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# ON ISOMORPHISM OF SIMPLICIAL COMPLEXES AND THEIR RELATED ALGEBRAS

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Abstract. Here, we provide a simple proof for the fact that two simplicial complexes are isomorphic if and only if their associated Stanley-Reisner rings, or their associated facet rings are isomorphic as $K$-algebras. As a consequence, we show that two graphs are isomorphic if and only if their associated edge rings are isomorphic as $K$-algebras. Based on an explicit $K$-algebra isomorphism of two Stanley-Reisner rings, or facet rings or edge rings, we present a fast algorithm to find explicitly the isomorphism of the associated simplicial complexes, or graphs.

## 1. **Introduction**

Let $X$ be a finite nonempty set. A simplicial complex $\Delta$ on $X$ is a set of subsets of $X$ such that, for any $x \in X$, $\{x\} \in \Delta$, and if $E \in \Delta$ and $F \subseteq E$, then $F \in \Delta$. A set in $\Delta$ is called a face and a maximal face in $\Delta$ is called a facet.

Let $X = \{x_1, \ldots, x_n\}$, and $\Delta$ be a simplicial complex on $X$. Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ indeterminates and with coefficients in a field $K$. Let $I(\Delta)$ be the ideal in $R$ generated by all square-free monomials $x_{i_1} \ldots x_{i_s}$, provided that $\{x_{i_1}, \ldots, x_{i_s}\} \notin \Delta$. The

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quotient ring $R/I(\Delta)$ is called the Stanley-Reisner ring of the simplicial complex $\Delta$.

It is easy to see that any quotient of a polynomial ring over an ideal, generated by square-free monomials of degree greater than 1, is the Stanley-Reisner ring of a simplicial complex. A natural question arises: If two Stanley-Reisner rings are isomorphic, are their corresponding simplicial complexes isomorphic? In 1996, W. Bruns and J. Gubeladze proved that if the isomorphism of the rings is a $K$-algebra isomorphism, then the corresponding simplicial complexes are isomorphic [1]. Here, we provide an alternative and constructive proof for this result.

In 2002, S. Faridi in [2] defined the notion of facet ideal for a simplicial complex which is a generalization of the notion of edge ideal for a graph defined by R. Villarreal [7].

Let $\Delta$ be a simplicial complex on the set $X = \{x_1, \dots, x_n\}$. Let $F(\Delta)$ be the ideal of $R = K[x_1, \dots, x_n]$ generated by all square-free monomials $x_{i_1} \dots x_{i_s}$, provided that $\{x_{i_1}, \dots, x_{i_s}\}$ is a facet in $\Delta$. The quotient ring $R/F(\Delta)$ is called the facet ring of the simplicial complex $\Delta$.

Let $G$ be a finite, simple and undirected graph with vertex set $\{x_1, \dots, x_n\}$. The ideal $E(G)$ of $R = K[x_1, \dots, x_n]$ generated by all square-free monomials $x_i x_j$, provided that $x_i$ is adjacent to $x_j$ in $G$, is called the edge ideal of $G$. The quotient ring $R/E(G)$ is called the edge ring of the graph $G$.

A question similar to the case of simplicial complexes can be stated for facet rings and edge rings. In 1997, H. Hajiabolhassan and M. L. Mehrabadi in [3] proved that two graphs are isomorphic if and only if their corresponding edge rings are isomorphic as $K$-algebras. Here, we prove the statement for the facet rings and conclude it for the edge rings.

Finally, we will present a fast algorithm which admits a $K$-algebra isomorphism of two Stanley-Reisner rings, or two facet rings, or two edge rings as the input and returns explicitly the isomorphism of the corresponding simplicial complexes or graphs, as the output.

## 2. **The isomorphism**

Let $I$ be an ideal of $R = K[x_1, \dots, x_n]$ and $J$ be an ideal of $S = K[y_1, \dots, y_m]$, both generated by monomials of degree greater than 1. Let $\phi : R/I \to S/J$ be a $K$-algebra isomorphism. Let $f$ be a monomial in $R$. Denote the image of $f$ in $R/I$ by $\bar{f}$. If $\bar{f}$ is a zero-divisor in $R/I$, then it is easy to check that the constant term of $\phi(\bar{f})$ in $S/J$ is zero. As discussed in [1], in the $K$-algebra isomorphism arguments of Stanley-Reisner rings of two simplicial complexes, without loss of generality, it is enough to reduce our attention to the case that all variables are zero divisor.

For any $i$, $1 \leq i \leq n$, let $L(x_i)$ denote the set of $y_j$'s such that $\bar{y}_j$ appears in the linear part of $\phi(\bar{x}_i)$ with nonzero coefficient.

**Lemma 2.1.** *With the above notations, if $\phi : R/I \to S/J$ is a $K$-algebra isomorphism, then $m = n$ and $L(x_i) \neq \emptyset$, for each $i$, $1 \leq i \leq n$.*

**Proof.** The ideal $J$ is generated by monomials of degree greater than one, and therefore, $(S/J)_1$, the degree one homogeneous component of $S/J$, is an $m$-dimensional $K$-vector space with basis $\{\bar{y}_1, \ldots, \bar{y}_m\}$. The map $\phi$ is surjective, and thus the set $\{(\phi(\bar{x}_1))_1, \ldots, (\phi(\bar{x}_n))_1\}$ generates $(S/J)_1$ as a vector space over $K$. Therefore, $n \geq m$. The map $\phi$ is isomorphism, and thus $\phi^{-1}$ is surjective too and therefore, $m \geq n$. For the last claim, note that if for some $i$, $1 \leq i \leq n$, $L(x_i) = \emptyset$, then $(\phi(\bar{x}_i))_1 = 0$ and the set $\{(\phi(\bar{x}_1))_1, \ldots, (\phi(\bar{x}_n))_1\}$ can not generate an $n$-dimensional vector space. $\square$

Let $\Delta_1$ and $\Delta_2$ be two simplicial complexes on sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$, respectively. Let $K[\Delta_1] = K[x_1, \ldots, x_n]/I(\Delta_1)$ and $K[\Delta_2] = K[y_1, \ldots, y_m]/I(\Delta_2)$ be the Stanley-Reisner rings associated with $\Delta_1$ and $\Delta_2$.

**Theorem 2.2.** *With the above notations, $\Delta_1$ and $\Delta_2$ are isomorphic as simplicial complexes if and only if $K[\Delta_1]$ and $K[\Delta_2]$ are isomorphic as $K$-algebras.*

**Proof.** It is obvious that if $\Delta_1$ and $\Delta_2$ are isomorphic as simplicial complexes, then $K[\Delta_1]$ and $K[\Delta_2]$ are isomorphic as $K$-algebras. For the converse, assume that $\phi : K[\Delta_1] \to K[\Delta_2]$ is a $K$-algebra isomorphism. By Lemma 2.1, $m = n$. An isomorphism as $\phi$, can be uniquely determined by images of $\bar{x}_i$, $i = 1, \ldots, n$, and

$$\phi(\bar{f}(x_1, \ldots, x_n)) = \phi(f(\bar{x}_1, \ldots, \bar{x}_n)) = f(\phi(\bar{x}_1), \ldots, \phi(\bar{x}_n))$$

for any polynomial $\bar{f}$ in $K[\Delta_1]$. Let $\phi(\bar{x}_i) = f_i(\bar{y}_1, \ldots, \bar{y}_n)$, $i = 1, \ldots, n$. By Lemma 2.1, for each $i$, $f_i$ has no nonzero constant term and has nonzero linear part. Suppose $\phi^{-1}$ is the inverse of $\phi$ and let $\phi^{-1}(\bar{y}_i) = g_i(\bar{x}_1, \ldots, \bar{x}_n)$, $i = 1, \ldots, n$. Let $f_{i1}$ and $g_{i1}$ denote the linear homogeneous components of $f_i$ and $g_i$, respectively:

$$\begin{aligned} f_{i1} &= a_{i1}\bar{y}_1 + \cdots + a_{in}\bar{y}_n, \qquad i = 1, \ldots, n \\ g_{j1} &= b_{j1}\bar{x}_1 + \cdots + b_{jn}\bar{x}_n, \qquad j = 1, \ldots, n. \end{aligned}$$

The equalities $\phi \circ \phi^{-1}(\bar{y}_i) = \bar{y}_i$ and $\phi^{-1} \circ \phi(\bar{x}_i) = \bar{x}_i$, for $i = 1, \ldots, n$, imply that

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ b_{21} & \ldots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ b_{21} & \ldots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} = I,$$

where $I$ is the identity matrix of order $n$. Therefore, in the following matrix, sum of entries of each row and each column is 1:

$$M(\phi) = \begin{bmatrix} a_{11}b_{11} & a_{21}b_{12} & \cdots & a_{n1}b_{1n} \\ a_{12}b_{21} & a_{22}b_{22} & \cdots & a_{n2}b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n}b_{n1} & a_{2n}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix}.$$

It is well known that, there is a transversal of length $n$ with nonzero elements in the matrix $M(\phi)$. See for instance, [4] or [6]. By a transversal, we mean a sequence of entries of the matrix with no common columns or rows. In other words, a transversal is a term in expansion of determinant of the matrix. Let $1j_1, 2j_2, \ldots, nj_n$ be indices of a transversal with nonzero elements in $M(\phi)$. By a change of indices of $y_i$'s and permuting corresponding columns of $M(\phi)$, suppose that the nonzero transversal is the main diagonal. Under this assumption, $y_i \in L(x_i)$ and $x_i \in L^{-1}(y_i)$, for $i = 1, \ldots, n$, where $L^{-1}(y_i)$ is the set of variables with nonzero coefficients in the linear part of $\phi^{-1}(\bar{y}_i)$. For a subset of $\{x_1, \ldots, x_n\}$ as $F$, which is not in $\Delta_1$, there is a minimal set $E \subseteq F$, where $E \notin \Delta_1$ and any proper subset of $E$ belongs to $\Delta_1$. Let $\{x_{i_1}, \ldots, x_{i_r}\}$ be a minimal set not belonging to $\Delta_1$. Then, $x_{i_1} \cdots x_{i_r}$ is a generator of $I(\Delta_1)$, that is, $\bar{x}_{i_1} \cdots \bar{x}_{i_r} = 0$ in $K[\Delta_1]$, and therefore $\phi(\bar{x}_{i_1} \cdots \bar{x}_{i_r}) = 0$ in $K[\Delta_2]$. This means that $\phi(\bar{x}_{i_1})\cdots\phi(\bar{x}_{i_r}) \in I(\Delta_2)$. The ideal $I(\Delta_2)$ is homogeneous and generated by monomials and therefore each homogeneous component and each monomial of $\phi(\bar{x}_{i_1})\cdots\phi(\bar{x}_{i_r})$ belongs to $I(\Delta_2)$. Therefore, the product $f_{i_1 1}\cdots f_{i_r 1}$ is in $I(\Delta_2)$. In this case, there are two possibilities:

- $y_{i_1}\cdots y_{i_r} \in I(\Delta_2)$, which means that $\{y_{i_1},\cdots,y_{i_r}\} \notin \Delta_2$;
- $y_{i_1}\cdots y_{i_r}$ is canceled by another monomial in $f_{i_1 1}\cdots f_{i_r 1}$.

We prove that the second case is not possible. The second condition implies that, for some $j$, $1 \leq j \leq r$, $y_{i_j}$ appears with nonzero coefficients in more than one $f_{i_l 1}$. Let $y_{t_1}^{\alpha_1}\cdots y_{t_s}^{\alpha_s}$ be the smallest monomial with lexicographic order in the set of all monomials with nonzero coefficients in the expansion of $f_{i_1 1}\cdots f_{i_r 1}$ with $\{t_1,\ldots,t_s\} \subseteq \{i_1,\ldots,i_r\}$. This monomial appears once in the expansion and so can not be canceled by another monomial. Therefore, $y_{t_1}^{\alpha_1}\cdots y_{t_s}^{\alpha_s} \in I(\Delta_2)$ and since $I(\Delta_2)$ is a radical ideal, then $y_{t_1}\cdots y_{t_s} \in I(\Delta_2)$ and so, $\bar{y}_{i_1}\cdots\bar{y}_{i_r} \in I(\Delta_2)$. Therefore, $\{x_{i_1},\ldots,x_{i_r}\} \notin \Delta_1$ implies that $\{y_{i_1},\ldots,y_{i_r}\} \notin \Delta_2$. With a similar argument for $\phi^{-1}$, $\{y_{i_1},\ldots,y_{i_r}\} \notin \Delta_2$ implies that $\{x_{i_1},\ldots,x_{i_r}\} \notin \Delta_1$, that is, $\Delta_1 \cong \Delta_2$. $\square$

Note that, for any simplicial complex $\Delta$, the ideal $I(\Delta)$ has no monomial of degree one, but in the case of facet ideals, zero dimensional facets correspond to degree one monomials in $F(\Delta)$. To use Lemma 2.1 in the proof of the next theorem, we will assume that there is not any zero dimensional facet in simplicial complexes. This does not reduce the generality of the theorem. Because, two simplicial complexes are isomorphic if and only if they have the same number of zero dimensional facets and the parts without any zero dimensional facet are isomorphic.

**Theorem 2.3.** *Any two simplicial complexes $\Delta_1$ and $\Delta_2$ are isomorphic if and only if their corresponding facet rings are isomorphic as $K$-algebras.*

**Proof.** The "only if" part is obvious. To prove the "if" part, let

$$K[x_1,\ldots,x_n]/F(\Delta_1) \xrightarrow{\phi} K[y_1,\ldots,y_m]/F(\Delta_2)$$

be a $K$-algebra isomorphism between the facet rings corresponding to $\Delta_1$ and $\Delta_2$. Similar to the proof of Theorem 2.2, it follows that $m = n$. By an appropriate change of indices, we may assume that $\{y_1,\ldots,y_n\}$ is the set corresponding to the main diagonal which we assumed to be a transversal with nonzero elements in the matrix $M(\phi)$. Then, $y_i \in L(i)$ and $x_i \in L^{-1}(y_i)$, for $i = 1, \ldots, n$. Let $\{x_{i_1}, \ldots, x_{i_s}\}$ be a facet in $\Delta_1$. Then, $x_{i_1} \cdots x_{i_s}$ is in the minimal generating set of $F(\Delta_1)$ and $\phi(\bar{x}_{i_1}) \cdots \phi(\bar{x}_{i_s}) \in F(\Delta_2)$. The same argument as the proof of Theorem 1, implies that $y_{i_1} \cdots y_{i_s} \in F(\Delta_2)$. If $y_{i_1} \cdots y_{i_s} \in F(\Delta_2)$ is not in the minimal generating set of $F(\Delta_2)$, then, without loss of generality, we may assume that $y_{i_1} \cdots y_{i_s - t} \in F(\Delta_2)$, for some $t$, $1 \leq t \leq s-1$. This means that, $\phi^{-1}(\bar{y}_{i_1}) \cdots \phi^{-1}(\bar{y}_{i_s - t}) \in F(\Delta_1)$ and therefore, $x_{i_1} \cdots x_{i_s - t} \in F(\Delta_1)$, which is a contradiction to minimality of $x_{i_1} \cdots x_{i_s}$ in $F(\Delta_1)$. Therefore, $\{y_{i_1}, \ldots, y_{i_s}\}$ is a facet in $\Delta_2$, and this gives a bijection between $\Delta_1$ and $\Delta_2$ as an isomorphism of simplicial complexes. $\square$

A simple and undirected graph $G$ can be regarded as a simplicial complex with facets $\{x_i, x_j\}$, where $x_i$ is adjacent to $x_j$ in $G$. With this interpretation, the edge ideal of $G$ is the same as its facet ideal. Therefore, we have the following result.

**Corollary 2.4.** *Let $G_1$ and $G_2$ be two graphs. $G_1$ and $G_2$ are isomorphic as graphs if and only if their edge rings are isomorphic as $K$-algebras.*

**Corollary 2.5.** *Let $\Delta_1$ and $\Delta_2$ be two simplicial complexes, $F(\Delta_1)$ and $F(\Delta_2)$ be their facet ideals. Let $\Gamma_1$ and $\Gamma_2$ be two simplicial complexes such that $F(\Delta_1)$ and $F(\Delta_2)$ are their Stanley-Reisner ideals. By the above theorems, $\Delta_1$ and $\Delta_2$ are isomorphic if and only if $\Gamma_1$ and $\Gamma_2$ are isomorphic.*

Corollary 2.5 can be proved directly and then one can assume that Theorem 2.3 as a corollary of Theorem 2.2. Note that Corollary 2.5 leads us to define a new dual for a given simplicial complex $\Delta$ as indicated in the corollary by $\Gamma$. It would be another project to investigate properties of this duality.

## 3. The algorithm

Here, we present a fast algorithm to construct explicitly an isomorphism between two simplicial complexes or two graphs, when a $K$-algebra isomorphism of their associated rings is given.

**Algorithm 1.** Let $R = K[x_1, \ldots, x_n]/I$ and $S = K[y_1, \ldots, y_n]/J$ be two $K$-algebras and $I$ and $J$ be ideals generated by some square-free monomials of degree greater than one. Any $K$-algebra homomorphism $\phi : R \to S$ can be uniquely determined by the images of $\bar{x}_1, \ldots, \bar{x}_n$. In the following, we assume that $R$ and $S$ are Stanley-Reisner rings of some simplicial complexes.

*Input*: $K$-algebras $R$ and $S$, and a $K$-algebra isomorphism $\phi : R \to S$,
*Output*: A simplicial complex $\Delta_1$ associated with $R$ and a simplicial complex $\Delta_2$ associated with $S$ as their Stanley-Reisner rings and a bijection $\psi : \{x_1, \ldots, x_n\} \to \{y_1, \ldots, y_n\}$ which determines an isomorphism of $\Delta_1$ and $\Delta_2$.

*Step 1.* Construct a simplicial complex $\Delta_1$ with the underlying set $\{x_1, \ldots, x_n\}$ and faces $\{x_{i_1}, \ldots, x_{i_r}\}$, where $x_{i_1} \cdots x_{i_r}$ is not divided by any of generators of $I$. Construct $\Delta_2$ on the set $\{y_1, \ldots, y_n\}$, and faces $\{y_{i_1}, \ldots, y_{i_s}\}$ where $y_{i_1} \cdots y_{i_s} \notin J$.
*Step 2.* Find the matrix $M(\phi)$.
*Step 3.* Find a transversal with nonzero elements in $M(\phi)$.
*Step 4.* Use the transversal in Step 3 to construct the map $\psi : \Delta_1 \to \Delta_2$.

The proof of Theorem 2.3 guaranties the correctness of the algorithm. It is known that finding a nonzero transversal in a matrix which has such a transversal, has a polynomial time algorithm [5], and so, the above algorithm is polynomial time too.

**Algorithm 2.** In Algorithm 1, we may consider the $K$-algebras $R$ and $S$ as facet rings of simplicial complexes, or if $I$ and $J$ are generated by some square-free monomials of degree 2, as edge rings of graphs. Then, in Step 1, we must construct a simplicial complexes $\Delta_1$ and $\Delta_2$ such that $R$ and $S$ are their facet rings, respectively. Following steps 2, 3, and 4, we finally obtain an isomorphism of simplicial complexes or graphs.

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