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# THE $nc$-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS†

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Abstract. A subgroup $H$ is said to be $nc$-supplemented in a group $G$ if there exists a subgroup $K \leq G$ such that $HK \lhd G$ and $H \cap K$ is contained in $H_G$, the core of $H$ in $G$. We characterize the supersolubility of finite groups $G$ with that every maximal subgroup of the Sylow subgroups is $nc$-supplemented in $G$.

## 1. Introduction

In this paper the word group always means finite group.

A subgroup $H$ is said to be complemented in $G$ if there exists a subgroup $K$ such that $G = HK$ and $H \cap K = 1$. Hall proved that a group is soluble if and only if every Sylow subgroup is complemented [7]. Ramadan in [13] proved that if $G/H$ is supersoluble and all maximal subgroups of the Sylow subgroups of $H$ are normal in $G$, then $G$ is supersoluble. A subgroup $H$ is $c-$normal in $G$ if there exists a normal subgroup $N$ of $G$ such that $HN = G$ and $H \cap N$ is contained in $H_G$, the core of $H$ in $G$ (see [17]). Obviously $c-$normality is weaker than normality. A subgroup $H$ is said to be $c-$supplemented in a group $G$ if

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there exists a subgroup $K$ such that $HK = G$ and $H \cap K$ is contained in $H_G$, the core of $H$ in $G$ (see [3]). The notion of $c$-supplementation is a generalization of the notions of complement and $c$-normality. Li et al. in [12] defined the following concept: A subgroup $H$ is said to be $nc$-supplemented in a group $G$ if there exists a subgroup $K \leq G$ such that $HK \lhd G$ and $H \cap K$ is contained in $H_G$, the core of $H$ in $G$.

In this note, we give some generalization of supersolubility based on the concept of $nc$-supplementation.

We will prove the following theorem:

**Theorem 1.1.** *Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H$ is supersoluble. If every maximal subgroup of every Sylow subgroup of $H$ is nc-supplemented in $G$, then $G$ is supersoluble.*

A class of finite group $\mathfrak{F}$ is said to be a formation if every epimorphic image of an $\mathfrak{F}$-group is an $\mathfrak{F}$-group and if $G/N_1 \cap N_2$ belongs to $\mathfrak{F}$ whenever $G/N_1$ and $G/N_2$ belong to $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if a finite group $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$ (see [14, p. 277]). The class of supersoluble group is a saturated formation (see [14, 9.4.5]). Let $\mathfrak{U}$ denote the class of all supersoluble groups.

Also we prove:

**Theorem 1.2.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathfrak{F}$. If every maximal subgroup of all Sylow subgroups of $H$ is nc-supplemented in $G$, then $G \in \mathfrak{F}$.*

Further definitions and notations are standard, please refer to [11] and [9].

## 2. **Preliminaries**

In this section, we give some concepts and some lemmas.

**Definition 2.1.** ([3]) *A subgroup $H$ is said to be $c-$supplemented in group $G$ if there exists a subgroup $K$ such that $HK = G$ and $H \cap K$ is contained in $Core_G(H)$. Then we say that $K$ is a $c-$supplement of $H$ in $G$.*

**Definition 2.2.** ([12]) *Let $G$ be a group and $H$ a subgroup of $G$. Then $H$ is said to be $nc-$supplemented in $G$ if there is a subgroup $K$ of $G$ such that $HK \trianglelefteq G$ and $H \cap K \leq H_G$. We say that $K$ is a $nc-$supplement of $H$ in $G$.*

**Remark 2.3.** *If $H$ is a maximal subgroup of $G$, then an nc-supplement of $H$ in $G$ is a c-supplement of $H$ in $G$.*

*Proof.* If $H$ is $nc$-supplemented in $G$, then there exists a subgroup $K$ such that $HK \lhd G$ and $H \cap K \leq H_G$. The maximality of $H$ implies that $HK = G$ or $HK = H$. In the former case, $H$ is $c$-supplemented in $G$. In the latter case, $H \lhd G$ and so $H$ is also $c$-supplemented in $G$. □

**Remark 2.4.** *Being nc-supplement is weaker than c-supplementation and normality.*

$nc$-supplemented is a generalized $c$-supplemented. In general, $nc$-supplementation does not imply $c$-supplementation. For example (see [12, Example 3]), let $G = A_4$ and $B = \{(1), (12)(34), (13)(24), (14)(23)\}$. Let $C = \{(1), (12)(34)\}$ and $H = \{(1), (13)(24)\}$. Then $B = CH \trianglelefteq G$ and $C$ is $nc-$supplemented in $G$ but not $c$-supplemented in $G$ since $C_G = 1$ and $G$ has no subgroup of order 6.

**Lemma 2.5.** *( [12, Lemma 4]) If $H$ is $nc-$supplemented in $G$, then there exists a subgroup $C$ of $G$ such that $H \cap C = H_G$ and $HC \trianglelefteq G$.*

**Lemma 2.6.** *( [12, Lemma 5]) Let $G$ be a group. Then*

*(1) If $H \leq M \leq G$ and $H$ is $nc-$supplemented in $G$, then $H$ is $nc-$supplemented in $M$.*

*(2) If $N \trianglelefteq G$ and $N \leq H$, then $H$ is $nc-$supplemented in $G$ if and only if $H/N$ is $nc-$supplemented in $G/N$.*

*(3) If $N \trianglelefteq G$ and $(|N|, |H|) = 1$. If $H$ is $nc-$supplemented in $G$ , then $HN/N$ is $nc-$supplemented in $G/N$.*

**Lemma 2.7.** *([16, 2.16]) Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$ and let $G$ be a group with a normal subgroup $H$ such that $G/H \in \mathfrak{F}$. If $H$ is cyclic, then $G \in \mathfrak{F}$.*

## 3. Main results and their applications

In this section, we give the proofs of the main theorems.

**The proof of Theorem 1.1**

*Proof.* Suppose that $G$ is a counter-example of minimal order. We have:

**Step 1.** Every proper subgroup $M$ of $G$ containing $H$ is supersoluble and $G$ is soluble.

Since $H \leq M$, it follows that $M/H$ is a proper subgroup of $G/H$. Since $G/H$ is supersoluble, it follows that $M/H$ is supersoluble. Thus

$M$ satisfies the hypotheses of the theorem, and by the minimality of $G$, $M$ is supersoluble. In particular, $H$ is supersoluble and so $G$ is soluble by [4].

**Step 2.** $\Phi(G) < H$ and $\Phi(G) = 1$.

Since the class of supersoluble group is a saturated formation by [14, 9.4.5], it is easy to get the result.

In the following, let $L$ be a minimal normal subgroup of $G$ contained in $H$. Then, by Step 1 and [10, Lemma 8. 6, p. 102] $L$ is an elementary abelian $p$-group for some prime divisor $p$ of $|G|$.

**Step 3.** $G/L$ is supersoluble and $L$ is the unique minimal normal subgroup of $G$ which is contained in $H$.

First, we check that $(G/L, H/L)$ satisfies the hypothesis as $(G, H)$. Let $\overline{Q} = QL/L$ be a Sylow $q$-subgroup of $H/L = \overline{H}$. Then $\overline{G} = G/L$. Hence we assume that $Q$ is a Sylow $q$-subgroup of $H$.

**Case a.** If $p = q$, we assume that $L < P$, then $P = Q > L$. Let $P_1$ be a maximal subgroup of $P$. By hypothesis $P_1$ is $nc$-supplemented in $G$, and by Lemma 2.6, $\overline{P_1}$ is $nc$-supplemented in $\overline{G}$. The minimality of $G$ implies that $\overline{G}$ is supersoluble.

**Case b.** Assume that $p \neq q$. Let $\overline{Q_1}$ be a maximal subgroup of a Sylow $q$-subgroup $\overline{Q}$ of $\overline{H}$. Without loss of generality, we assume that $\overline{Q_1} = Q_1L/L$. Since $Q_1$ is $nc$-supplemented in $G$, it follows, by Lemma 2.6, that $\overline{Q_1}$ is $nc$-supplemented in $\overline{G}$. The minimality of $G$ implies that $\overline{G}$ is supersoluble.

Now, let $R$ be another minimal normal subgroup of $G$ contained in $H$. Then $G/R$ is supersoluble by Step 3. Since $G/R \cap L \lesssim G/R \times G/L$, it follows, from [1, Theorem 3] that, $G/R \cap L$ is supersoluble. On the other hand, $R \cap L \leq L$ and so $R \cap L = 1$ or $R \cap L = L$ by the minimality of $L$. In the former case, $G/1 \cong G$ is supersoluble, a contradiction. In the latter, $L$ is unique.

**Step 4.** $L = F(H) = C_H(L)$.

Since $L$ is an elementary abelian normal subgroup of $G$, $L \leq H$. So by [11, 6.5.4], $F(H)$, the Fitting subgroup of $H$ contains every minimal normal subgroup of $H$. By [6, Theorem 1.9.17] and Step 2, $F(H)$ is the direct product of minimal normal subgroups of $G$ contained in $H$. Then $L = F(H)$ by Step 3. Since $G$ is soluble by Step 1, $F(H) \leq C_H(L) = C_H(F(H)) \leq F(H)$ by [19, Lemma 2.3].

**Step 5.** $L$ is a Sylow subgroup of $H$.

Let $q$ be the largest prime divisor of $|H|$ and let $Q$ be a Sylow $q$-subgroup of $H$. Since $H/L$ is supersoluble, it follows, by [9, VI-9.1(c)], that $LQ/L$ is characteristic in $G/L$ and so $LQ \trianglelefteq G$. Thus we have:

**Case a.** If $p = q$, then $L \leq P = Q \lhd G$. Therefore, by Step 1 and [4, Hilfssatz C], $L = Q$ is a Sylow subgroup of $H$.

**Case b.** If $p < q$, then $L \leq P$ and $PQ = PLQ$ is a subgroup of $G$. Since every maximal subgroup of all Sylow subgroups of $PQ$ is $nc$-supplemented in $PQ$ by Lemma 2.2(1), $PQ$ satisfies the hypothesis of the theorem. Then we have:

**Subcase a.** If $PQ < G$, then, by Step 1, $PQ$ is supersoluble and so $Q \lhd PQ$ by [9, VI-9.1]. Hence $LQ = L \times Q$ and so $Q \leq C_G(L) \leq L$ by [19, Lemma 2.3], a contradiction.

**Subcase b.** Assume that $PQ = H = G$ and $L < P$ in the case $Q \ntrianglelefteq G$. Since $L \cap N_G(Q) = 1$ and $LQ$ is characteristic in $H = PQ = G$, it follows that $G = [L]N_G(Q)$. Let $P_2$ be a Sylow $p$-subgroup of $N_G(Q)$. Then $LP_2$ is a Sylow $p$-subgroup of $G$. Choose a maximal subgroup $P_1$ of $LP_2$ with $P_2 \leq P_1$. Obviously, $L \nleq P_1$ and $P_{1G} = 1$. Otherwise, $L = P_{1G}$, which contradicts that $L \cap N_G(Q) = 1$. By hypotheses, $P_1$ is $nc$-supplemented in $G$, then there exists a subgroup $K$ such that $P_1K \lhd G$ and so $P_1 \cap K \leq P_{1G} = 1$. Hence if $K$ is a $q$-subgroup of a Sylow $q$-subgroup $Q$ of $G$, then $P_1K$ is supersoluble by Step 1 and $K$ is characteristic in $P_1K$ which is normal in $G$. Then $LK = L \times K$ and so, by [19, Lemma 2.3], $K \leq C_G(L) \leq L$, a contradiction. Thus we assume that $K$ is not a $q$-group. Since $|K|_p = |G : P_1|_p = p$, it follows that $K$ has a normal $p$-complement $Q^*$. Obviously, $P_1Q^*$ is a subgroup of $G$. By Step 1, $P_1Q^*$ is supersoluble. And so, by [9, VI-9.1], $Q^* \lhd P_1Q^*$. Thus $LQ^* = L \times Q^*$ and $Q^* \leq C_{P_1Q^*}(L) \leq L$ by [19, Lemma 2.3], a contradiction. So we have $P_1K = G$. Now $|K|_p = |G : P_1|_p = p$ implies that $K$ has a normal $p$-complement $Q_1$ which is also a Sylow $q$-subgroup of $G$. By [8, Theorem 4.2.2], there exists a $g \in LP_2 = P$ such that $Q_1^g = Q$. Since $P_1 \lhd P$, we have $G = P_1K = (P_1K)^g = P_1K^g$ and $P_1 \cap K^g = 1$. Since $K^g \cong K$ has a normal $p$-complement and $Q_1^g = Q \leq K^g$, it follows that $K^g \leq N_G(Q)$. Since $P = LP_2 = P_1LP_2 = P_1LP_2 \cap G = P_1(LP_2 \cap K^g)$, if $P_1(LP_2 \cap K^g) \leq P_2$, then $LP_2 \leq P_1P_2 \leq P_2$, a contradiction. So $P_1(LP_2 \cap K^g) \nleq P_2$ and $P_2$ must be a proper subgroup of $P_3 = \langle P_2, LP_2 \cap K^g \rangle$, where $P_3$ is a subgroup of a Sylow $p$-subgroup $P$. Thus $P_2$ and $K^g$ are contained in $N_G(Q)$ and so $P_3$ is a $p$-subgroup of $G$ containing a proper Sylow $p$-subgroup $P_2$ of $N_G(Q)$, a contradiction.

Thus $L$ is a Sylow subgroup of $H$.

**Step 6.** $|L| = p$.

Let $L_1$ be a maximal subgroup of $L$. Then, by hypothesis, $L_1$ is *nc*-supplemented in $G$ and so, by Lemma 2.5, there exists a subgroup $K$ of $G$ such that $L_1K \trianglelefteq G$ and $L_1 \cap K \leq L_{1G}$. By Step 3, $L_1K \geq L$, and so $L = L \cap (L_1K) = L_1(L \cap K)$. It follows that $L \cap K = L$ or $L \cap K < L$. In the first case, it is easy to get $L \cap K \triangleleft G$. In the second case, $L_1 \cap K < L_1 < L$, and so $L_1 \cap K = L_1 \cap K \cap K < L \cap K < L$. Since $L_1 \cap K \triangleleft G$ and $L \triangleleft G$, it follows that $L(L_1 \cap K) \triangleleft G$. As $L(L_1 \cap K) = (LL_1) \cap K = L \cap K$, we have $L \cap K \triangleleft G$ and so $L \cap K \geq L$ by the minimality and uniqueness of $L$. Then $L \cap K = L$ and so $L \leq K$. Hence $L_1 \cap K \leq L \cap K = L$ and so $L_1 \cap K = 1$. Thus $L_1 = 1$ and $|L| = p$.

**Step 7.** The final contradiction.

By Step 3, $G/L$ is supersoluble. By Step 6, $L$ is a cyclic subgroup of prime order. Then by Lemma 2.7, $G$ is supersoluble, a contradiction.

The final contradiction completes the proof. □

**Remark 3.1.** *The condition of Theorem 1.1 " $G/H$ is supersoluble " cannot be replaced by " $G/H$ is soluble ". Let $G = A_4 \times C_5$, where $A_4$ is the alternating group of degree 4 and $C_5$ is a cyclic group of order 5. Then $G/C_5 \cong A_4$ is soluble. Obviously, $C_5$ satisfies the hypotheses, but $G$ is not supersoluble.*

**Corollary 3.2.** ([3, Theorem 3.3]) *Let $G$ be a finite group and let $N$ be a normal subgroup of $G$ such that $G/N$ is supersoluble. If every maximal subgroup of every Sylow subgroup of $N$ is c-supplemented in $G$, then $G$ is supersoluble.*

**Corollary 3.3.** ([17, Theorem 1.1]) *Let $G$ be a finite group. Suppose $P_1$ is c-normal in $G$ for every Sylow subgroup $P$ of $G$ and every maximal subgroup $P_1$ of $P$. Then $G$ is supersoluble.*

**Corollary 3.4.** ([2, Theorem 3.2]) *Let $G$ be a finite solvable group. Then $G$ is supersoluble if and only if $G/H$ is supersoluble and all maximal subgroups of every Sylow subgroup of $F(H)$ are normal in $G$.*

**Corollary 3.5.** ([15, Theorem 1]) *Let $G$ be a finite group such that all maximal subgroups of Sylow subgroups are normal in $G$. Then $G$ is supersoluble.*

**Corollary 3.6.** ([13, Theorem 3.5]) *Assume that $G/H$ is supersolvable and all maximal subgroups of the Sylow subgroups of $H$ are normal in $G$. Then $G$ is supersolvable.*

## The proof of the theorem 1.2

*Proof.* Assume that the theorem is false. And suppose that $G$ is a counter-example of minimal order. By Lemma 2.6, we have that every maximal subgroup of the Sylow subgroups of $H$ is $nc$-supplemented in $H$ and so $G$ is soluble. Then by [12, Theorem 11], $H$ is soluble. We consider the following two cases:

**Case 1.** $H$ is a $p$-group for some prime number $p$.

**Step 1.** Let $N$ be the $\mathfrak{F}$-residual subgroup of $G$. Then $N = C_H(N) = F(H)$.

Let $M$ be a nontrivial normal subgroup of $G$ and let $B$ be a maximal subgroup of $MH$ with $M \leq B$. Then $B = M(H \cap B)$. Since $p = |MH : B| = |MH : M(H \cap B)| = |H : H \cap B|$, it follows that $H \cap B$ is a maximal subgroup of $H$. By hypothesis, $H \cap B$ is $nc$-supplemented in $G$ and so is $B$. Thus $B/M$ is $nc$-supplemented in $G/M$ by Lemma 2.6(2). The minimal choice of $G$ implies that $G/M \in \mathfrak{F}$. Since $N$ is the $\mathfrak{F}$-residual subgroup of $G$, it follows that $\Phi(G) = 1$ and $N$ is an elementary abelian subgroup of $G$ since $\mathfrak{F}$ is a saturated formation. Obviously $N \leq H$. Let $F(H)$ be the Fitting subgroup of $H$. Then $N = F(H)$ since $\mathfrak{F}$ is a saturated formation. Then $F(H) \leq C_H(N) \leq N$ since $H$ is solvable. Thus $N = C_H(N) = F(H)$ is a minimal normal nontrivial $p$-subgroup of $G$.

**Step 2.** $H$ is a Sylow $p$-subgroup of $G$.

Suppose that $H$ is not a Sylow $p$-subgroup of $G$ and $G$ is soluble. It follows, from [5, Theorem 3.5, p. 229], that there exists a Hall $\{p, q\}$-subgroup of $G$, where $q$ is a prime which is not equal to $p$, and that $HQ$ is a subgroup of $G$ since $H$ is normal in the Sylow $p$-subgroup of $G$ and $H \lhd G$. Since $G/H$ is supersoluble, $HQ/H$ is supersoluble . If $HQ < G$, then $HQ$ is supersoluble and so is $NQ$. Then $N \cap Q = 1$, and $NQ = N \times Q$ since $N \lhd NQ$ and $NQ$ is supersoluble. By [5, Theorem 1.3, p. 218], $Q \leq C_G(N) \leq N$, a contradiction. So $H$ is a Sylow $p$-subgroup of $G$.

**Step 3.** $|N| = p$.

Let $H_1$ be a maximal subgroup of $H$. Then $N < H_1$. Otherwise, $N = H_1 \lhd G$, it follows, from [17, Theorem 1.1], that $G \in \mathfrak{F}$. $H_1$ is $nc$-supplemented in $G$ by hypothesis and so there exists a subgroup $K$ of $G$ such that $H_1K \lhd G$ and $H_1 \cap K \leq H_{1G}$. Thus we have that $H_1 \cap K = 1$ or $H_1 \cap K = N$. If the former, $H_1K \geq H$ or $H_1K = H_1$ and so $K \geq H$ or $H_1 \geq K$, which contradicts $H_1 \cap K = 1$.

Hence $N \leq K$ and $N$ is a Sylow $p$-subgroup of $K$. If $N$ is not a Sylow $p$-subgroup of $K$, then there is a Sylow $p$-subgroup $P_K$ of $G$ with $N < P_K$, and so $H_1P_K = H$ or $H_1P_K = H_1$. In the former case, $P_K = H$ and so $H_1 \cap K = H_1 \cap H = H_1 \lhd G$. It follows, from [13, Theorem 3.5], that $G$ is supersoluble, a contradiction. In the latter, $N < P_K \leq H_1$ and so $N = H_1 \cap K = H_1 \cap P_K = P_K > N$, another contradiction. Thus $N$ is a normal Sylow $p$-subgroup of $K$. By Step 2, $K < G$ and so $HK < G$. Since $HK/H$ is supersoluble and every maximal subgroup of $H$ is $nc$-supplemented in $HK$, it follows, from the minimal choice of $G$ that, $HK$ is supersoluble and so $K$ is supersoluble. Let $Q$ be a Sylow $q$-subgroup of $K$, where $q$ is the largest prime of $|K|$. Thus $Q$ is normal in $K$, and $NQ = N \times Q$. This means $Q \leq C_K(N) \leq N$, a contradiction. Hence there does not exist non-trivial maximal subgroup of $H$, that is, $H$ is a Sylow $p$-subgroup of $G$ of order $p$. Namely, $|H| = |N| = p$.

**Step 4.** The final contradiction.

By Step 3, $H$ is a cyclic subgroup. By Lemma 2.7, $G \in \mathfrak{F}$, a contradiction.

**Case 2.** $H$ is not of prime power order.

Let $P$ be a Sylow $p$-subgroup of $H$. Then by hypothesis and Lemma 2.6(1), the maximal subgroups of every Sylow subgroup of $H$ are $nc$-supplemented in $H$. Then by Theorem 1.1, $H$ is supersoluble, and so by [4, Hillssatz C] $H$ has a normal Sylow subgroup $P$

Since $P$ is characteristic in $H$ and $H \lhd G$, it follows that $P \lhd G$. Clearly, $(G/P)/(H/P) \cong G/H \in \mathfrak{F}$. By the minimality of $G$, $G/P \in \mathfrak{F}$. But now $G \in \mathfrak{F}$ by Case 1, a contradiction.

So the minimal counter-example does not exist.

This completes the proof. □

**Remark 3.7.** *The condition of Theorem 1.2, "* $\mathfrak{U}$ *" cannot be replaced by "* $\mathfrak{N}$*", where* $\mathfrak{N}$ *is the class of all nilpotent groups. Let* $G = S_3$ *the symmetric group of degree 3. Then* $G$ *is supersoluble, but* $G$ *not nilpotent.*

**Corollary 3.8.** *(*[18, Theorem 1]*) Let* $\mathfrak{F}$ *be a saturated formation containing* $\mathfrak{U}$*. Suppose that* $G$ *is a group with a soluble normal subgroup* $H$ *such that* $G/H \in \mathfrak{F}$*. If all maximal subgroups of all Sylow subgroups of* $F(H)$ *are c-normal in* $G$*, then* $G \in \mathfrak{F}$*.*

**Corollary 3.9.** *(*[19, Theorem 3.1]*) Let* $\mathfrak{F}$ *be a saturated formation containing* $\mathfrak{U}$*. Suppose that* $G$ *is a group with a normal subgroup* $H$ *such* *that $G/H \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F^*(H)$ are c-normal in $G$, then $G \in \mathfrak{F}$.*

**Corollary 3.10.** *(*[20, Theorem 1.2]*) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a group $G$ with a normal subgroup $H$ such that $G/H \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F^*(H)$ are c-supplemented in $G$, then $G \in \mathfrak{F}$.*



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