(r,f), r\to\infty$ if $\Theta(\infty,f)=0$.

## 1. **Introduction**

Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$. Assume that basic definitions, theorems and standard notations of the Nevanlinna theory for meromorphic function (see [3], [10] or [12]) are known. We use the following notations of frequent use of value distribution (see [3]) with their usual meaning:

$$m(r,f), N(r,a), \overline{N}(r,a), \delta(a,f), \Theta(a,f), \cdots$$

As usual, if $a=\infty$, we write $N(r,\infty)=N(r,f), \overline{N}(r,\infty)=\overline{N}(r,f)$. We denote by $S(r,f)$ any quantity such that

$$S(r,f)=o(T(r,f)), \quad r\to+\infty$$

MSC(2010): Primary: 30D30; Secondary: 30D35.
Keywords: Characteristic function, Nevanlinna's deficiency, maximum deficiency sum.
Received: 20 July 2011, Accepted: 21 February 2012.
∗Corresponding author

without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of $r$ of finite linear measure. The well known Nevanlinna's deficiency relation states that

$$\sum_a \delta(a,f) \le \sum_a \Theta(a,f) \le 2.$$

If $\sum_a \delta(a,f) = 2$, then we say that $f(z)$ has maximum deficiency sum (see [2]).

Let $f(z)$ be a meromorphic scalar valued function in $\mathbb{C}$. On the characteristic function of derivative of $f(z)$ with maximum deficiency sum has been studied by Shan, Singh, Gopalakrishna, Edrei and Weitsman [1], [4]-[8]. For example, Edrei [1] and Weitsman [8] have proved

Theorem A Let $f(z)$ be a transcendental meromorphic function of finite order and assume $\sum_{a\in\mathbb{C}} \delta(a,f) = \eta \ge 1$ and $\delta(\infty) = 2-\eta$. Then

$$T(r,f') \sim \eta T(r,f), r \to +\infty.$$

If $\sum_a \delta(a,f) = 2$ is replaced by $\sum_a \Theta(a,f) = 2$, Singh, Gopalakrishna [6] and Singh, Kulkarni [7] have proved

Theorem B Let $f(z)$ be a transcendental meromorphic scalar valued function of finite order and assume $\sum_a \Theta(a,f) = 2$. Then

$$\lim_{r\to+\infty} \frac{T(r,f')}{T(r,f)} = 2 - \Theta(\infty).$$

Hence

(1) if $\Theta(\infty,f) = 1$, $T(r,f) \sim T(r,f')$ as $r \to \infty$;

(2) if $\Theta(\infty,f) = 0$, $T(r,f') \sim 2T(r,f)$ as $r \to \infty$.

We extend the above result to higher order derivatives as follows:

**Theorem 1.1.** *Suppose that $f$ is a transcendental meromorphic function of finite order and $\sum_a \Theta(a,f) = 2$. Then for any positive integer $k$, we have*

(1) *if $\Theta(\infty,f) = 1$, $T(r,f) \sim T(r,f^{(k)})$ as $r \to \infty$;*

(2) *if* $\Theta(\infty, f) = 0$, $T(r, f^{(k)}) \sim (k+1)T(r, f)$ *as* $r \to \infty$.

From Theorem 1.1, we can get

**Corollary 1.2.** [11] *Suppose that* $f$ *is a transcendental meromorphic function of finite order and* $\sum_a \delta(a, f) = 2$. *Then for any positive integer* $k$*, we have*

(1) *if* $\delta(\infty, f) = 1$, $T(r, f) \sim T\left(r, f^{(k)}\right)$ *as* $r \to \infty$;

(2) *if* $\delta(\infty, f) = 0$, $T(r, f^{(k)}) \sim (k+1)T\left(r, f\right)$ *as* $r \to \infty$.

## 2. **Proof of Theorem 1.1**

*Proof.* (1) We prove Theorem 1.1 (1) by induction. Since $\Theta(\infty, f) = 1$, by Theorem B, we have $T(r, f) \sim T(r, f')$ as $r \to \infty$. Assume that

$$T(r, f) \sim T(r, f^{(k)}), r \to \infty. \tag{2.1}$$

Now we prove $T(r, f) \sim T(r, f^{(k+1)})$ as $r \to \infty$.

Without loss of generality we can assume that $q \geq 2$. Put

$$F(z) = \sum_{i=1}^{q} \frac{1}{f(z) - a_i}, \quad a_i \in \mathbb{C}.$$

Then (See [6])

$$\sum_{i=1}^{q} m(r, a_i) \leq m\left(r, F\right) + O(1).$$

So

$$\begin{aligned}
\sum_{i=1}^{q} m(r, a_i) &\leq m\left(r, F\right) + O(1) \\
&= m\left(r, \frac{1}{f^{(k+1)}} F f^{(k+1)}\right) + O(1) \\
&\leq m\left(r, \frac{1}{f^{(k+1)}}\right) + m\left(r, \sum_{i=1}^{q} \frac{f^{(k+1)}}{f(z) - a_i}\right) + O(1) \\
&= m\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f).
\end{aligned}$$

Hence

$$\begin{aligned} qT(r,f) &\leq \sum_{i=1}^{q} N(r,a_i) + m\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &= \sum_{i=1}^{q} N(r,a_i) + T\left(r, f^{(k+1)}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \sum_{i=1}^{q} N(r,a_i) + T\left(r, f^{(k+1)}\right) - N\left(r, \frac{1}{f'}\right) + S(r,f) \\ &= T\left(r, f^{(k+1)}\right) + \sum_{i=1}^{q} \overline{N}(r,a_i) - N_0\left(r, \frac{1}{f'}\right) + S(r,f). \end{aligned}$$

where $N_0\left(r, \frac{1}{f'}\right)$ is formed with the zeros of $f'$ which are not zeros of any of the $f - a_i, i = 1, 2, \cdots, q$. Since $N_0\left(r, \frac{1}{f'}\right) \geq 0$, we have

$$qT(r,f) \leq T\left(r, f^{(k+1)}\right) + \sum_{i=1}^{q} \overline{N}(r,a_i) + S(r,f).$$

Thus

$$\sum_{i=1}^{q} \left(1 - \frac{\overline{N}(r,a_i)}{T(r,f)}\right) \leq \frac{T\left(r, f^{(k+1)}\right)}{T(r,f)} + \frac{S(r,f)}{T(r,f)}.$$

So

$$\sum_{i=1}^{q} \Theta(a_i, f) \leq \liminf_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r,f)},$$

holds for any $q \geq 2$. Letting $q \to \infty$, we obtain

$$1 = \sum_{a\neq\infty} \Theta(a,f) \leq \liminf_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r,f)} \tag{2.2}$$

Combining (2.1) and (2.2) we have

$$1 \leq \liminf_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r, f^{(k)})} \leq \limsup_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r, f^{(k)})}. \tag{2.3}$$

On the other hand, since $\overline{N}\left(r, f^{(k)}\right) = \overline{N}(r,f), \Theta(\infty, f) = 1$ and (2.1), we have

$$\limsup_{r\to\infty} \frac{\overline{N}\left(r, f^{(k)}\right)}{T\left(r, f^{(k)}\right)} \le \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} = 0.$$

So

$$\Theta\left(\infty, f^{(k)}\right) = 1.$$

Thus

$$\begin{aligned} T\left(r, f^{(k+1)}\right) = &\ m\left(r, f^{(k+1)}\right) + N\left(r, f^{(k+1)}\right) \\ \le &\, m\left(r, f^{(k)}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + N\left(r, f^{(k)}\right) + \overline{N}\left(r, f^{(k)}\right) \\ = &\ T\left(r, f^{(k)}\right) + \overline{N}\left(r, f^{(k)}\right) + S(r,f). \end{aligned}$$

Hence

$$\limsup_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T\left(r, f^{(k)}\right)} \le 2 - \Theta(\infty, f^{(k)}) = 1. \tag{2.4}$$

(2.1) and (2.3)-(2.4) together imply $T(r,f) \sim T\left(r, f^{(k+1)}\right)$ as $r \to \infty$.

We can prove (2) of Theorem 1.1 by using the same method as that in [11]. As [11] may not be abundantly available, we give the following proof. From Nevanlinna's second fundamental theorem, we have

$$(q-1)T(r,f) \quad \le \quad T(r,f) + \sum_{i=1}^{q} \overline{N}(r, a_i) + \overline{N}(r,f) + S(r,f).$$

Thus

$$\sum_{i=1}^{q} \Theta(a_i, f) \le 1 + \liminf_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} \le 1 + \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} \le 2.$$

Letting $q \to \infty$, we obtain

$$2 = \sum_{a\neq\infty} \Theta(a,f) \le 1 + \liminf_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} \le 1 + \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} \le 2$$

So

$$T(r,f) \sim N(r,f) \sim \overline{N}(r,f), r \to \infty. \tag{2.5}$$

Since

$$\begin{aligned}(k+1)\overline{N}(r,f) &\leq N(r,f)+k\overline{N}(r,f)=N\left(r,f^{(k)}\right)\\ &\leq T\left(r,f^{(k)}\right)\\ &\leq m(r,f)+m\left(r,\frac{f^{(k)}}{f}\right)+N\left(r,f^{(k)}\right)\\ &= T(r,f)+k\overline{N}(r,f)+S(r,f).\end{aligned}$$

From this and (2.5), we get $T(r,f^{(k)})\sim(k+1)T(r,f)$ as $r\to\infty$. □

## 3. Proof of Corollary 1.2

*Proof.* Since $\delta(a,f)\leq\Theta(a,f)$ for every $a\in\mathbb{C}\cup\{\infty\}$, if $\sum_a\delta(a,f)=2$, then $\sum_a\Theta(a,f)=2$ and $\delta(a,f)=\Theta(a,f)$ for every $a\in\mathbb{C}\cup\{\infty\}$. Hence Corollary 1.2 follows by Theorem 1.1. □

## Acknowledgments

This research was partially supported by the NNSF of China (Grant No. 11201395) and by NSF of Educational Department of the Hubei Province (Grant No. Q20132801, D20132804).

## References

[1] A. Edrei, Sums of deficiencies of meromorphic functions II, *J. Analyse Math.* **19** (1967) 53–74.
[2] M. L. Fang, A note on a result of Singh and Kulkarni, *Int. J. Math. Math. Sci.* **23** (2000) 285–288.
[3] W. K. Hayman, Meromorphic Functions, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.
[4] S. M. Shah and S. K. Singh, Borel's theorem on $a$-points and exceptional values of entire and meromorphic functions, *Math. Z.* **59** (1953) 88–93.
[5] S. M. Shah and S. K. Singh, On the derivative of a meromorphic function with maximum defect, *Math. Z.* **65** (1956) 171–174.
[6] S. K. Singh and H. S. Gopalakrishna, Exceptional values of entire and meromorphic functions, *Math. Ann.* **191** (1971) 121–142.
[7] S. K. Singh and V. N. Kulkarni, Characteristic function of a meromorphic function and its derivative, *Ann. Polon. Math.* **28** (1973) 123–133.

[8] A. Weitsman, Meromorphic functions with maximal deficiency sum and a conjecture of F. Nevanlinna, *Acta Math.* **123** (1969) 115–139.

[9] H. Wittich, Neuere Untersuchungen Über eindeutige Analytische Funktionen, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.

[10] L. Yang, Value Distribution Theory and its New Research, Science Press Beijing, Beijing, 1993.

[11] L. Z. Yang, Sum of deficiencies of meromorphic function and its characteristic function, *J. Shandong Univ. Nat. Sci.* **24** (1989) 7–11.

[12] J. H. Zheng, Value Distribution of Meromorphic Functions, Tsinghua University Press Beijing, Beijing; Springer, Heidelberg, 2010.

**Jia Wu**
Xianning Vocational and Technical College, P.O. Box 437100, Xianning, P. R. China
Email: 44976882@qq.com

**Zhaojun Wu**
School of Mathematics and Statistics, Hubei University of Science and Technology, P.O. Box 437100, Xianning, P. R. China
Email: wuzj52@hotmail.com