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THE STREAMLINE DIFFUSION METHOD WITH IMPLICIT INTEGRATION FOR THE MULTI-DIMENSIONAL FERMI PENCIL BEAM EQUATION

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ABSTRACT. We derive error estimates in the appropriate norms, for the streamline diffusion (SD) finite element methods for steady state, energy dependent, Fermi equation in three space dimensions. These estimates yield optimal convergence rates due to the maximal available regularity of the exact solution. High order SD method together with implicit integration are used. The formulation is strongly consistent in the sense that the derivative in the penetration is included in the stabilization term. Here our focus is on theoretical aspects of the h and hp approximations in SD settings.

1. **Introduction**

In this paper we shall consider a pencil beam of particles normally incident on a slab of finite thickness. The particles enter at a single point, say at $(x, y, z) = (0, 0, 0)$, in the direction of positive x-axis ($\mu = 1$). We assume the mean scattering angle is small ($\bar{\mu}_0 \approx 1$) and that large-scattering is negligible. (This is often valid assumption for charged particle, such as electrons, protons, or heavy ions.) Thus the beam will gradually broaden as it advances into slab. The problem of determining quantitatively how such a beam broadens was first considered in 1940

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by Fermi([15]) who, using physical reasoning, derived a monoenergetic model equation with a closed form solution. Fermi's work was motivated by the study of cosmic rays in the atmosphere. This physical problem has applications in such diverse fields as astrophysics, material science, electron microscopy, and radiation therapy. The Fermi equation is obtained either as an asymptotic limit of the Fokker-Planck equation as the transport cross-section (σ_{tr}) gets smaller or as an asymptotic limit of the transport (linear Boltzmann) equation for vanishing transport cross-section and high (tends to ∞) total cross-section (σ_t). It can be shown that under appropriate conditions, the linear Boltzmann and Fokker-Planck equations do in fact have the same leading-order approximation (the Fermi equation) for pencil beam problems. For details in derivation of Fermi equation we refer to [11]. (The physical quantities σ_{tr} and σ_t are defined below). The Boltzmann equation for the basic pencil beam transport problem with no absorption and no energy-dependencies is given by

$$
\begin{aligned}
&\mu\frac{\partial\psi}{\partial\psi}+\eta\frac{\partial\psi}{\partial y}+\xi\frac{\partial\psi}{\partial z}=\int_{4\pi}\sigma_s(\underline{\Omega}\cdot\underline{\Omega}')\left[\psi(\underline{x},\underline{\Omega}')-\psi(x,\underline{\Omega})\right]d^2\underline{\Omega}',\\
&\qquad 0<x<1,\\
&\psi(0,y,z,\underline{\Omega})=\delta(y)\delta(y)\frac{\delta(1-\mu)}{2\pi},\qquad 0<\mu\le 1,\\
&\psi(1,y,z,\underline{\Omega})=0,\qquad -1\le\mu<0.
\end{aligned}
\tag{1.1}
$$

Here we follow the conventional notations. We use dimensionless spatial variables, scaled so that the slab width is unity. The slab width in units of mean free paths is σ_t^{-1}, with $\sigma_t = 2\pi\int_{-1}^{1}\sigma_s(\mu_0)d\mu_0$. We use the notation

$$
\underline{\Omega}=(\mu,\eta,\xi),\qquad \eta=\sqrt{1-\mu^2}\cos\phi,\qquad \xi=\sqrt{1-\mu^2}\sin\phi.
$$

The differential scattering cross section, for the case of no absorption, has the expansion

$$
\sigma_s(\mu_0)=\sigma_t\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}f_nP_n(\mu_0),\qquad f_0=1,\qquad f_1=\bar{\mu}_0. \tag{1.2}
$$

with P_n being the nth Legendre polynomial. The Fokker-Planck approximation to this transport equation is given by

(1.3)
$$\mu\frac{\partial\psi}{\partial\psi}+\eta\frac{\partial\psi}{\partial y}+\xi\frac{\partial\psi}{\partial z}=\frac{\sigma_{tr}}{2}\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}+\frac{1}{1-\mu^2}\frac{\partial^2}{\partial\phi^2}\right]\psi(x,\underline{\Omega}),\quad 0<x<1,$$
$$\psi(0,y,z,\underline{\Omega})=\delta(y)\delta(z)\frac{\delta(1-\mu)}{2\pi},\qquad 0<\mu\le 1,$$
$$\psi(1,y,z,\underline{\Omega})=0,\qquad -1\le\mu<0.$$

where $\sigma_{tr}=\sigma_t(1-\bar{\mu}_0)$. The Fokker-Planck equation (1.3) is usually derived from transport equation (1.1) by assuming that small-angle scattering dominates large-angle scattering, and by expanding the angular flux in the integral in equation (1.1) into a Taylor series about $\underline{\Omega}'=\underline{\Omega}$, retaining only the second order terms. Fermi proposed the following model to approximate pencil beam problems for $\sigma_{tr}\ll 1$:

$$\frac{\partial\psi}{\partial x}+\eta\frac{\partial\psi}{\partial y}+\xi\frac{\partial\psi}{\partial z}=\frac{\sigma_{tr}}{2}\left(\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\xi^2}\right)\psi(x,\underline{\Omega}),\qquad 0<x<1,\tag{1.4}$$
$$\psi(0,y,z,\eta,\xi)=\delta(y)\delta(z)\delta(\eta)\delta(\xi).$$

Fermi obtained this model using physical reasoning, not as an approximation to the transport or Fokker-Planck equations. The main virtue of the Fermi equation is that by artificially extending the range of η and ξ to the entire real line and by Fourier transforming with respect to y,z,η and ξ, one can obtain the exact solution for $\sigma_{tr}=\sigma_{tr}(x)$. However, it is not generally possible to derive an exact form solution for $\sigma_{tr}=\sigma_{tr}(x,y,z)$. In this paper we study the approximate solution for the three-dimensional Fermi pencil beam equation using high order stabilization methods. We prove stability estimates and derive optimal convergence rates for the weighted current function, as in the convection dominated convection diffusion problems. This work extends the results introduced in [2] to the case of the multidimensional Fermi equation. The SD-method and DG-method for Fermi equation in two space dimensions are studied there and error bounds of order $\mathcal{O}(h^{k+1/2})$ are given for the weighted current function. A posteriori error estimates are also studied in [3]. We refer to [6] which considers different stabilization techniques for Vlasov-Poisson-Fokker-Planck System using continuous and discontinuous space-time elements. The *hp*-analysis are also investigated in [8]. Some fullydiscrete schemes with numerical results can be found in [7]. There are several points of concern with this type of problems: The Fermi equation considered in this paper is degenerate in both convection and diffusion in the sense that drift and diffusion are taking place in, physically, different domains. Besides the problem is convection dominated since the diffusion term has a very small coefficient compared to the coefficient of the convection term. Furthermore, the problem is associated with a boundary condition in form of product of certain δ functions, which are not suitable for numerical consideration involving L_2 norms. We have therefore considered model problems with somewhat smoother data approaching Dirac δ function. Finally, in spite of the assumption of no back-scattering, i.e., the scattering angle $-\pi/2 \leq \theta \leq \pi/2$, we still need to restrict the range of θ, through focusing or filtering, and avoid small intervals in vicinity of the endpoints $\pm\pi/2$, in order to get, after scaling, bounded computational domains relevant in numerical considerations. The streamline diffusion method (SD-method) is a generalized form of the standard Galrekin method designed for the finite element studies of the hyperbolic problems, giving good stability and high accuracy. The SD-method which is used for our purpose in this paper is obtained by modifying the test function through adding a multiple of the "drift-terms" involved in the equation to the usual test function. This yields a weighted least square control of the residual of the finite element solution. See, e.g., [17] and [18] and the references therein for further details in the SD method. Here we have considered both h and hp versions of SD methods. As for numerical implementation, a characteristic method, as well as a semi-streamline diffusion for Fermi pencil beam equation have been studied in [2] and [7], respectively. An outline of this paper is as follows: In Section 2, we introduce the model problem and present some notations. Section 3 is devoted to the study of stability estimates and proof of the convergence rates for the, h, streamline diffusion approximation of the Fermi equation. In Section 4 the hp analysis of streamline diffusion with implicit integration over x variable are illustrated and studied.

2. **Notations and preliminaries**

By introducing the new angular variables

$$
\begin{aligned}
v_1 &= \frac{\eta}{\mu} = \frac{\eta}{\sqrt{1-\eta^2-\xi^2}},\\
v_2 &= \frac{\xi}{\mu} = \frac{\xi}{\sqrt{1-\eta^2-\xi^2}},
\end{aligned}
\tag{2.1}
$$

in (1.4) we consider a model problem for three dimensional Fermi equation on a bounded polygonal domains $\Omega_{\mathbf{x}} \subset \mathbb{R}^3$ with velocities $v \in \Omega_v \subset \mathbb{R}^2$:

$$
\begin{cases}
\frac{\partial f}{\partial x} + v\cdot\nabla_\perp f = \frac{\sigma_{tr}}{2}(\Delta_v f), & \text{in } (0,L]\times\Omega,\\
f(0,x_\perp,v) = f_0(x_\perp,v), & \text{in } \Omega = \Omega_{x_\perp}\times\Omega_v,\\
f(x,x_\perp,v) = 0, & \\
\qquad \text{in } (0,L]\times([\Gamma_v^-\times\Omega_v]\cup[\Omega_{x_\perp}\times\partial\Omega_v]), &
\end{cases}
\tag{2.2}
$$

where $f_0 \in L_2(\Omega)$, and the outflow boundary is given by

$$
\Gamma_v^- = \{x_\perp \in \partial\Omega_{x_\perp} : \mathbf{n}(x_\perp)\cdot v < 0\}, \quad \text{for } v \in \Omega_v. \tag{2.3}
$$

Here $\mathbf{n}(x_\perp)$ is the outward unit normal to $\partial\Omega_{x_\perp}$ at the point $x_\perp \in \partial\Omega_{x_\perp}$, $x_\perp = (y,z)$, $v = (v_1,v_2)$, $\nabla_\perp = (\frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ and, $\sigma_{tr} = \sigma_{tr}(x,y,z)$ is the transport cross-section (actually $\sigma_{tr} = \sigma_{tr}[E(x,y,z)]$ is energy dependent).
We shall use a finite element structure on $\Omega_{x_\perp}\times\Omega_v$: by letting $T_h^{x_\perp} = \{\tau_{x_\perp}\}$ and $T_h^v = \{\tau_v\}$ be finite element subdivisions of $\Omega_{x_\perp}$ and Ω_v, into the elements $\tau^{x_\perp}$ and τ^v, respectively. Thus, $T_h = T_h^{x_\perp}\times T_h^v = \{\tau_{x_\perp}\times\tau_v\} = \{\tau\}$ will be a subdivision of $\Omega = \Omega_{x_\perp}\times\Omega_v$ with elements $\{\tau_{x_\perp}\times\tau_v\} = \{\tau\}$. We also use the partition $0 = x_0 < x_1 < \ldots < x_M = L$ of the interval $I = (0,L]$ into subintervals $I_m = (x_{m-1}, x_m)$, $m = 1,...,M$. Now, let $\mathcal{C}_h$ be the corresponding subdivision of $Q_L := (0,L]\times\Omega$ into elements $K = I_m\times\tau$ with the mesh parameter $h = \text{diam } K$. Let $P_p(K)$ be the set of all polynomials of degree at most p on K; in $x, x_\perp$ and v, and define the finite element space

$$
V_h = \{g \in \widetilde{\mathcal{H}}_0 : g\circ F_K \in P_p(\hat{K});\ \forall K \in \mathcal{C}_h\}, \tag{2.4}
$$

with

$$
\widetilde{\mathcal{H}}_0 = \prod_{m=1}^{M} H_0^1(S_m), \qquad S_k = I_k\times\Omega, \qquad k = 1,\cdots,M. \tag{2.5}
$$

and

$$H_0^1(S_m) = \{g \in H^1(S_m) : g \equiv 0 \quad \text{on } \partial\Omega_v\}. \tag{2.6}$$

Moreover

$$\begin{array}{ll} (f,g)_m = (f,g)_{S_m}, & \|g\|_m^2 = (g,g)_m, \\ \langle f,g\rangle_m = (f(x_m,.,.),g(x_m,.,.))_\Omega, & |g|_m^2 = \langle g,g\rangle_m, \\ \langle f,g\rangle_{\Gamma^-} = \int_{\Gamma^-} fg(\beta\cdot\mathbf{n})ds, & \langle f,g\rangle_{\Gamma_m^-} = \int_{I_m}\langle f,g\rangle_{\Gamma^-}ds, \\ \langle f,g\rangle_{\Gamma_I^-} = \int_I \langle f,g\rangle_{\Gamma^-}ds, & \end{array} \tag{2.7}$$

where

$$\Gamma^- = \{(x_\perp, v) \in \Gamma = \partial(\Omega_{x_\perp} \times \Omega_v) : \beta\cdot\mathbf{n} < 0\},$$

$\beta = (v, \mathbf{0})$ and $\mathbf{n} = (\mathbf{n}_{x_\perp}, \mathbf{n}_v)$ with $\mathbf{n}_{x_\perp}$ and $\mathbf{n}_v$ being outward unit normals to $\partial\Omega_{x_\perp}$ and $\partial\Omega_v$, respectively. Throughout the paper C will denote a constant not necessarily the same at each occurrence and independent of the parameters and functions involved in the problem, unless otherwise specifically specified. Finally, for piecewise polynomials w_i defined on the triangulation $\mathcal{C}_h' = \{K\}$ with $\mathcal{C}_h' \subset \mathcal{C}_h$ and for D_i being some differential operators, we use the notation,

$$(D_1w_1, D_2w_2)_{Q'} = \sum_{K\in\mathcal{C}_h'} (D_1w_1, D_2w_2)_K, \qquad Q' = \bigcup_{K\in\mathcal{C}_h'} K, \tag{2.8}$$

where $(.,.)_Q$ is the usual $L_2(Q)$ scalar product and $\|.\|_Q$ is the corresponding $L_2(Q)$-norm.

3. Streamline diffusion method

3.1. **Streamline diffusion method.** For σ_{tr} constant or $\sigma_{tr} = \sigma_{tr}(x)$ one can obtain closed form analytic solution for the Fermi equation. Below we use a variational formulation, with a test functions consisting of the sum of a trial function g and an extra streaming term: $h(\partial_x g + v\cdot\partial_{x_\perp} g)$, i.e., we use test functions different from the trial functions, therefore we are dealing with a kind of Petrov-Galerkin method. We prove a stability lemma for the discrete problem in general three dimensional case, i.e., with $\sigma = \sigma_{tr} = \sigma_{tr}(x,y,z)$, using also the corresponding variational formulation we derive our first a priori error estimate. In our studies the parameter σ is, basically, of the order of mesh size or smaller. For Fermi equation (2.2) we define continuous variational formulation as: Find $f \in H^1(Q_L)$ such that for all $g \in H^1(Q_L)$,

(3.1)
$$(f_x + v\cdot\nabla_\perp f, g + \delta(g_x + v\cdot\nabla_\perp g))_{Q_L} + \sigma(\nabla_v f, \nabla_v g)_{Q_L}$$
$$-\delta\sigma(\Delta_v f, g_x + v\cdot\nabla_\perp g)_{Q_L} + \langle f, g\rangle_0 - \langle f, g\rangle_{\Gamma^-} = \langle f_0, g\rangle_0.$$

where δ is of order of mesh size. To proceed, we introduce the corresponding bilinear form

(3.2)
$$B(f,g) = (f_x + v\cdot\nabla_\perp f, g + h(g_x + v\cdot\nabla_\perp g))_{Q_L} + \sigma(\nabla_v f, \nabla_v g)_{Q_L}$$
$$- h\sigma(\Delta_v f, g_x + v\cdot\nabla_\perp g)_{Q_L} + \langle f, g\rangle_0 - \langle f, g\rangle_{\Gamma^-}.$$

Now our objective is to solve the following discrete variational problem: Find $f^h \in V^h$ such that

$$B(f^h, g) = L(g), \qquad \forall g \in V_h, \tag{3.3}$$

where

$$L(g) = \langle f_0, g\rangle_0. \tag{3.4}$$

Below we shall show that the bilinear form B is coercive:

Lemma 3.1. *There is a constant C such that*

$$B(g,g) \ge C|||g|||^2, \qquad \forall g \in V_h, \tag{3.5}$$

where

$$|||g|||^2 = [\sigma\|\nabla_v g\|^2_{Q_L} + |g|^2_M + |g|^2_0 + h \parallel g_x + v\cdot\nabla_\perp g \parallel^2_{Q_L}$$
$$+ \int_{I\times\partial\Omega} g^2 \mid \beta.\mathbf{n} \mid dvds].$$

Proof. We let $f^h = g$ in (3.2). Then,

(3.6)
$$B(g,g) = h\|g_x + v\cdot\nabla_\perp g\|^2_{Q_L} + \sigma\|\nabla_v g\|^2_{Q_L} - h\sigma(\Delta_v g, g_x + v\cdot\nabla_\perp g)_{Q_L}$$
$$- \langle g, g\rangle_{\Gamma_I^-} + (g_x, g)_{Q_L} + \langle g, g\rangle_0 + (v\cdot\nabla_\perp g, g)_{Q_L}.$$

By a partial integration, we have that

$$(g_x, g) = \tfrac{1}{2}\langle g, g\rangle \mid_{x_0}^{x_M} = \tfrac{1}{2}\int_\Omega [g^2(x_M) - g^2(x_0)], \tag{3.7}$$

and also since $\beta = (v, \mathbf{0})$,

$$(3.8)\quad \begin{aligned}(v \cdot \nabla_\perp g, g) - \langle g, g\rangle_{\Gamma_I^-} &= \frac{1}{2}\int_{I\times\partial\Omega} g^2(\beta\cdot\mathbf{n})dv - \int_{I\times\Gamma^-} g^2(\beta\cdot\mathbf{n})dv \\ &= \frac{1}{2}\int_{I\times\partial\Omega} g^2|\beta\cdot\mathbf{n}|dv.\end{aligned}$$

Now since $\sigma = \sigma_{tr}$ is independent of velocity variable v, we may use the inverse estimate and assumption on σ to obtain

$$h\sigma(\Delta_v g, g_x + v\cdot\nabla_\perp g)_{Q_L} \le \tfrac{1}{2}(\sigma\|\nabla_v g\|_{Q_L}^2 + h\|g_x + v\cdot\nabla_\perp g\|_{Q_L}^2).$$

Thus our bilinear form will satisfy

$$(3.9)\quad \begin{aligned}B(g,g) \ge{} & h\|g_x + v\cdot\nabla_\perp g\|_{Q_L}^2 + \sigma\|\nabla_v g\|_{Q_L}^2 + \frac{1}{2}\langle g,g\rangle_0 \\ & + \frac{1}{2}\langle g,g\rangle_M - \frac{1}{2}h\|g_x + v\cdot\nabla_\perp g\|_{Q_L}^2 - \frac{1}{2}\sigma\|\nabla_v g\|_{Q_L}^2 = \frac{1}{2}|||g|||^2,\end{aligned}$$

which gives the desired result. □

We shall also need the following interpolation error estimates, see Ciarlet [14]: Let $f \in H^{r+1}(\Omega)$ then there exists an interpolant $\tilde{f}^h \in V_h$ of f such that

$$(3.10)\quad \|f - \tilde{f}^h\| \le Ch^{r+1}\|f\|_{r+1},$$

$$(3.11)\quad \|f - \tilde{f}^h\|_1 \le Ch^{r}\|f\|_{r+1},$$

$$(3.12)\quad |f - \tilde{f}^h| \le Ch^{r+1/2}\|f\|_{r+1}.$$

Let $\eta = f - \tilde{f}^h$ be the interpolation error and set $\xi = f^h - \tilde{f}^h$. We may write the error as

$$(3.13)\quad e = f - f^h = \eta - \xi.$$

The convergence theorem is now:

Theorem 3.2. *Let f and f^h be the solutions of the continuous and discrete Fermi equation satisfying* (2.2) *and* (3.3), *respectively. Then there is a constant $C = C(\Omega)$ such that we have*

$$(3.14)\quad |||f - f^h||| \le Ch^{k+1/2}\|f\|_{k+1}.$$

Proof. Using the relation $B(e,\xi)=0$ (since $\xi \in V_h$), we have that

$$\begin{aligned}|||\xi|||^2 &\le B(\xi,\xi) = B(\eta - e,\xi) = B(\eta,\xi) \\ &= (\eta_x + v\cdot\nabla_\perp\eta, \xi + h(\xi_x + v\cdot\nabla_\perp\xi))_{Q_L} + \sigma(\nabla_v\eta,\nabla_v\xi)_{Q_L} \\ &\quad - h\sigma(\Delta_v\eta, \xi_x + v\cdot\nabla_\perp\xi)_{Q_L} + \langle\eta,\xi\rangle_0 - \langle\eta,\xi\rangle_{\Gamma_I^-}.\end{aligned} \tag{3.15}$$

Integrating by parts,

$$\begin{aligned}&(\eta_x + v\cdot\nabla_\perp\eta,\xi)_{Q_L} + \langle\eta,\xi\rangle_0 - \langle\eta,\xi\rangle_{\Gamma_I^-} \\ &\quad = -(\eta, \xi_x + v.\nabla_\perp\xi) + \langle\eta,\xi\rangle_M + \frac{1}{2}\int_{I\times\partial\Omega}\eta\xi|\beta\cdot\mathbf{n}|.\end{aligned} \tag{3.16}$$

By the inverse estimate and assumption on σ

$$\sigma(\nabla_v\eta,\nabla_v\xi)_{Q_L} \le \sigma\|\nabla_v\eta\|_{Q_L}\|\nabla_v\xi\|_{Q_L} \le \sigma\|\nabla_v\eta\|^2_{Q_L} + \frac{\sigma}{4}\|\nabla_v\xi\|^2_{Q_L}, \tag{3.17}$$

and

$$\begin{aligned}h\sigma(\Delta_v\eta, \xi_x + v\cdot\nabla_\perp\xi)_{Q_L} &\le h\sigma\|\Delta_v\eta\|_{Q_L}\|\xi_x + v\cdot\nabla_\perp\xi\|_{Q_L} \\ &\le Ch^{-1}\|\eta\|^2_{Q_L} + \frac{h}{4}\|\xi_x + v\cdot\nabla_\perp\xi\|^2_{Q_L}.\end{aligned} \tag{3.18}$$

Combining the estimates (3.15)-(3.18) gives

$$\begin{aligned}|||\xi|||^2 \le B(\eta,\xi) \le \frac{1}{4}|||\xi|||^2 + C\Big[h^{-1}\parallel\eta\parallel^2_{Q_L} &+ h\|\eta_x + v\cdot\nabla_\perp\eta\|^2_{Q_L} \\ &+\sigma\|\nabla_v\eta\|^2_{Q_L} + |\eta|^2_M + \int_{I\times\partial\Omega}\eta^2|\beta\cdot\mathbf{n}|dvds\Big].\end{aligned} \tag{3.19}$$

Using (3.10)-(3.12) and a kick-back argument we obtain the desired result. □

Remark 3.3. *Here are some features of problem* (2.2)*: (i) The lack of pure current term for the beam problem, i.e., no absorption on the left hand side of the equation, will lead to stability with no explicit L_2-norm control. Besides, in all the above estimates the semi-norms, (L_2-norms of partial derivatives), appear with a small coefficients of order $\mathcal{O}(\sqrt{h})$. Since the test functions are zero on part of $\partial\Omega$ with positive Lebesgue measure, we could again use a version of the Poincare-Friedricks inequality and obtain an estimate for the L_2-norm with the same coefficients as for the semi-norms involved in the weighted stability norm, i.e., we add a L_2-norm with a coefficient of order $\mathcal{O}(\sqrt{h})$ to the $|||.|||$ norm in*

Lemma 3.1. However, a better approach would be through Lemma 3.4 (cf. [2]*) below, in a situation where jump discontinuities are introduced and included in the stability norm* $|||.|||$*. This approach improves the* L_2*-norm estimate regaining the factor* $h^{1/2}$*.*

Lemma 3.4. *For any constant* $C_1 > 0$*, we have for* $g \in V_h$*,*

$$\|g\|_{Q_L} \leq \left[\frac{1}{C_1}\|g_x + v.\nabla_\perp g\|^2_{Q_L} + \sum_{m=1}^{M}|g_-|^2_m + \int_{I\times\partial\Omega} g^2|\beta\cdot\mathbf{n}|\right] he^{C_1 h} \tag{3.20}$$

Proof. See the argument in the proof of Lemma 4.2 in [2]. □

4. Stabilized high order SD method

In this section, we address the full discretization of Fermi equation (2.2) by considering both the backward Euler for variable x and the high order streamline diffusion method for remaining variables. We derive a stability estimate in a general framework that may be easily extended to include theta-scheme. For notational simplicity, we use the uniform partition $0 = x_0 < x_1 < \ldots < x_M = L$ of the interval $I = (0, L]$ into subintervals $I_m = (x_{m-1}, x_m)$, where $x_m - x_{m-1} = k$ for $m = 1, ..., M$. Now, let $\mathcal{T}_h$ be the subdivision of Ω into elements $\{K\}$ with the mesh parameter $h = \text{diam } K$. We assume that each $K \in \mathcal{T}_h$ is the image under a family of bijective affine maps $\{F_K\}$ of a fixed standard master element $\hat{K}$ into K, where $\hat{K}$ is purely the open unit hypercube in $\mathbb{R}^4$. Let $P_p(K)$ be the set of all polynomials of degree at most p on K.

$$V_h^p = \chi_h^p \cap H_0^1(\Omega), \tag{4.1}$$

where

$$\chi_h^p = \{g \in \mathcal{C}^0(\Omega) : g \circ F_K \in P_p(\hat{K});\ \forall K \in \mathcal{T}_h\}. \tag{4.2}$$

In the previous section we descretized all variables with finite element method, assuming f^h to be the approximate solution and using test functions of the form $g + \delta(g_x + v \cdot \nabla_\perp g)$ where δ as a small parameter of order h (or h^α, $\alpha > 1$), would supply us with a necessary (missing) diffusion term of order h in the direction of streamlines: $(1, v, \mathbf{0})$. More specifically, in the stability estimates we have been able to control an extra term of the form $h\|g_x + v \cdot \nabla_\perp g\|$. In this section, however, the choice of δ is somewhat involved and in addition to the equation type, it also depends on the choice of the parameters h and p which are chosen locally (elementwise) in an optimal manner. Therefore, in hp-analysis, δ would appropriately appear as an elementwise (local) parameter.

4.1. **The semidiscrete problem.** Now the semidiscrete problem reads: Find $f_h(x) \in V_h^p$ such that for all $g \in V_h^p$,

$$(\partial_x f_h(x), g) + \sum_{K\in\mathcal{T}_h} (\partial_x f_h(x), \delta_K(v\cdot g))_K + \hat{B}_\delta(f_h(x), g) = 0, \tag{4.3}$$

where bilinear form $\hat{B}_\delta(.,.)$ is defined by

$$\begin{aligned} \hat{B}_\delta(f,g) = \sum_{K\in\mathcal{T}_h} &[(v\cdot\nabla_\perp f, g + \delta_K(v\cdot\nabla_\perp g))_K + \sigma(\nabla_v f, \nabla_v g)_K \\ &-\delta_K\sigma(\Delta_v f, v\cdot\nabla_\perp g)_K] - \langle f, g\rangle_{\Gamma^-} \ . \end{aligned} \tag{4.4}$$

Here δ is the non-negative piecewise constant function which satisfies

$$\delta|_K = \delta_K, \qquad \delta_K = \text{constant for } K \in \mathcal{T}_h.$$

The precise choice of δ depends on the nature of the coefficients in the partial differential equation and will be discussed in more details later. Note that in the hp version of the SD-approach we interpret $(.,.)_\Omega$ as $\sum_{K\in\mathcal{T}_h}(.,.)_K$ counting for the local character of parameter δ_K. We also define the norm $[||.||]_\delta$, in a natural way obtained from (4.4) by considering the local effects of δ_K,

$$[||g||]_\delta^2 = \frac{1}{2}\sum_{K\in\mathcal{T}_h}\left(\sigma\|\nabla_v g\|_K^2 + \delta_K\|v\cdot\nabla_\perp g\|_K^2\right). \tag{4.5}$$

Further, we assume that the family of partitions $\{\mathcal{T}_h\}_{h>0}$ is shape regular, in the sense that there is a positive constant C_0, independent of h, such that

$$C_0 h_K^4 \le \text{meas}(K), \qquad \forall K \in \bigcup_{h>0}\{\mathcal{T}_h\}, \tag{4.6}$$

where meas(K) is the diameter of four dimensional sphere inscribed in K. In order to analyze the semi-discrete method (4.3), we define the Ritz projection R_h associated with the stabilized bilinear form $\hat{B}_\delta$ as $R_h f \in V_h^p$ such that

$$\hat{B}_\delta(R_h f, g) = \hat{B}_\delta(f, g), \qquad \forall g \in V_h^p. \tag{4.7}$$

For the problem (4.7) we have the following stability lemma:

Lemma 4.1. *Assume that the local SD-parameter δ_K is selected in the range*

$$0 < \delta_K \leq \frac{h_K^2}{\sigma C_I^2 p^4}, \qquad \forall K \in \mathcal{T}_h, \tag{4.8}$$

where C_I is the constant in an inverse estimate. Then the bilinear form $\hat{B}_\delta(.,.)$ is coercive on $V_h^p \times V_h^p$, i.e.,

$$\hat{B}_\delta(g,g) \geq \frac{1}{2}[||g||]_\delta^2, \qquad \forall g \in V_h^p. \tag{4.9}$$

Proof. We use the definition of $hatB_\delta$ in (4.4) and write

$$\begin{aligned} \hat{B}_\delta(g,g) = {} & \delta_K \|v \cdot \nabla_\perp g\|_K^2 + \sigma \|\nabla_v g\|_K^2 \\ & - \delta_K \sigma (\Delta_v g, v \cdot \nabla_\perp g)_K + (v \cdot \nabla_\perp g, g)_K. \end{aligned} \tag{4.10}$$

Using Green's formula we also have

$$(v \cdot \nabla_\perp g, g) = \tfrac{1}{2} \textstyle\int_{\partial\Omega^+} g^2 (\beta \cdot \mathbf{n}) dv. \tag{4.11}$$

The estimate of the term involving $\delta_K \sigma$, where we apply Cauchy-Schwarz and inverse inequalities together with the assumption on δ_K, implies that

$$\begin{aligned} & \delta_K \sigma (\Delta_v g, v \cdot \nabla_\perp g)_K \\ & \leq \frac{1}{2} C_I h_K^{-1} p^2 \sqrt{\sigma \delta_K} \left[\sigma \|\nabla_v g\|_K^2 + \delta_K \|v \cdot \nabla_\perp g\|_K^2 \right] \\ & \leq \frac{1}{2} \left[\sigma \|\nabla_v g\|_K^2 + \delta_K \|v \cdot \nabla_\perp g\|_K^2 \right]. \end{aligned} \tag{4.12}$$

Combining (4.10)-(4.12) will give the desired result. $\square$

In what follows we shall use the following approximation property: Let $g \in H^s(K)$ and $\|.\|_{s,K}$ be the usual Sobolev norm on K; there exists a constant C depending on s and r but independent of g, h_K and p, and a polynomial $\Pi_p g$ of degree p, such that for any $0 \leq r \leq s$ the following estimate holds true (see [9]),

$$\|g - \Pi_p g\|_{r,K} \leq C \frac{h_K^{\mu - r}}{p^{s-r}} \|g\|_{s,K}, \tag{4.13}$$

where $s \geq 0$, and $\mu = \min(p+1, s)$. We shall also require a global counterpart of the above approximation result for the finite element space V_h^p, so in the sequel we adopt the following:

Lemma 4.2. *Let $g \in H_0^1(\Omega) \cap H^r(\Omega)$, $r > 2$ such that $g\,|_K \in H^s(K)$, with a positive integer $s \geq r$ and $K \in \mathcal{C}_h$. Then there exists an interpolant $\Pi_p g \in V_h^p$ of g which is continuous on Ω such that*

$$\|g - \Pi_p g\|_{1,K} \leq C \frac{h_K^{\mu-1}}{p^{s-1}} \|g\|_{s,K}, \tag{4.14}$$

where, $C > 0$ is a constant independent of h and p, and $\mu = \min(p+1, s)$.

Proof. See, e.g., [16] where a proof is outlined, assuming certain regularity degree. More elaborated proof can be found in [23], [12] and the references therein. □

We shall also need the following trace inequality:

$$\|\eta\|_{\partial K}^2 \leq C(\|\nabla\eta\|_K \|\eta\|_K + h_K^{-1} \|\eta\|_K^2), \qquad \forall K \in \mathcal{T}_h. \tag{4.15}$$

Theorem 4.3. *Let $\mathcal{T}_h$ be a shape regular mesh on Ω and let f be the exact solution of (2.2) that satisfies the assumptions of Lemma 4.2. Let $R_h f$ be the solution of (4.7) and assume that the SD-parameter δ_K satisfies $0 < \delta_K \leq \frac{h_K^2}{\sigma C_I^2 p^4}$ for each $K \in \mathcal{T}_h$. Then the following error bound holds true*

$$[||f - R_h f||]_\delta^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-2}} \Big(\frac{1}{p^2} + \frac{1}{p} + \sigma h_K^{-1} + \delta_K h_K^{-1} + \frac{h_K}{\delta_K p^2}\Big) \|f\|_{s,K}^2. \tag{4.16}$$

Proof. We first decompose the error in a discrete and interpolation error

$$f - R_h f = \eta - \xi, \tag{4.17}$$

where $\eta = f - \Pi_p f$ and $\xi = R_h f - \Pi_p f$. Here $\Pi_p f \in V_h^p$ is the conforming interpolant of f in Lemma 4.2. Using triangle inequality we get

$$[||f - R_h f||]_\delta \leq [||\eta||]_\delta + [||\xi||]_\delta, \tag{4.18}$$

Using (4.7) and Lemma 4.1 we get

$$\begin{aligned}
\frac{1}{2}[||\xi||]_\delta^2 &\le \hat{B}_\delta(\xi,\xi) = \hat{B}_\delta(\eta,\xi) \\
&= \sigma(\nabla_v\eta, \nabla_v\xi) - \sigma \sum_{K\in\mathcal{T}_h} \delta_K(\Delta_v\eta, v\cdot\nabla_\perp\xi)_K \\
&\quad + (v\cdot\nabla_\perp\eta, \xi) + \sum_{K\in\mathcal{T}_h} \delta_K(v\cdot\nabla_\perp\eta, v\cdot\nabla_\perp\xi)_K \\
&= \sum_{i=1}^{5} T_i.
\end{aligned} \tag{4.19}$$

The terms T_1 and T_3 to T_7 can be estimated by the same techniques as in the proof of Theorem 3.2. Further, using the inverse inequality and assumptions on σ and δ_K we get

$$\begin{aligned}
|T_2| &\le \delta_K\sigma\|\Delta_v\eta\|_K\|v\cdot\nabla_\perp\xi\|_K \\
&\le C_I\delta_K\sigma p^2 h_k^{-1}\|\nabla_v\eta^n\|_K\|v\cdot\nabla_\perp\xi\|_K \\
&\le 2\sigma\|\eta\|_K^2 + \frac{\delta_K}{8}\|v\cdot\nabla_\perp\xi\|_K^2.
\end{aligned}$$

We shall rewrite the estimates above concisely as

$$[||\xi||]_\delta^2 \le C(I_1+I_2), \tag{4.20}$$

where

$$\begin{aligned}
I_1 &= \textstyle\sum_{K\in\mathcal{T}_h}\left(\|\eta\|_K^2 + \delta_K^{-1}\|\eta\|_K^2 + \delta_K\|v\cdot\nabla_\perp\eta\|_K^2 + \sigma\|\nabla_v\eta\|^2\right), \\
I_2 &= \int_{\partial\Omega^+}\eta^2|\beta\cdot\mathbf{n}|dvds.
\end{aligned}$$

Below we estimate I_1 and I_2 separately. As for I_1, using Lemma 4.2 and assumption on δ_K we have,

$$I_1 \le C\sum_{K\in\mathcal{T}_h}\frac{h_K^{2\mu-2}}{p^{2s-2}}(\delta_K^{-1}\frac{h_K^2}{p^2} + \delta_K + \sigma)\|f\|_{s,K}^2. \tag{4.21}$$

As, for the term I_2, we have from trace estimate (4.15),

$$I_2 \le \sum_{K\in\mathcal{T}_h}(h_K^{-1}\frac{h_K^{2\mu}}{p^{2s}})\|f\|_{s,K}^2. \tag{4.22}$$

Hence from (4.20)-(4.22) we get that
(4.23)

$$[||\xi|||]_\delta^2 \le C \sum_{K\in\mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-2}}(\frac{1}{p^2}+\frac{1}{p}+\sigma h_K^{-1}+\delta_K h_K^{-1}+\frac{h_K}{\delta_K p^2})\|f\|_{s,K}^2.$$

Finally, the term $[||\eta|||]_\delta$ can be estimated in the same way and we get,

$$[||\eta|||]_\delta^2 \le C \sum_{K\in\mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-2}}(\frac{1}{p}+\sigma h_K^{-1}+\delta_K h_K^{-1})\|f\|_{s,K}^2. \tag{4.24}$$

Substituting the estimates (4.23)-(4.24) into (4.18), we get the desired result and the proof is complete. □

Remark 4.4. *In Theorem 4.3, we chose δ_K for all $K \in \mathcal{C}_h$ when σ is small compared to h_k and $\frac{1}{p}$. The parameter C_δ is selected in a way that δ_K satisfies the hypothesis of Theorem 4.3. This particular choice of δ_K is motivated by our analysis in the discretization error* (4.16) *in the norm $[||.|||]_\delta$, in order to give hp-error bound as,*

$$[||f-R_hf|||]_\delta^2 \le C \sum_{K\in\mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-1}}\|f\|_{s,K}^2. \tag{4.25}$$

We note that our assumption on σ has a key role on obtaining the optimality of the error bound simultaneously in h and p.

Remark 4.5. *For notational simplicity we have not chosen to allow an element-by-element variation of the polynomial degree p and the local Sobolev smoothness parameter s of the analytical solution f. However our analysis can be extended easily to this case by replacing p by p_K, s by s_K and $\|f\|_s$ by $\|f\|_{s,K}$ for $K \in T_h$. Subsequently, in the local approximation* (4.13), *$\mu = \min(p+1, s)$ is replaced by $\mu_K = \min(p_K + 1, s_K)$.*

For the semi-discrete problem (4.3) we have the following stability result:

Lemma 4.6. *Suppose that f is the exact solution of problem* (2.2), *f_h is the solution of problem* (4.3) *and δ_K satisfies the hypothesis of Theorem 4.3. Then there is a constant C independent of x, h and p such that for* *all $x \in [0, L]$ we have*

$$\begin{aligned}\|f_h(x) - f(x)\|^2 \leq &\|f_{h,0} - f_0\|^2 + C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-1}} \Big[\|f_0\|_{s,K}^2 \\ &+ \int_0^x \|\partial_x f\|_{s,K}^2 dx\Big],\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}\int_0^x [||f_h(s) - f(s)||]_\delta ds \leq &C\Big\{\|f_{h,0} - f_0\| + \sum_{K \in \mathcal{T}_h} \frac{h_K^{\mu-1/2}}{p^{s-1/2}} \big[\|f_0\|_{s,K} \\ &+ \int_0^x (\|f(s)\|_{s,K} + \|\partial_x f(s)\|_{s,K} ds)\big]\Big\}.\end{aligned} \tag{4.27}$$

Proof. We decompose the error in two partes

$$f_h(x) - f(x) = f_h(x) - R_h f(x) + R_h(x) - f(x) = \theta(x) + \eta(x), \tag{4.28}$$

where

$$\theta(x) = f_h(x) - R_h f(x),$$

and

$$\eta(x) = R_h f(x) - f(x).$$

To bound the projection error $\eta(x)$ from Theorem 4.3, we have

$$\|\eta(x)\|^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-1}} \left[\|f_0\|_{s,K}^2 + \int_0^x \|\partial_x f(s)\|_{s,K}^2 ds\right] \tag{4.29}$$

where we have used

$$\|f(x)\|_{s,K}^2 \leq C \left[\|f_0\|_{s,K}^2 + \int_0^x \|\partial_x f(s)\|_{s,K}^2 ds\right]. \tag{4.30}$$

Since the projection R_h does not depend on x, using (2.2), (4.3) and definition (4.7) of R_h we have

$$\begin{aligned}(\partial_x \theta(x), g) + \hat{B}_\delta(\theta(x), g) = &-(\partial_x \eta(x), g) \\ &- \sum_{K \in T_h} (\partial_x \eta(x) + \partial_x \theta(x), \delta_K(v \cdot \nabla_\perp \theta))_K \qquad \forall g \in V_h^p.\end{aligned} \tag{4.31}$$

By choosing $g = \theta(x)$ and using Lemma 4.1 we obtain

$$
\begin{aligned}
\frac{1}{2}\left[\|\partial_x\theta(x)\|^2 + [|| \theta ||]_\delta^2\right] &\leq (\partial_x\theta(x), \theta(x)) + \hat{B}_\delta(\theta(x), \theta(x)) \\
&= -(\partial_x\eta(x), \theta(x)) - \sum_{K\in\mathcal{T}_h} (\partial_x\eta(x) + \partial_x\theta(x), \delta_K(v\cdot\nabla_\perp\theta))_K,
\end{aligned} \tag{4.32}
$$

To bound the second term on the right hand side the inequality above, we have

$$
(\partial_x\theta(x), \delta_K(v\cdot\nabla_\perp\theta))_K \leq \delta_K\|\partial_x\theta(x)\|^2 + \frac{\delta_K}{4}\|(v\cdot\nabla_\perp\theta)\|^2. \tag{4.33}
$$

The term $(\partial_x\eta(x), \delta_K(v\cdot\nabla_\perp\theta))_K$ may be estimated in the same fashion. We proceed by choosing $\delta_K \leq \frac{1}{4}$ and absorbing these terms in the left hand side of (4.32), to obtain

$$
\|\partial_x\theta(x)\|^2 \leq C\|\partial_x\eta(x)\|^2. \tag{4.34}
$$

By integrating from 0 to x we have

$$
\|\theta(x)\|^2 \leq \|\theta(0)\|^2 + C\int_0^x \|\partial_x\eta(s)\|^2 ds. \tag{4.35}
$$

Using Remark 4.4 we have

$$
\begin{aligned}
\|\theta(0)\|^2 &\leq 2\|f_{h,0} - f_0\|^2 + 2\|f(0) - R_h(0)\|^2 \\
&\leq 2\|f_{h,0} - f_0\|^2 + 2\sum_{K\in\mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-1}}\|f_0\|_{s,K}^2
\end{aligned} \tag{4.36}
$$

and also

$$
\|\partial_x\eta(s)\|^2 \leq \sum_{K\in\mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-1}}\|\partial_x f\|_{s,K}^2. \tag{4.37}
$$

Thus, from (4.35)-(4.37) we deduce

$$
\begin{aligned}
\|\theta(x)\|^2 &\leq \|f_{h,0} - f_0\|^2 \\
&\quad + C\left\{\sum_{K\in\mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-1}}\left[\|f_0\|_{s,K}^2 + \int_0^x \|\partial_x f(s)\|_{s,K}^2 ds\right]\right\}.
\end{aligned} \tag{4.38}
$$

Combining (4.31), (4.29) and (4.43) we obtain the desired result. To prove the estimate (4.27), using (4.32) and the coercivity Lemma 4.1 we have

$$
\frac{1}{2}\|\partial_x\theta(x)\|^2 + [||\theta(x)||]_\delta^2 \leq \|\partial_x\eta(x)\|\|\theta(x)\|, \tag{4.39}
$$

and thus by integration over x we obtain

$$
\begin{aligned}
\|\theta(x)\| + \int_0^x [||\theta(s)||]_\delta ds &\le \|f_{h,0} - f_0\| \\
&+ C\left\{\sum_{K\in\mathcal{T}_h} \frac{h_K^{\mu-1/2}}{p^{s-1/2}}\left[\|f_0\|_{s,K} + \int_0^x \|\partial_x f(s)\|_{s,K} ds\right]\right\},
\end{aligned}
\tag{4.40}
$$

which proves the estimate (4.27). □

4.2. **The discretization in x variable.** Below we discretize in x by the backward Euler scheme which leads to a sequence of boundary value problem. Find $f_h^n \in V^h$ such that

$$
(\bar{\partial}_x f_h^n, g) + \sum_{K\in\mathcal{T}_h} (\bar{\partial}_x f_h^n, \delta_K(v\cdot g))_K + \hat{B}_\delta(f_h^n, g) = 0 \qquad \forall g \in V_h^p, \tag{4.41}
$$

where f_h^0 is some suitable approximation of f_0 and and $\bar{\partial}_x$ is a first step backward Euler operator defined by

$$
\bar{\partial}_x f_h^n = \tfrac{f_h^n - f_h^{n-1}}{k}.
$$

Based on discretization, we consider the following variational formulation: Find $f_h^n \in V_h^p$, such that

$$
(f_h^n, g) + \sum_{K\in\mathcal{T}_h} (f_h^n, \delta_K(v\cdot g))_K + k\hat{B}_\delta(f_h^n, g) = \hat{L}^n(g) \qquad \forall g \in V_h^p,
$$

where

$$
\hat{L}^n(g) = (f_h^{n-1}, g) + \sum_{K\in\mathcal{T}_h} (f_h^{n-1}, \delta_K(v\cdot g))_K. \tag{4.42}
$$

We shall now prove the following error estimate for the fully discrete problem (4.41):

Theorem 4.7. *Suppose f_h^n and f be the solutions of* (4.41) *and* (2.2), *respectively. Then for $n \geq 0$ we have*

(4.43)
$$\|f_h^n - f(x_n)\|^2 + \sum_{j=1}^{n}[||f_h^j - f(x_j)||]_\delta^2 \leq \|f_{h,0} - f_0\|^2$$
$$+ C\Big\{ \sum_{K\in\mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-1}} \left[\|f_0\|_{s,K}^2 + \int_0^x \|\partial_x f(s)\|_{s,K}^2 ds\right]$$
$$+ k^2 \int_0^{x_n} \|\partial_{xx} f(s)\|^2 ds\Big\}.$$

Proof. In an analogy with (4.44) we write

$$f_h^n - f(x_n) = f_h^n - R_h f(x_n) + R_h f(x_n) - f(x_n) = \theta^n + \eta^n. \tag{4.44}$$

It follows from variational formulation (4.41) and (2.2) that for $n \geq 1$

(4.45)
$$(\bar{\partial}_x \theta^n, g) + \hat{B}_\delta(\theta^n, g) = -(\omega^n, g)$$
$$- \sum_{K\in\mathcal{T}_h} (\omega^n + \bar{\partial}_x \theta^n, \delta_K (v \cdot \nabla_\perp g))_K \qquad \forall g \in V_h^p,$$

where

$$\omega^n = (R_h - I)\bar{\partial}_x f(t_n) + \bar{\partial}_x f(t_n) - \partial_x f(t_n) = \omega_1^n + \omega_2^n. \tag{4.46}$$

By Cauchy's inequality we have

$$(\bar{\partial}_x \theta^n, \delta_K(v\cdot\nabla_\perp g))_K \leq \frac{\delta_K}{k}(\|\theta^n\|_K^2 + \|\theta^{n-1}\|_K^2) + \frac{\delta_K}{4}\|(v\cdot\nabla_\perp g)\|_K^2. \tag{4.47}$$

We also use a similar argument to estimate the term $(\omega^n, \delta_K(v \cdot g))$. By choosing $\delta_K \leq \frac{1}{4}$, $g = \theta^n$, and using standard kick-back arguments to (4.45) we obtain

$$\|\theta^n\|^2 - \|\theta^{n-1}\|^2 + k[||\theta^n||]_\delta^2 \leq C\left[k\|\omega_1^n\|^2 + k\|\omega_2^n\|^2\right]. \tag{4.48}$$

Now sum over $j = 1, \ldots, n$. Then by application of standard dissipation relations, we have

$$\|\theta^n\|^2 + k\sum_{j=1}^{n}[||\theta^j||]_\delta^2 \leq \|\theta^0\|^2 + C\left[k\sum_{j=1}^{n}\|\omega_1^j\|^2 + k\sum_{j=1}^{n}\|\omega_2^j\|^2\right]. \tag{4.49}$$

We may write

(4.50)
$$\omega_1^j = (R_h - I)k^{-1} \int_{x_{j-1}}^{x_j} \partial_x f ds = k^{-1} \int_{x_{j-1}}^{x_j} (R_h - I)\partial_x f ds \leq k^{-\frac{1}{2}} \left(\int_{x_{j-1}}^{x_j} |(R_h - I)\partial_x f ds| \right)^{\frac{1}{2}},$$

whence by Remark 4.4,

(4.51)
$$k \sum_{j=1}^{n} \|\omega_1^j\|^2 \leq \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |(R_h - I)\partial_x f|^2 ds \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-1}} \int_0^{x_n} \|\partial_x f\|_{s,K}^2 ds.$$

The second term is estimated by Taylor's formula,

(4.52)
$$\omega_2^j = k^{-1}(f(x_j) - f(x_{j-1})) - \partial_x f(x_j) = - k^{-1} \int_{x_{j-1}}^{x_j} (s - x_{j-1})\partial_{xx} f(s) ds \leq k^{\frac{1}{2}} \left(\int_{x_{j-1}}^{x_j} |\partial_{xx} f(s)| ds \right)^{\frac{1}{2}},$$

so that

$$k \sum_{j=1}^{n} \|\omega_2^j\|^2 \leq k^2 \int_0^{x_n} \|\partial_{xx} f(s)\|^2 ds \ . \tag{4.53}$$

It remains to estimate the approximation of the Ritz-projection, i.e., $[||\eta^n||]_\delta$. By equation (4.29) we have

(4.54)
$$\sum_{j=1}^{n} [||\eta(x_j)||]_\delta^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\mu-1}}{p^{2s-1}} \left[\|f_0\|_{s,K}^2 + \int_0^{x_n} \|\partial_x f(s)\|_{s,K}^2 ds \right].$$

Combining all the estimates, we complete the proof of the theorem. □

Conclusion: Our analysis extends the result of [2] to a three dimensional degenerate type convection-dominated convection-diffusion problem with a small and variable diffusion coefficient. To enhance stability while keeping accuracy, we used a consistent stabilized method using space-time elements. We also considered semidiscretization as an intermediate step and combined the method with backward Euler scheme and obtained optimal error estimates in appropriate norms for sufficiently smooth solutions. However, as it is known, the main drawback of using hp-SD method is that, the stabilization terms involve coupling with the second order term, the source term and the x-derivative. This can cause to severe computational costs. Regarding the numerical aspects, the dimension of the discretized problem is 4-dimensional in transversal domain which is difficult to handle. One remedy shall be considering the discrete velocity model of the Fermi equation which we address in our future work.

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